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Mittag-Leffler stability analysis of multiple equilibrium points in impulsive fractional-order quaternion-valued neural networks

K. UDHAYAKUMAR[1], R. RAKKIYAPPAN[1], Jin-de CAO[‡2], Xue-gang TAN[3]

[1]*Department of Mathematics, Bharathiar University, Coimbatore 641046, India*

[2]*School of Mathematics, Southeast University, Nanjing 210096, China*

[3]*School of Automation, Southeast University, Nanjing 210096, China*

E-mail: udhai512@gmail.com; rakkigru@gmail.com; jdcao@seu.edu.cn; xgtan_sde@163.com

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Abstract: In this study, we investigate the problem of multiple Mittag-Leffler stability analysis for fractional-order quaternion-valued neural networks (QVNNs) with impulses. Using the geometrical properties of activation functions and the Lipschitz condition, the existence of the equilibrium points is analyzed. In addition, the global Mittag-Leffler stability of multiple equilibrium points for the impulsive fractional-order QVNNs is investigated by employing the Lyapunov direct method. Finally, simulation is performed to illustrate the effectiveness and validity of the main results obtained.

Key words: Mittag-Leffler stability; Fractional-order; Quaternion-valued neural networks; Impulse
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1 Introduction

Recently, neural networks (NNs) have attracted attention from various fields due to their wide applications in image processing, system identification, propagation, pattern recognition, associative memory, combinational optimization, etc. Most of these applications depend on the dynamical properties of NNs. Therefore, different kinds of stability analysis and properties including bifurcation and chaos in NNs have attracted much attention (Cao and Xiao, 2007; Rakkiyappan et al., 2014, 2015a, 2015b, 2016; Stamova, 2014; Wang H et al., 2015; Li XD and Wu, 2016; Li XD and Ding, 2017; Li XD et al., 2017; Wu and Zeng, 2017; Yang et al., 2018; Huang YJ and Li, 2019; Khan et al., 2019; Li X et al., 2019; Nie et al., 2019; Pang et al., 2019; Qi et al., 2019; Wang JJ and Jia, 2019). Quaternion-valued NNs (QVNNs) are a generic extension of real- and complex-valued NNs (RVNNs and CVNNs), and they inhibit the non-commutative property of the quaternion algebra (Chen XF et al., 2017; Hu et al., 2017; Liu Y et al., 2017, 2018; Song and Chen, 2018; Li N and Zheng, 2020). The quaternion problems are more difficult than those in real- or complex-valued systems, which is the reason for the slow development in quaternion fields.

It is well known that two-dimensional data can be processed well in CVNNs and many RVNNs. If the data is three- or four-dimensional, such as in the cases with body images, color images, and four-dimensional signals, it can be directly encoded in terms of quaternions in quaternion networks, showing QVNNs to be more important than CVNNs and

‡ Corresponding author
ORCID: K. UDHAYAKUMAR, https://orcid.org/0000-0001-5764-1990; Jin-de CAO, https://orcid.org/0000-0003-3133-7119

RVNNs. Moreover, quaternions have attracted attention in a wide range of applications, including rotation, image comparison, and color night vision. Recently, a few researchers have considered theoretical investigations of the dynamical properties of QVNNs, concentrating mainly on the global behavior such as global stability. Decomposition of quaternions is a useful method of dealing with the noncommutativity of quaternion fields. Using plural decomposition and the Lipschitz technique, the stability results of QVNNs were obtained in terms of continuous and discrete time cases by Liu Y et al. (2018). In the past decade, the incorporation of fractional calculus into NNs has achieved better results than the integer-order NNs investigated by Cao and Xiao (2007) and Abdurahman et al. (2015). This is because the fractional-order derivatives inherently have excellent memory and hereditary properties in representing the network model. As is well known, there are many advantages over the integer-order NNs and the corresponding fractional-order NNs. However, the main difference is that fractional-order systems are more accurate than integer-order systems; i.e, there are more degrees of freedom in fractional-order systems. Moreover, compared with classical integer-order systems, fractional-order systems are characterized by infinite memory. Considering all the above-mentioned reasons, the incorporation of a memory term into an NN model is unavoidable. On the other hand, some researchers have attempted to investigate the advantages of both quaternions and fractional derivatives in NNs, and proposed fractional-order QVNNs.

