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Low complexity robust adaptive beamforming for general-rank signal model with positive semidefinite constraint*

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Abstract: We propose a low complexity robust beamforming method for the general-rank signal model, to combat against mismatches of the desired signal array response and the received signal covariance matrix. The proposed beamformer not only considers the norm bounded uncertainties in the desired and received signal covariance matrices, but also includes an additional positive semidefinite constraint on the desired signal covariance matrix. Based on the worst-case performance optimization criterion, a computationally simple closed-form weight vector is obtained. Simulation results verify the validity and robustness of the proposed beamforming method.

Key words: Beamforming, General-rank, Low complexity, Positive semidefinite (PSD) constraint, Model mismatches
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1 Introduction

Adaptive beamforming has found numerous applications in radar, sonar, radio astronomy, wireless communications, medical imaging, and other areas (van Trees, 2002; Vorobyov, 2012). One of the most commonly used adaptive beamformers is the minimum variance distortionless response (MVDR) beamformer, also known as the Capon beamformer (Capon, 1969). However, the MVDR beamformer is sensitive to model mismatches, such as the desired signal array response mismatch and the received signal covariance matrix mismatch, especially when the desired signal is present in the training snapshots. Over the last decades, many robust adaptive beamforming (RAB) methods (Cox *et al.*, 1987; Li *et al.*, 2003; Vorobyov *et al.*, 2003; Hassanien *et al.*, 2008; Gu and Leshem, 2012) have been developed to improve the robustness against various model mismatches. Although these methods are efficient for the point signal source model (i.e., rank-one model), most of them cannot be directly extended to the case in which the rank of the desired signal covariance matrix is higher than one. Recently, a class of general-rank RAB methods has been developed to overcome this difficulty (Shahbazpanahi *et al.*, 2003; Chen and Gershman, 2008; Chen and Gershman, 2011; Zhang and Liu, 2012a; 2012b; Khabbazibasmenj and Vorobyov, 2013). Based on the explicit modeling of uncertainties in the desired and received signal covariance matrices, Shahbazpanahi *et al.* (2003) proposed the general-rank RAB method for the first time and derived a closed-form solution. However, they ignored the positive semidefinite (PSD) requirement for the desired signal covariance matrix, leading to an overly conservative beamformer design (Chen and Gershman, 2008). To address this problem, less conservative beamforming methods have been developed by imposing an additional PSD constraint on the desired signal covariance matrix (Chen and Gershman, 2008; Chen and Gershman, 2011; Khabbazibasmenj

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and Vorobyov, 2013). The beamforming method in Khabbazibasmenj and Vorobyov (2013) is proven to be able to find the globally optimal solution, provided that the mismatch of the desired signal covariance matrix is sufficiently small. Although the beamforming methods in Chen and Gershman (2008), Chen and Gershman (2011), and Khabbazibasmenj and Vorobyov (2013) outperform the one in Shahbazpanahi *et al.* (2003), they all suffer from high computational complexity since iterative semidefinite programming (SDP) algorithms have to be employed to solve these beamforming problems.

In this paper, the RAB problem is considered for the general-rank signal model. A low complexity RAB method is proposed to improve the beamformer's robustness. The proposed RAB method considers the PSD constraint on the desired signal covariance matrix, which is constructed by a product of a matrix and its conjugate transpose so that the PSD constraint can be met indirectly. Based on the explicit models, a simple closed-form expression for the beamforming weight vector is obtained by using the worst-case performance optimization criterion. Simulation results show that the proposed RAB method can achieve superior performance compared with state-of-the-art counterparts at a low computational cost.

2 Signal model

The narrowband signal received by an array consisting of M sensors at time instant k can be written as

$$\boldsymbol{x}(k)=\boldsymbol{s}(k)+\boldsymbol{i}(k)+\boldsymbol{n}(k), \tag{1}$$

where $\boldsymbol{s}(k)$, $\boldsymbol{i}(k)$, and $\boldsymbol{n}(k)$ are the statistically independent $M\times1$ vectors of the desired signal, interference, and noise, respectively. The beamformer output is given by

$$y(k)=\boldsymbol{w}^{\mathrm{H}}\boldsymbol{x}(k), \tag{2}$$

where $\boldsymbol{w}=[w_1, w_2, \ldots, w_M]^{\mathrm{T}}$ is a complex weight vector, and $(\cdot)^{\mathrm{T}}$ and $(\cdot)^{\mathrm{H}}$ denote the transpose and complex conjugate transpose, respectively. The output signal-to-interference-plus-noise ratio (SINR) of the beamformer is defined as

$$\text{SINR}=\frac{\boldsymbol{w}^{\mathrm{H}}\boldsymbol{R}_s\boldsymbol{w}}{\boldsymbol{w}^{\mathrm{H}}\boldsymbol{R}_{i+n}\boldsymbol{w}}, \tag{3}$$