It is well known that, many factors such as time delay, chaos, bifurcation, and system complexities influence the fractional-order NNs, resulting in instability at certain time instants. Therefore, the integration of impulsivity into the proposed fractional-order QVNNs becomes essential. Combining the memory and hereditary properties of fractional-order systems and the impulsive effect, the resulting impulsive fractional-order QVNNs guarantee better outcomes when compared with usual integer-order NNs. As one of the classical phenomena of dynamic NNs, multistability analysis has been studied extensively by Zeng et al. (2010), Huang Y et al. (2012), Zeng and Zheng (2012), Liu P et al. (2017, 2018), and Zhang FH and Zeng (2018). In Popa and Kaslik (2018), periodic solutions for the Hopfield-type integer-order NNs were studied in the presence of both time-dependent and distributed delays, taking the impulsive effects into account. In this study, we investigate the multistability problem of impulsive fractional-order QVNNs in the Mittag-Leffler sense. To the best of our knowledge, this is the first time the Mittag-Leffler stability theory has been developed thoroughly for the case of impulsive fractional-order QVNNs. Mittag-Leffler stability analysis in fractional-order systems is still an open problem.

Motivated by the above discussions, we conduct the study of multiple Mittag-Leffler stability results on impulsive fractional-order QVNNs. First, n-dimensional QVNNs are converted into a $4n$-dimensional RVNN system using the decomposition and non-commutative properties of quaternions. Then, sufficient conditions for Mittage-Leffler stability are discussed for fractional-order nonlinear systems. Finally, two numerical examples are given to demonstrate the effectiveness of the theoretical results.

2 Preliminaries

In this section, we present some important definitions and lemmas of fractional calculus which help prove the main results.

Definition 1 (Podlubny, 1998; Kilbas et al., 2006) The Caputo fractional derivative of order $0 < \alpha$ for $f(t) \in \mathcal{C}^n([t_0, +\infty], \mathbb{R})$ is

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-s)^{n-(\alpha+1)} f^{(n)}(s)\mathrm{d}s, $$

where $\Gamma(\cdot)$ is a Gamma function defined as $\Gamma(\alpha) = \int_{t_0}^{\infty} \frac{\mathrm{e}^{-t}}{t^{1-\alpha}}\mathrm{d}t$ $(n-1 < \alpha < n)$. If we choose $0 < \alpha < 1$,

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-s)^{-\alpha} \frac{\mathrm{d}f^{(1)}(s)}{\mathrm{d}s}\mathrm{d}s. $$

For convenience, in the rest of this paper, we adopt D^{α} to denote Caputo's fractional derivative operator ${}_{t_0}^{C}D_t^{\alpha}$.

Definition 2 (Podlubny, 1998; Kilbas et al., 2006) For any $\alpha, \beta > 0$ and real number ν, the Mittag-Leffler function $E_{\alpha,\beta}(\nu)$ with two parameters is defined as

$$E_{\alpha,\beta}(\nu)=\sum_{q=0}^{\infty}\frac{\nu^q}{\Gamma(q\alpha+\beta)}.$$

If $\beta=1$, we can obtain the one-parameter form of the Mittag-Leffler function $E(\nu)=\sum\limits_{q=0}^{\infty}\frac{\nu^q}{\Gamma(q\alpha+1)}$; if $\alpha=\beta=1$, then $E_{1,1}(\nu)=e^{\nu}$.

Consider the following fractional-order QVNNs with impulses:

$$\begin{cases} D^{\alpha}h_p(t)=-c_ph_p(t)+\sum\limits_{q=1}^{n}a_{pq}f_q\left(h_q(t)\right)+\mathcal{R}_p,\\ \delta h_p(t_k)=h_p(t_k^+)-h_p(t_k^-)=\beta_{kp}\left(h_p(t_k)\right), \end{cases} \tag{1}$$

where $k=1,2,\ldots,m$ and $p=1,2,\ldots,n$, or, equivalently, in the vector form

$$\begin{cases} D^{\alpha}h(t)=-\mathcal{C}h(t)+\mathcal{A}f\big(h(t)\big)+\mathcal{R},\\ \delta h(t_k)=h(t_k^+)-h(t_k^-)=\beta_k\big(h(t_k)\big), \end{cases} \tag{2}$$