where $\boldsymbol{R}_s=E\{\boldsymbol{s}(k)\boldsymbol{s}^{\mathrm{H}}(k)\}$ and $\boldsymbol{R}_{i+n}=E\{[\boldsymbol{i}(k)+\boldsymbol{n}(k)][\boldsymbol{i}(k)+\boldsymbol{n}(k)]^{\mathrm{H}}\}$ are the desired signal covariance matrix and interference-plus-noise covariance matrices, respectively, and $E\{\cdot\}$ denotes the statistical expectation. Note that the desired signal matrix can be of arbitrary rank, i.e.,

$$1\le\text{rank}\{\boldsymbol{R}_s\}\le M, \tag{4}$$

where $\text{rank}\{\cdot\}$ represents the rank operator. In the particular case of point signal source, the rank of matrix $\boldsymbol{R}_s$ is one. However, in many practical situations, the rank of matrix $\boldsymbol{R}_s$ is always higher than one, for example, in the scenarios with incoherently scattered signal sources or signals with fluctuating waveforms (Shahbazpanahi *et al.*, 2003).

3 Problem formulation

The problem of maximizing formula (3), known as MVDR beamforming, is mathematically equivalent to

$$\min_{\boldsymbol{w}}\ \boldsymbol{w}^{\mathrm{H}}\boldsymbol{R}_x\boldsymbol{w}\ \ \text{s.t.}\ \boldsymbol{w}^{\mathrm{H}}\boldsymbol{R}_s\boldsymbol{w}=1, \tag{5}$$

where $\boldsymbol{R}_x=E\{\boldsymbol{x}(k)\boldsymbol{x}^{\mathrm{H}}(k)\}$ is the received signal covariance matrix. If $\boldsymbol{R}_s$ and $\boldsymbol{R}_x$ are known exactly, the optimal solution can be found as (Shahbazpanahi *et al.*, 2003)

$$\boldsymbol{w}_{\text{opt}}=P\{\boldsymbol{R}_x^{-1}\boldsymbol{R}_s\}, \tag{6}$$

which is the MVDR beamformer for the general-rank signal model. Herein, $P\{\cdot\}$ denotes the principal eigenvector of a matrix.

In practice, the true values of $\boldsymbol{R}_s$ and $\boldsymbol{R}_x$ are unavailable, and they are often replaced by their estimates. So, $\boldsymbol{R}_s$ and $\boldsymbol{R}_x$ can be modeled as $\boldsymbol{R}_s=\widehat{\boldsymbol{R}}_s+\boldsymbol{\varDelta}_1$ and $\boldsymbol{R}_x=\widehat{\boldsymbol{R}}_x+\boldsymbol{\varDelta}_2$, respectively, where $\widehat{\boldsymbol{R}}_s$ and $\widehat{\boldsymbol{R}}_x$ are the estimates of $\boldsymbol{R}_s$ and $\boldsymbol{R}_x$, respectively, and $\boldsymbol{\varDelta}_1$ (or $\boldsymbol{\varDelta}_2$) is the corresponding estimation uncertainty. It is known that these uncertainties will lead to a significant loss in the performance of the MVDR beamformer (Shahbazpanahi *et al.*, 2003). To alleviate the performance degradation, Shahbazpanahi *et al.* (2003) introduced the explicit models of uncertainty, i.e., $\|\boldsymbol{\Delta}_1\|\le\varepsilon$ and $\|\boldsymbol{\Delta}_2\|\le\gamma$, where $\|\cdot\|$ denotes the Fresenius norm, and ε and γ are the uncertainty bounds on $\boldsymbol{\Delta}_1$ and $\boldsymbol{\Delta}_2$, respectively. Consequently, problem (5) is transformed into (Shahbazpanahi *et al.*, 2003)

$$\begin{aligned}&\min_{\boldsymbol{w}}\ \max_{\|\boldsymbol{\Delta}_2\|\le\gamma}\boldsymbol{w}^{\mathrm{H}}(\widehat{\boldsymbol{R}}_x+\boldsymbol{\Delta}_2)\boldsymbol{w}\\&\text{s.t.}\ \min_{\|\boldsymbol{\Delta}_1\|\le\varepsilon}\boldsymbol{w}^{\mathrm{H}}(\widehat{\boldsymbol{R}}_s+\boldsymbol{\Delta}_1)\boldsymbol{w}\ge1.\end{aligned}\tag{7}$$

Using the worst-case performance optimization criterion, the solution can be derived by

$$\boldsymbol{w}=P\{(\widehat{\boldsymbol{R}}_x+\gamma\boldsymbol{I})^{-1}(\widehat{\boldsymbol{R}}_s-\varepsilon\boldsymbol{I})\},\tag{8}$$

where $\boldsymbol{I}$ is the identity matrix.

Obviously, $\boldsymbol{R}_s=\widehat{\boldsymbol{R}}_s+\boldsymbol{\Delta}_1$ is PSD, since it is a covariance matrix. However, the PSD requirement for matrix $\widehat{\boldsymbol{R}}_s+\boldsymbol{\Delta}_1$ is not considered in the RAB problem (7), which may easily cause the worst-case desired signal matrix $\widehat{\boldsymbol{R}}_s-\varepsilon\boldsymbol{I}$ to be indefinite or negative definite (Chen and Gershman, 2008). Note that the worst-case sample covariance matrix $\widehat{\boldsymbol{R}}_x+\gamma\boldsymbol{I}$ is always positive definite. If constraint $\widehat{\boldsymbol{R}}_s+\boldsymbol{\Delta}_1\succeq0$ is added in problem (7) to ensure matrix $\widehat{\boldsymbol{R}}_s+\boldsymbol{\Delta}_1$ being PSD, the optimization problem will be hard to solve. In Chen and Gershman (2008), the PSD constraint $\widehat{\boldsymbol{R}}_s+\boldsymbol{\Delta}_1\succeq0$ was achieved by first factorizing $\widehat{\boldsymbol{R}}_s$ into $\widehat{\boldsymbol{R}}_s=\boldsymbol{Q}^{\mathrm{H}}\boldsymbol{Q}$, and then constructing matrix $\widehat{\boldsymbol{R}}_s+\boldsymbol{\Delta}_1$ such that $\widehat{\boldsymbol{R}}_s+\boldsymbol{\Delta}_1=(\boldsymbol{Q}+\boldsymbol{\Delta})^{\mathrm{H}}(\boldsymbol{Q}+\boldsymbol{\Delta})$, where $\boldsymbol{\Delta}$ is a norm bounded mismatch uncertainty in the square-root matrix $\boldsymbol{Q}$ with $\|\boldsymbol{\Delta}\|\le\eta$. Accordingly, the RAB problem for the general-rank signal model becomes

$$\begin{aligned}&\min_{\boldsymbol{w}}\ \max_{\|\boldsymbol{\Delta}_2\|\le\gamma}\boldsymbol{w}^{\mathrm{H}}(\widehat{\boldsymbol{R}}_x+\boldsymbol{\Delta}_2)\boldsymbol{w}\\&\text{s.t.}\ \min_{\|\boldsymbol{\Delta}\|\le\eta}\ \boldsymbol{w}^{\mathrm{H}}(\boldsymbol{Q}+\boldsymbol{\Delta})^{\mathrm{H}}(\boldsymbol{Q}+\boldsymbol{\Delta})\boldsymbol{w}\ge1.\end{aligned}\tag{9}$$

In problem (9), the maximum of the quadratic term $\boldsymbol{w}^{\mathrm{H}}(\widehat{\boldsymbol{R}}_x+\boldsymbol{\Delta}_2)\boldsymbol{w}$ in the objective function with respect to $\boldsymbol{\Delta}_2$ can be easily derived as $\boldsymbol{w}^{\mathrm{H}}(\widehat{\boldsymbol{R}}_x+\gamma\boldsymbol{I})\boldsymbol{w}$ by using the Lagrange multiplier. After some algebraic simplifications, the minimum of the quadratic term $\boldsymbol{w}^{\mathrm{H}}(\boldsymbol{Q}+\boldsymbol{\Delta})^{\mathrm{H}}(\boldsymbol{Q}+\boldsymbol{\Delta})\boldsymbol{w}$ in the constraint with respect to $\boldsymbol{\Delta}$ can be obtained as $(\|\boldsymbol{Q}\boldsymbol{w}\|-\eta\|\boldsymbol{w}\|)^2$ with the condition of $\|\boldsymbol{Q}\boldsymbol{w}\|>\eta\|\boldsymbol{w}\|$. For a detailed derivation, see Appendix I-A in Khabbazibasmenj and Vorobyov (2013). Thus, the constraint in problem (9) can be equivalently replaced by $\|\boldsymbol{Q}\boldsymbol{w}\|-\eta\|\boldsymbol{w}\|\ge1$. Based on the results described above, problem (9) is equivalent to

$$\begin{aligned}&\min_{\boldsymbol{w}}\ \ \boldsymbol{w}^{\mathrm{H}}(\widehat{\boldsymbol{R}}_x+\gamma\boldsymbol{I})\boldsymbol{w}\\&\text{s.t.}\ \ \|\boldsymbol{Q}\boldsymbol{w}\|-\eta\|\boldsymbol{w}\|\ge1.\end{aligned}\tag{10}$$

It is obvious that this optimization problem is non-convex because of the non-convex constraint. Many iterative SDP algorithms have been developed to solve problem (10) (Chen and Gershman, 2008; Chen and Gershman, 2011; Khabbazibasmenj and Vorobyov, 2013). However, high computational cost makes the online and real-time processing difficult to achieve. A computationally efficient algorithm is required to solve problem (10).