where $k=1,2,\ldots,m$, $0<\alpha<1$, $h(t)=\big(h_1(t),h_2(t),\ldots,h_n(t)\big)^{\mathrm{T}}\in\mathbb{Q}^n$ is the state variable at time t, $\mathcal{C}=\mathrm{diag}(c_1,c_2,\ldots,c_n)$ with $c_p>0$, $f\big(h(t)\big)$ denotes the neuron activation functions, $\mathcal{A}\in\mathbb{Q}^{n\times n}$ is the interconnection matrix, $\mathcal{R}=(\mathcal{R}_1,\mathcal{R}_2,\ldots,\mathcal{R}_n)^{\mathrm{T}}\in\mathbb{Q}^n$ is an external input, and β_k denotes the impulsive operator. The time sequence is represented by $\{t_k\}$ for all $k\in\mathbb{Z}$ and satisfies $0=t_0<t_1<\ldots<t_k<\ldots,\lim_{k\to+\infty}t_k=+\infty$.

Definition 3 If a vector $h^*\in\mathbb{R}^n$ satisfies

$$\begin{cases} -\mathcal{C}h^*+\mathcal{A}f(h^*)+\mathcal{R}=0,\\ \beta_k(h^*)=0, \end{cases}$$

where $k=1,2,\ldots,n$, then h^* is called an equilibrium of NNs (2).

Definition 4 (Chen JJ et al., 2014) The equilibrium point $h^*=(h_1^*,h_2^*,\ldots,h_n^*)^{\mathrm{T}}$ of system (2) is said to be globally Mittag-Leffler stable. There exist positive constants $\mathcal{U}$ and $\mathcal{G}$, such that for any solution $h(t)$ of system (2) with initial value h_0, we have

$$||h(t)-h^*||\le\mathcal{G}\,||h_0-h^*||\,E_q\left(-\mathcal{U}(t-t_0)^q\right),$$

where $t\ge t_0$. If the equilibrium point h^* of system (2) is globally Mittag-Leffler stable, then QVNNs (2) are globally Mittag-Leffler stable.

System (2) follows from the non-commutative property of quaternion algebra, and uses the Hamilton rules: ij $=$ k, ji $=-$k, jk $=$ i, kj $=-$i, ki $=$ j, ik $=-$j, ijk $=\mathrm{i}^2=\mathrm{j}^2=\mathrm{k}^2=-1$, and $\nu\in\{R,I,J,K\}$; thus, we can rewrite NNs (2) as the following four real-valued NNs:

$$\begin{aligned} D^{\alpha}h^R(t)=&-\mathcal{C}h^R(t)+\mathcal{A}^Rf^R\big(h^R(t)\big)-\mathcal{A}^If^I\big(h^I(t)\big)\\ &-\mathcal{A}^Jf^J(h^J(t))-\mathcal{A}^Kf^K(h^K(t))+\mathcal{R}^R,\\ D^{\alpha}h^I(t)=&-\mathcal{C}h^I(t)+\mathcal{A}^Rf^I(h^I(t))+\mathcal{A}^If^R(h^R(t))\\ &+\mathcal{A}^Jf^K(h^K(t))-\mathcal{A}^Kf^J(h^J(t))+\mathcal{R}^I,\\ D^{\alpha}h^J(t)=&-\mathcal{C}h^J(t)+\mathcal{A}^Rf^J(h^J(t))\\ &-\mathcal{A}^If^K(h^K(t))+\mathcal{A}^Jf^R(h^R(t))\\ &+\mathcal{A}^Kf^I(z^I(t))+\mathcal{R}^J,\\ D^{\alpha}h^K(t)=&-\mathcal{C}h^K(t)+\mathcal{A}^Rf^K(h^K(t))\\ &+\mathcal{A}^If^J(h^J(t))-\mathcal{A}^Jf^I(h^I(t))\\ &+\mathcal{A}^Kf^R(h^R(t))+\mathcal{R}^K,\\ h^{\nu}(t_k^-)=&h^{\nu}(t_k),h^{\nu}(t_k^+)-h^{\nu}(t_k^-)=\beta_k^{\nu}(h(t_k)), \end{aligned} \tag{3}$$

where $t\neq t_k$ and $p=1,2,\ldots,n$.

Assumption 1 The impulsive operator $\beta_k=[\beta_{k1},\beta_{k2},\ldots,\beta_{kn}]^{\mathrm{T}}$ is defined on $\{\varphi:(-\infty,t_k]\to\mathbb{L}^n|\varphi$ is piecewise continuous on $(-\infty,0]$, left continuous on $[0,t_k]$, with a first-kind discontinuity at t_r, and differentiable on every interval $(t_{r-1},t_r),1\le r\le k\}$. This will require $\beta_k(\varphi_0)=0$ for any constant function φ_0.

Assumption 2 $G(1)=(1,\infty),G(-1)=(-\infty,-1)$, and $G_{\mathbb{L}}(h)=G(h^R)+\mathrm{i}G(h^I)+\mathrm{j}G(h^J)+\mathrm{k}G(h^K)$ for every $\varsigma\in\{\pm1,\pm\mathrm{i},\pm\mathrm{j},\pm\mathrm{k}\}^n$, and we define the set $\varPhi_{\varsigma}=G_{\mathbb{L}}(\varsigma_1)\times G_{\mathbb{L}}(\varsigma_2)\times\ldots\times G_{\mathbb{L}}(\varsigma_n)$. For example, we take $G_{\mathbb{L}}(-1+\mathrm{i}-\mathrm{j}+\mathrm{k})=(-\infty,-1)+(1,\infty)\mathrm{i}+(-\infty,-1)\mathrm{j}+(1,\infty)\mathrm{k}$.

Assumption 3 The components of the activation functions f_p^R, f_p^I, f_p^J, and f_p^K are bounded and globally Lipschitz continuous; then, for any constants $\vartheta_q^{\nu R}$, $\vartheta_q^{\nu I}$, $\vartheta_q^{\nu J}$, and $\vartheta_q^{\nu K}$, $\nu=R,I,J,K,|f_p^{\nu}(h)|\le 1$, and $|f_q^{\nu}(h_1^R,h_1^I,h_1^J,h_1^K)-f_q^{\nu}(h_2^R,h_2^I,h_2^J,h_2^K)|\le\vartheta_q^{\nu R}|h_1^R-h_2^R|+\vartheta_q^{\nu I}|h_1^I-h_2^I|+\vartheta_q^{\nu J}|h_1^J-h_2^J|+\vartheta_q^{\nu K}|h_1^K-h_2^K|$.

Assumption 4 There exists an $l\in(0,1)$ such that functions $f_q^{\nu}(h^{\nu})$ satisfy $f_q^{\nu}(h^{\nu})\ge l$ if $h^{\nu}\ge1$, and $f_q^{\nu}(h^{\nu})\le -l$ if $h^{\nu}\le-1$, for $q=1,2,\ldots,n$ and $\nu\in\{R,I,J,K\}$.

Assumption 5 The external input vector satisfies

$$\begin{aligned} ||\mathcal{R}_p(t)||<&a_{pp}^Rl-d_p-\left(|a_{pp}^I|+|a_{pp}^J|+|a_{pp}^K|\right)\\ &-\sum_{q\neq p}^{n}\left(|a_{pq}^R|+|a_{pq}^I|+|a_{pq}^J|+|a_{pq}^K|\right), \end{aligned}$$

where $t \in \mathbb{R}$ and $p = 1, 2, \ldots, n$.

Assumption 6 For any $k \in \mathbb{Z}^+$ and $\varsigma \in \{\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{k}\}^n$, if $\sigma(t) \in \Phi_\varsigma$, then $\sigma(t_k) + \beta_k(\sigma) \in \Phi_\varsigma$.

For theoretical investigations, we need the following lemmas:

Lemma 1 (Wu and Zeng, 2017) Let $\chi_1 > 0$, $\chi_2 > 0$, $\chi_3 > 1$, $\chi_4 > 1$, and $\frac{1}{\chi_3} + \frac{1}{\chi_4} = 1$. Then for any $\delta > 0$, we have $\chi_1\chi_2 \leq \frac{1}{\chi_3}(\chi_1\delta)^{\chi_3} + \frac{1}{\chi_4}(\chi_2\frac{1}{\delta})^{\chi_4}$. The inequality holds if and only if $(\chi_1\delta)^{\chi_3} = (\chi_2\frac{1}{\delta})^{\chi_4}$.

Lemma 2 (Popa and Kaslik, 2018) Let Eq. (4) (at the bottom of this page) hold, where $M_p^\nu = \dfrac{1}{d_p}\Big(||\mathcal{R}_p^\nu|| + \sum\limits_{q=1}^{n}\left(|a_{pq}^R| + |a_{pq}^I| + |a_{pq}^J| + |a_{pq}^K|\right)\Big)$, $p = 1, 2, \ldots, n$, and $\nu \in \{R, I, J, K\}$.