4 The proposed algorithm

In this section, a low-complexity RAB approach is proposed for the general-rank signal model with the PSD constraint. Specifically, a closed-form solution to the considered beamforming problem is derived in the following.

It is obvious that problem (10) can be rewritten as

$$\begin{aligned}&\min_{\boldsymbol{w}}\ \ \boldsymbol{w}^{\mathrm{H}}(\widehat{\boldsymbol{R}}_x+\gamma\boldsymbol{I})\boldsymbol{w}\\&\text{s.t.}\ \ (\|\boldsymbol{Q}\boldsymbol{w}\|-\eta\|\boldsymbol{w}\|)^2\ge1,\\&\qquad\|\boldsymbol{Q}\boldsymbol{w}\|>\eta\|\boldsymbol{w}\|.\end{aligned}\tag{11}$$

To solve this non-convex problem, we temporarily drop constraint $\|\boldsymbol{Q}\boldsymbol{w}\|>\eta\|\boldsymbol{w}\|$, and then problem (11) reduces to

$$\begin{aligned}&\min_{\boldsymbol{w}}\ \ \boldsymbol{w}^{\mathrm{H}}(\widehat{\boldsymbol{R}}_x+\gamma\boldsymbol{I})\boldsymbol{w}\\&\text{s.t.}\ \ (\|\boldsymbol{Q}\boldsymbol{w}\|-\eta\|\boldsymbol{w}\|)^2\ge1.\end{aligned}\tag{12}$$

Applying the Cauchy-Schwarz inequality to the constraint in problem (12), we have

$$\begin{aligned}&(\|\boldsymbol{Q}\boldsymbol{w}\|-\eta\|\boldsymbol{w}\|)^2\\&=\boldsymbol{w}^{\rm H}\widehat{\boldsymbol{R}}_s\boldsymbol{w}+\eta^2\|\boldsymbol{w}\|^2-2\eta\|\boldsymbol{Q}\boldsymbol{w}\|\cdot\|\boldsymbol{w}\|\\&\geq\boldsymbol{w}^{\rm H}\widehat{\boldsymbol{R}}_s\boldsymbol{w}+\eta^2\|\boldsymbol{w}\|^2-2\eta\|\boldsymbol{Q}\|\cdot\|\boldsymbol{w}\|^2\\&=\boldsymbol{w}^{\rm H}\left[\widehat{\boldsymbol{R}}_s+\left(\eta^2-2\eta\sqrt{{\rm tr}(\widehat{\boldsymbol{R}}_s)}\right)\boldsymbol{I}\right]\boldsymbol{w},\end{aligned}\tag{13}$$

where the property $\|\boldsymbol{Q}\|=\sqrt{\boldsymbol{Q}^{\rm H}\boldsymbol{Q}}$ and the definition $\widehat{\boldsymbol{R}}_s=\boldsymbol{Q}^{\rm H}\boldsymbol{Q}$ are used, and ${\rm tr}(\cdot)$ denotes the trace operator. Involving a worst-case problem approximation in the constraint, problem (12) can therefore be approximated as

$$\begin{aligned}&\min_{\boldsymbol{w}}\quad \boldsymbol{w}^{\rm H}(\widehat{\boldsymbol{R}}_x+\gamma\boldsymbol{I})\boldsymbol{w}\\&\text{s.t.}\quad \boldsymbol{w}^{\rm H}\left[\widehat{\boldsymbol{R}}_s+(\eta^2-2\eta\sqrt{{\rm tr}(\widehat{\boldsymbol{R}}_s)})\boldsymbol{I}\right]\boldsymbol{w}\geq 1.\end{aligned}\tag{14}$$

Problem (14) is equivalent to problem (12) only when $\|\boldsymbol{Qw}\|=\|\boldsymbol{w}\|$. Otherwise, problem (14) will be an approximation of problem (12) by shrinking the feasible region, and thus it may not provide a globally optimal solution to problem (12). In problem (14), the inequality constraint is satisfied by equality if the optimal solution is achieved, which can be proved by contradiction (see Appendix A). Thus, problem (14) can be further recast as

$$\begin{aligned}&\min_{\boldsymbol{w}}\quad \boldsymbol{w}^{\rm H}(\widehat{\boldsymbol{R}}_x+\gamma\boldsymbol{I})\boldsymbol{w}\\&\text{s.t.}\quad \boldsymbol{w}^{\rm H}\left[\widehat{\boldsymbol{R}}_s+(\eta^2-2\eta\sqrt{{\rm tr}(\widehat{\boldsymbol{R}}_s)})\boldsymbol{I}\right]\boldsymbol{w}=1.\end{aligned}\tag{15}$$