If Assumption 3 is satisfied, then the following statements hold:

1. There exists at least one equilibrium point of QVNNs (1) corresponding to input vector $\mathcal{R}$ in set $\Phi_{\mathcal{R}}$.

2. Every equilibrium point of QVNNs (1) belongs to set $\Phi_{\mathcal{R}}$.

Lemma 3 (Popa and Kaslik, 2018) Suppose that Assumptions 1–5 hold. Then the following conditions are true:

1. In every set Φ_ς ($\varsigma \in \{\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{k}\}^n$), there exists at least one equilibrium point of QVNNs (1) corresponding to the external input vector $\mathcal{R}$.

2. If Assumption 6 is satisfied, then set Φ_ς ($\varsigma \in \{\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{k}\}^n$) is a positively invariant set.

3. If Assumption 3 holds, then the equilibrium point of QVNNs (1) corresponding to input vector $\mathcal{R}$ in set Φ_ς ($\varsigma \in \{\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{k}\}^n$) is unique and exponentially stable.

Lemma 4 (Zhang XX et al., 2017) For the following fractional-order impulsive system:

$$\begin{cases} D^\alpha h(t) = -\mathcal{C}h(t) + \mathcal{A}f(t, h(t)) + \mathcal{R}, \\ \delta h(t_k) = h(t_k^+) - h(t_k^-) = \beta_k(h(t_k)), \end{cases} \tag{5}$$

where $k = 1, 2, \ldots, m$, assume that the following conditions hold: (1) $f(t, 0) = 0$ ($t > 0$); (2) $\beta_k = 0$ ($k = 1, 2, \ldots, m$); (3) there exists a positive definite function $V(t)$ that satisfies $D^\alpha V(t, e(t)) \leq -\xi V(t, e(t))$ and $V(t^+, e(t) + E_k(h)) \leq V(t, e(t))$, where $t = t_k$ and $k = 1, 2, \ldots, m$. Then the equilibrium point of QVNNs (1) is Mittag-Leffler stable.

Remark 1 Based on Brouwer's and Leray-Schauder's fixed point theories (Schauder, 1930), Lemmas 1 and 2 can be easy to prove. For details, see Lemma 2 and Theorem 2 in Popa and Kaslik (2018).

3 Main results

In this section, the Mittag-Leffler stability analysis of multiple equilibrium points for the impulsive fractional-order QVNNs is investigated. Consider QVNNs (1) have the initial value $h^\nu(0) = h_0^\nu$. Let h^* be the equilibrium point of the impulsive fractional-order QVNNs (1) and thus make the transformation $e^\nu(t) = h^\nu(t) - h^*$. Then system (3) is transformed into

$$\begin{aligned} D^\alpha e^R(t) =& -\mathcal{C}e^R(t) + \mathcal{A}^R f^R(e^R(t)) - \mathcal{A}^I f^I(e^I(t)) \\ & - \mathcal{A}^J f^J(e^J(t)) - \mathcal{A}^K f^K(e^K(t)), \\ D^\alpha e^I(t) =& -\mathcal{C}e^I(t) + \mathcal{A}^R f^I(e^I(t)) + \mathcal{A}^I f^R(e^R(t)) \\ & + \mathcal{A}^J f^K(e^K(t)) - \mathcal{A}^K f^J(e^J(t)), \\ D^\alpha e^J(t) =& -\mathcal{C}e^J(t) + \mathcal{A}^R f^J(e^J(t)) - \mathcal{A}^I f^K(e^K(t)) \\ & + \mathcal{A}^J f^R(e^R(t)) + \mathcal{A}^K f^I(e^I(t)), \\ D^\alpha e^K(t) =& -\mathcal{C}e^K(t) + \mathcal{A}^R f^K(e^K(t)) + \mathcal{A}^I f^J(e^J(t)) \\ & - \mathcal{A}^J f^I(e^I(t)) + \mathcal{A}^K f^R(e^R(t)), \\ e^\nu(t_k^-) =& e^\nu(t_k), e^\nu(t_k^+) - e^\nu(t_k^-) = \Gamma_k^\nu(e(t_k)), \\ e(0) =& e_0, t \neq t_k, p = 1, 2, \ldots, n, \end{aligned} \tag{6}$$

where $e(t) = (e_1(t), e_2(t), \ldots, e_n(t))^{\mathrm{T}}$, $f(t, e(t)) = (f_1(t, e_1(t)), f_2(t, e_2(t)), \ldots, f_n(t, e_n(t)))^{\mathrm{T}}$, and $f_p(t, e_p(t)) = f_p(t, e_p + h_p^*) - f_p(t, h_p^*)$ ($p = 1, 2, \ldots, n$ and $e_0 = h_0 - h^*$).