The optimal solution to problem (15) can be obtained by using the Lagrange multiplier method, based on the following function:

$$\begin{aligned}G(\boldsymbol{w},\lambda)=&\,\boldsymbol{w}^{\rm H}(\widehat{\boldsymbol{R}}_x+\gamma\boldsymbol{I})\boldsymbol{w}\\&+\lambda\left\{1-\boldsymbol{w}^{\rm H}\left[\widehat{\boldsymbol{R}}_s+\left(\eta^2-2\eta\sqrt{{\rm tr}(\widehat{\boldsymbol{R}}_s)}\right)\boldsymbol{I}\right]\boldsymbol{w}\right\}.\end{aligned}\tag{16}$$

Nulling the derivative of Eq. (16) with respect to $\boldsymbol{w}$ results in

$$(\widehat{\boldsymbol{R}}_x+\gamma\boldsymbol{I})\boldsymbol{w}=\lambda\left[\widehat{\boldsymbol{R}}_s+\left(\eta^2-2\eta\sqrt{{\rm tr}(\widehat{\boldsymbol{R}}_s)}\right)\boldsymbol{I}\right]\boldsymbol{w}.\tag{17}$$

It follows that problem (15) can be viewed as a generalized eigenvalue problem, where the variable vector $\boldsymbol{w}$ is the generalized eigenvector of matrix pencil $\left\{\widehat{\boldsymbol{R}}_x+\gamma\boldsymbol{I},\widehat{\boldsymbol{R}}_s+(\eta^2-2\eta\sqrt{{\rm tr}(\widehat{\boldsymbol{R}}_s)})\boldsymbol{I}\right\}$ (λ is the corresponding generalized eigenvalue).

Premultiplying both sides of Eq. (17) with $\boldsymbol{w}^{\rm H}$ yields

$$\boldsymbol{w}^{\rm H}(\widehat{\boldsymbol{R}}_x+\gamma\boldsymbol{I})\boldsymbol{w}=\lambda\boldsymbol{w}^{\rm H}\left[\widehat{\boldsymbol{R}}_s+\left(\eta^2-2\eta\sqrt{{\rm tr}(\widehat{\boldsymbol{R}}_s)}\right)\boldsymbol{I}\right]\boldsymbol{w}.\tag{18}$$

Based on constraint $\boldsymbol{w}^{\rm H}[\widehat{\boldsymbol{R}}_s+(\eta^2-2\eta\sqrt{{\rm tr}(\widehat{\boldsymbol{R}}_s)})\boldsymbol{I}]\boldsymbol{w}=1$, we can obtain $\boldsymbol{w}^{\rm H}(\widehat{\boldsymbol{R}}_x+\gamma\boldsymbol{I})\boldsymbol{w}=\lambda$. Thus, the optimal solution minimizing the objective function of problem (15) is actually the generalized eigenvector of $\left\{\widehat{\boldsymbol{R}}_x+\gamma\boldsymbol{I},\widehat{\boldsymbol{R}}_s+(\eta^2-2\eta\sqrt{{\rm tr}(\widehat{\boldsymbol{R}}_s)})\boldsymbol{I}\right\}$ corresponding to the smallest positive generalized eigenvalue.

Eq. (17) can be easily reshaped as

$$(\widehat{\boldsymbol{R}}_x+\gamma\boldsymbol{I})^{-1}\left\{\widehat{\boldsymbol{R}}_s+\left[\eta^2-2\eta\sqrt{{\rm tr}(\widehat{\boldsymbol{R}}_s)}\right]\boldsymbol{I}\right\}\boldsymbol{w}=\frac{1}{\lambda}\boldsymbol{w},\tag{19}$$

which indicates that problem (15) can also be viewed as an eigenvalue problem, where $\boldsymbol{w}$ is the eigenvector of matrix $(\widehat{\boldsymbol{R}}_x+\gamma\boldsymbol{I})^{-1}\left[\widehat{\boldsymbol{R}}_s+(\eta^2-2\eta\sqrt{{\rm tr}(\widehat{\boldsymbol{R}}_s)})\boldsymbol{I}\right]$ and $1/\lambda$ is the corresponding eigenvalue. Since the smallest positive λ leads to the maximum value of $1/\lambda$, the optimal solution to problem (15) is given by the principle eigenvector of matrix $(\widehat{\boldsymbol{R}}_x+\gamma\boldsymbol{I})^{-1}\cdot(\widehat{\boldsymbol{R}}_{\rm s}+(\eta^2-2\eta\sqrt{{\rm tr}(\widehat{\boldsymbol{R}}_{\rm s})})\boldsymbol{I})$, i.e.,

$$\boldsymbol{w}_{\rm rob}=P\left\{(\widehat{\boldsymbol{R}}_x+\gamma\boldsymbol{I})^{-1}\left[\widehat{\boldsymbol{R}}_s+\left(\eta^2-2\eta\sqrt{{\rm tr}(\widehat{\boldsymbol{R}}_s)}\right)\boldsymbol{I}\right]\right\}.\tag{20}$$

This means that problem (12) has a closed-form solution given by Eq. (20). Note that problem (12) is a relaxed version of problem (11) without considering the constraint $\|\boldsymbol{Qw}\|>\eta\|\boldsymbol{w}\|$. Fortunately, the weight vector $\boldsymbol{w}_{\rm rob}$ is proved to satisfy constraint $\|\boldsymbol{Qw}\|>\eta\|\boldsymbol{w}\|$ (see Appendix B), and therefore it is also the solution to problem (11) or (10).

Here, we compare the computational complexities of the proposed beamforming method with that of the iterative SDP beamforming methods in Chen and Gershman (2008), Chen and Gershman (2011), and Khabbazibasmenj and Vorobyov (2013) for the general-rank signal model. In the batch processing mode, it can be seen from Eq. (20) that the computational complexity of the proposed beamforming method is dominated mainly by the matrix inversion and eigen-decomposition. For an $M\times M$ matrix, these two operations both require $O(M^3)$ computations. Hence, the overall complexity of the proposed beamforming method is $O(M^3)$. The algorithms proposed in Chen and Gershman (2008), Chen and Gershman (2011), and Khabbazibasmenj and Vorobyov (2013) are all iterative algorithms, and each iteration involves solving an SDP problem. Usually, the SDP problems are solved numerically using interior point methods with the computational complexity of $O(M^{3.5}\log(1/\varsigma))$ (de Maio *et al.*, 2010), where ς is a prefixed accuracy. Therefore, if K iterations are involved, the complexity of the iterative algorithms in Chen and Gershman (2008), Chen and Gershman (2011), and Khabbazibasmenj and Vorobyov (2013) will be $O(KM^{3.5}\log(1/\varsigma))$. Apparently, the proposed method is more computationally efficient than its counterparts.

5 Simulation results

A uniform linear array with M isotropic sensors spaced half a wavelength apart is considered in simulation. An interference source, which presents interference in each sensor with a 20 dB interference-to-noise ratio (INR), is assumed to impinge on the array. Both the desired and interference sources are locally incoherently scattered sources with the same angular spread of 4°. The desired and interference sources have Gaussian and uniform angular power densities characterized by the central angles of 30° and −30°, respectively. We assume that the presumed desired source also has a Gaussian shaped angular power density, but the central angle and angular spread are 32° and 6°, respectively. The estimate of the desired signal covariance matrix is calculated as $\widehat{\boldsymbol{R}}_s=(\boldsymbol{a}(\widehat{\theta})\boldsymbol{a}^{\mathrm{H}}(\widehat{\theta}))\odot\boldsymbol{B}(\widehat{\theta},\widehat{\sigma}_\phi)$ (Trump and Ottersten, 1996), where $\widehat{\theta}$ and $\widehat{\sigma}_\phi$ are the presumed central angle and angular spread, respectively, $\boldsymbol{a}(\widehat{\theta})$ is the steering vector associated with direction $\widehat{\theta}$, the kth row and lth column element of matrix $\boldsymbol{B}(\widehat{\theta},\widehat{\sigma}_\phi)$ is $B_{kl}=\exp\{-2[\pi\varDelta(k-l)]^2\widehat{\sigma}_\phi^2\cos^2\widehat{\theta}\}$ ($\varDelta$ is the separation between two adjacent sensors in wavelengths), and $\odot$ is the Schur-Hadamard inner product. The performance of the proposed beamforming method is compared with that of the general-rank RAB methods in Shahbazpanahi *et al.* (2003), Chen and Gershman (2008), and Khabbazibasmenj and Vorobyov (2013). As suggested in Shahbazpanahi *et al.* (2003), the diagonal loading parameter γ=30 is chosen for all the tested RAB methods, and the value $\varepsilon=9\mathrm{tr}(\boldsymbol{R}_s)/M$ is used for the RAB method in Shahbazpanahi *et al.* (2003). We select $\eta=0.75\sqrt{\mathrm{tr}(\boldsymbol{R}_s)}$ for the proposed RAB method and the RAB methods in Chen and Gershman (2008) and Khabbazibasmenj and Vorobyov (2013). For each scenario in the simulation, 200 independent Monte-Carlo trials are performed.

In the first simulation, the effects of SNR and the number of snapshots on the array output SINR are respectively investigated, where the number of sensors (M) is set to 20. Fig. 1a shows the output SINRs of the tested beamforming methods versus the input signal-to-noise ratio (SNR) for a fixed number of snapshots (N=50). The general-rank RABs in Chen and Gershman (2008), Khabbazibasmenj and Vorobyov (2013), and the proposed beamforming method all achieve higher output SINRs than that in Shahbazpanahi *et al.* (2003). This is mainly due to the consideration of the PSD constraint on the desired signal covariance matrix. Moreover, the output SINRs of the proposed beamformer are very close to to that of the general-rank RAB in Khabbazibasmenj and Vorobyov (2013). Interestingly, according to the analysis in Section 4, the computation cost of the proposed closed-form beamforming method is much lower than that of the general-rank RAB in Khabbazibasmenj and Vorobyov (2013) (i.e., iterative SDP formulation). Fig. 1b shows the output SINRs versus the number of snapshots for a fixed SNR of 15 dB. The results are similar to those in Fig. 1a, which also validates the analysis above.