Theorem 1 Assume that the conditions of Lemmas 2 and 3 hold, and $\Gamma_k^\nu(h^\nu(t_k)) = -\eta(h^\nu(t_k) - h^*)$ ($k = 1, 2, \ldots, m$, $\zeta > 0$, and $\nu \in \{R, I, J, K\}$) where h^* is the steady state of the impulsive QVNNs (1).

$$\begin{aligned} \Phi_{\mathcal{R}} =& \left(\left[-M_1^R, M_1^R\right] + \mathrm{i}\left[-M_1^I, M_1^I\right] + \mathrm{j}\left[-M_1^J, M_1^J\right] + \mathrm{k}\left[-M_1^K, M_1^K\right]\right) \\ & \cdot \left(\left[-M_2^R, M_2^R\right] + \mathrm{i}\left[-M_2^I, M_2^I\right] + \mathrm{j}\left[-M_2^J, M_2^J\right] + \mathrm{k}\left[-M_2^K, M_2^K\right]\right) \\ & \cdot \ldots \cdot \left(\left[-M_n^R, M_n^R\right] + \mathrm{i}\left[-M_n^I, M_n^I\right] + \mathrm{j}\left[-M_n^J, M_n^J\right] + \mathrm{k}\left[-M_n^K, M_n^K\right]\right). \end{aligned} \tag{4}$$

If $|1-\delta_{kp}^{\nu}|^{\zeta} \leq 1$ and there exists inequality (7) (at the bottom of this page), where

$$\begin{aligned}
\bar{A}_{pq}^{\nu} &= a_{pq}^{R}\vartheta_{q}^{R\nu} - a_{pq}^{I}\vartheta_{q}^{I\nu} - a_{pq}^{J}\vartheta_{q}^{J\nu} - a_{pq}^{K}\vartheta_{q}^{K\nu},\\
\tilde{A}_{pq}^{\nu} &= a_{pq}^{R}\vartheta_{q}^{I\nu} + a_{pq}^{I}\vartheta_{q}^{R\nu} + a_{pq}^{J}\vartheta_{q}^{K\nu} - a_{pq}^{K}\vartheta_{q}^{J\nu},\\
\check{A}_{pq}^{\nu} &= a_{pq}^{R}\vartheta_{q}^{J\nu} - a_{pq}^{I}\vartheta_{q}^{K\nu} + a_{pq}^{J}\vartheta_{q}^{R\nu} + a_{pq}^{K}\vartheta_{q}^{I\nu},\\
\hat{A}_{pq}^{\nu} &= a_{pq}^{R}\vartheta_{q}^{K\nu} + a_{pq}^{I}\vartheta_{q}^{J\nu} - a_{pq}^{J}\vartheta_{q}^{I\nu} - a_{pq}^{K}\vartheta_{q}^{R\nu},
\end{aligned}$$

then the impulsive fractional-order QVNN (1) is Mittag-Leffler stable.

Proof Consider the following Lyapunov function candidate:

$$V(t,e(t)) = V_1 + V_2 + V_3 + V_4, \tag{8}$$

where

$$\begin{aligned}
V_1 &= \sum_{p=1}^{n} \zeta^{-1}|e_p^R(t)|^{\zeta},\ V_2 = \sum_{p=1}^{n} \zeta^{-1}|e_p^I(t)|^{\zeta},\\
V_3 &= \sum_{p=1}^{n} \zeta^{-1}|e_p^J(t)|^{\zeta},\ V_4 = \sum_{p=1}^{n} \zeta^{-1}|e_p^K(t)|^{\zeta}.
\end{aligned}$$

When $t \neq t_k$ $(k=1,2,\ldots,m)$, calculating the fractional derivatives of $V(t,e(t))$ along the trajectories of NNs (3), we can find from Lemma 1 and NNs (3) that inequality (9) (on the next page) is valid.