In the second simulation, the influence of the number of sensors on algorithm complexity is considered. The computational complexities of the algorithms are measured in terms of average CPU time per Monte-Carlo run. The simulation has been conducted using MATLAB (version 2012b) on a PC with a

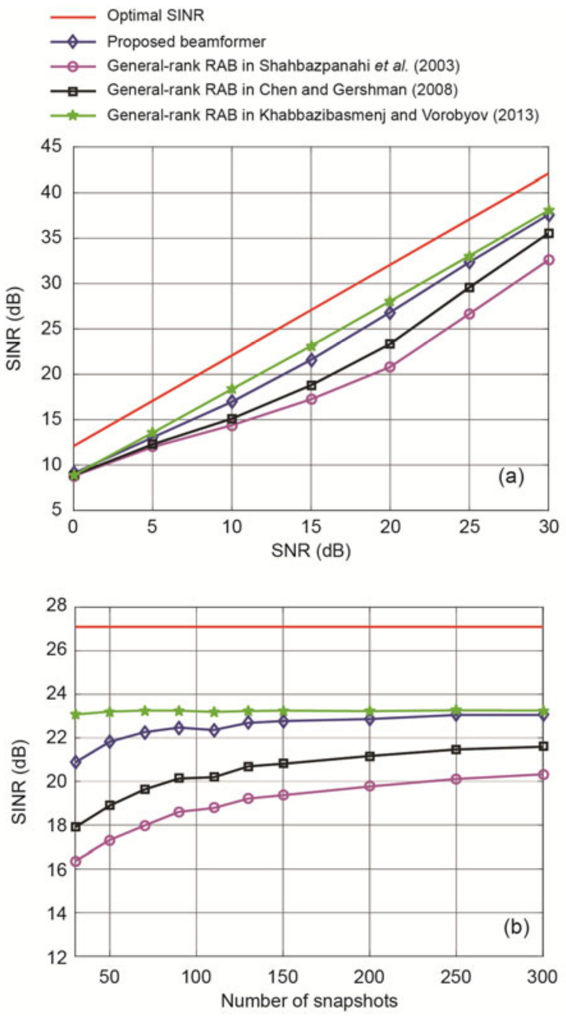

Fig. 1 Output SINR versus SNR (a) and the number of snapshots (b)

3.40 GHz Intel Core i5 processor and 4 GB RAM. Besides, the Matlab software, CVX (Grant *et al.*, 2015), is used to solve the iterative SDP algorithms in Chen and Gershman (2008) and Khabbazibasmenj and Vorobyov (2013). Fig. 2 shows the CPU time against the number of sensors, where SNR=10 dB and N=20. The complexity of the proposed algorithm is comparable to that in Shahbazpanahi *et al.* (2003), but it is far less than those in Chen and Gershman (2008) and Khabbazibasmenj and Vorobyov (2013).

6 Conclusions

In this paper, we proposed a low-complexity robust beamformer for the general-rank signal model with the PSD constraint. Using the worst-case

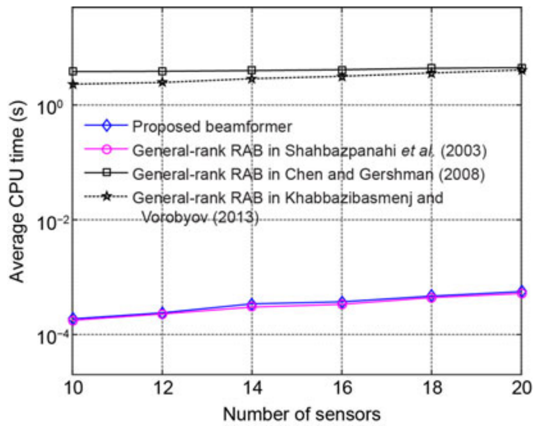

Fig. 2 Average CPU time versus the number of sensors

performance optimization criterion, a computationally simple closed-form weight vector was obtained. Simulation results demonstrated the superiority of the proposed method over its counterparts.

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Appendix A: Proof by contradiction

Assume that the objective function $\boldsymbol{w}^{\mathrm{H}}(\hat{\boldsymbol{R}}_x+\gamma\boldsymbol{I})\boldsymbol{w}$ in problem (14) is minimized under the constraint $\boldsymbol{w}^{\mathrm{H}}[\hat{\boldsymbol{R}}_s+(\eta^2-2\eta\sqrt{\mathrm{tr}(\hat{\boldsymbol{R}}_s)})\boldsymbol{I}]\boldsymbol{w}=\kappa>1$. Then, replacing $\boldsymbol{w}$ with $\boldsymbol{w}/\sqrt{\kappa}$, we can decrease the objective function $\boldsymbol{w}^{\mathrm{H}}(\hat{\boldsymbol{R}}_x+\gamma\boldsymbol{I})\boldsymbol{w}$ by a factor of κ, which is contrary to our assumption. Therefore, the minimum of the objective function is achieved at κ=1; i.e., the optimal solution to problem (14) is achieved when the inequality constraint is satisfied by equality.

Appendix B: Property of $\boldsymbol{w}_{\mathrm{rob}}$

Before proceeding, we prove that $0<\eta<\sqrt{\lambda_{\max}(\hat{\boldsymbol{R}}_s)}$, where $\lambda_{\max}(\cdot)$ stands for the maximum eigenvalue of a matrix. From the constraint in problem (10), we have

$$\|\boldsymbol{Qw}\|\geq 1+\eta\|\boldsymbol{w}\|. \tag{B1}$$

Since $\eta>0$, $1+\eta\|\boldsymbol{w}\|>0$. Then, the following relationship holds:

$$\boldsymbol{w}^{\mathrm{H}}\boldsymbol{Q}^{\mathrm{H}}\boldsymbol{Qw}\geq 1+2\eta\|\boldsymbol{w}\|+\eta^2\|\boldsymbol{w}\|^2>\eta^2\|\boldsymbol{w}\|^2, \tag{B2}$$

which implies that

$$\eta^2<\frac{\boldsymbol{w}^{\mathrm{H}}\boldsymbol{Q}^{\mathrm{H}}\boldsymbol{Qw}}{\|\boldsymbol{w}\|^2}\leq\lambda_{\max}(\hat{\boldsymbol{R}}_s), \tag{B3}$$

where we have used the definition $\hat{\boldsymbol{R}}_s=\boldsymbol{Q}^{\mathrm{H}}\boldsymbol{Q}$ and the result of the Rayleigh quotient in the second inequality (Cirrincione *et al.*, 2002). Therefore, we can obtain $0<\eta<\sqrt{\lambda_{\max}(\hat{\boldsymbol{R}}_s)}$. Turn now to display the property of $\boldsymbol{w}_{\mathrm{rob}}$.

From Eq. (18), we can obtain that

$$\begin{aligned}&\boldsymbol{w}^{\mathrm{H}}\hat{\boldsymbol{R}}_s\boldsymbol{w}\\&=\frac{\boldsymbol{w}^{\mathrm{H}}(\hat{\boldsymbol{R}}_x+\gamma\boldsymbol{I})\boldsymbol{w}}{\lambda}+\left(2\eta\sqrt{\mathrm{tr}(\hat{\boldsymbol{R}}_s)}-\eta^2\right)\|\boldsymbol{w}\|^2.\end{aligned} \tag{B4}$$

Thus,

$$\begin{aligned}&\boldsymbol{w}^{\mathrm{H}}\hat{\boldsymbol{R}}_s\boldsymbol{w}-\eta^2\|\boldsymbol{w}\|^2\\&=\frac{\boldsymbol{w}^{\mathrm{H}}(\hat{\boldsymbol{R}}_x+\gamma\boldsymbol{I})\boldsymbol{w}}{\lambda}+2\left(\eta\sqrt{\mathrm{tr}(\hat{\boldsymbol{R}}_s)}-\eta^2\right)\|\boldsymbol{w}\|^2.\end{aligned} \tag{B5}$$

Since $\hat{\boldsymbol{R}}_x+\gamma\boldsymbol{I}$ is positive and $0<\eta<\sqrt{\lambda_{\max}(\hat{\boldsymbol{R}}_s)}\leq$

$\sqrt{\mathrm{tr}(\hat{\boldsymbol{R}}_s)}$, it follows that

$$\boldsymbol{w}^{\mathrm{H}}(\hat{\boldsymbol{R}}_x+\gamma\boldsymbol{I})\boldsymbol{w}>0 \tag{B6}$$

and

$$\left(\eta\sqrt{\mathrm{tr}(\hat{\boldsymbol{R}}_s)}-\eta^2\right)\|\boldsymbol{w}\|^2>0. \tag{B7}$$

Applying $\lambda>0$ and inequalities (B6) and (B7) to Eq. (B5) leads to

$$\boldsymbol{w}^{\mathrm{H}}\hat{\boldsymbol{R}}_s\boldsymbol{w}-\eta^2\|\boldsymbol{w}\|^2>0. \tag{B8}$$

Therefore, the weight vector $\boldsymbol{w}_{\mathrm{rob}}$ obtained from Eq. (18) satisfies constraint $\|\boldsymbol{Q}\boldsymbol{w}\|>\eta\|\boldsymbol{w}\|$.