Using Lemma 1, we can obtain

$$\begin{aligned}
|e_p^R(t)|^{\zeta-1}|e_q^R(t)| &\leq \zeta^{-1}\Bigg((\zeta-1)|e_p^R(t)|^{\zeta}\varepsilon_1^R\\
&\quad + |e_q^R(t)|^{\zeta}\frac{1}{\varepsilon_1^{R^{\zeta-1}}}\Bigg),\\
|e_p^R(t)|^{\zeta-1}|e_q^I(t)| &\leq \zeta^{-1}\Bigg((\zeta-1)|e_p^R(t)|^{\zeta}\varepsilon_2^R\\
&\quad + |e_q^I(t)|^{\zeta}\frac{1}{\varepsilon_2^{R^{\zeta-1}}}\Bigg),\\
|e_p^R(t)|^{\zeta-1}|e_q^J(t)| &\leq \zeta^{-1}\Bigg((\zeta-1)|e_p^R(t)|^{\zeta}\varepsilon_3^R\\
&\quad + |e_q^J(t)|^{\zeta}\frac{1}{\varepsilon_3^{R^{\zeta-1}}}\Bigg),\\
|e_p^R(t)|^{\zeta-1}|e_q^K(t)| &\leq \zeta^{-1}\Bigg((\zeta-1)|e_p^R(t)|^{\zeta}\varepsilon_4^R\\
&\quad + |e_q^K(t)|^{\zeta}\frac{1}{\varepsilon_4^{R^{\zeta-1}}}\Bigg).
\end{aligned}$$

Substituting the above inequalities into inequality (9), then we have inequality (10) (on the next page).

Similar to the proof of $D^{\alpha}V_1$, we can estimate $D^{\alpha}V_2, D^{\alpha}V_3$, and $D^{\alpha}V_4$, which are omitted here to save space. As for $D^{\alpha}V_1$, we have inequalities (11)–(13), shown on page 240.

Adding inequalities (10)–(13), we obtain inequality (14) (shown on page 241).

From inequality (7), we can select a positive

$$\begin{cases}
\lambda^R = \min\limits_{1\leq p\leq n}\Bigg\{c_p\zeta - \sum\limits_{q=1}^{n}\Big(\bar{A}_{pq}^{R}\Big((\zeta-1)\varepsilon_1^R + \varepsilon_1^{R^{1-\zeta}}\Big) + \bar{A}_{pq}^{I}(\zeta-1)\varepsilon_2^R + \bar{A}_{pq}^{J}(\zeta-1)\varepsilon_3^R\\
\qquad + \bar{A}_{pq}^{K}(\zeta-1)\varepsilon_4^R + \tilde{A}_{pq}^{R}\varepsilon_1^{I^{1-\zeta}} + \check{A}_{pq}^{R}\varepsilon_1^{J^{1-\zeta}} + \hat{A}_{pq}^{R}\varepsilon_1^{K^{1-\zeta}}\Big)\Bigg\} > 0,\\
\lambda^I = \min\limits_{1\leq p\leq n}\Bigg\{c_p\zeta - \sum\limits_{q=1}^{n}\Big(\tilde{A}_{pq}^{I}\Big((\zeta-1)\varepsilon_2^I + \varepsilon_2^{I^{1-\zeta}}\Big) + \tilde{A}_{pq}^{R}\left((\zeta-1)\varepsilon_1^I\right) + \tilde{A}_{pq}^{J}(\zeta-1)\varepsilon_3^I\\
\qquad + \tilde{A}_{pq}^{K}(\zeta-1)\varepsilon_4^I + \bar{A}_{pq}^{I}\varepsilon_2^{R^{1-\zeta}} + \check{A}_{pq}^{I}\varepsilon_2^{J^{1-\zeta}} + \hat{A}_{pq}^{I}\varepsilon_2^{K^{1-\zeta}}\Big)\Bigg\} > 0,\\
\lambda^J = \min\limits_{1\leq p\leq n}\Bigg\{c_p\zeta - \sum\limits_{q=1}^{n}\Big(\check{A}_{pq}^{J}\Big((\zeta-1)\varepsilon_3^J + \varepsilon_3^{J^{1-\zeta}}\Big) + \check{A}_{pq}^{R}\left((\zeta-1)\varepsilon_1^J\right) + \check{A}_{pq}^{I}(\zeta-1)\varepsilon_2^J\\
\qquad + \check{A}_{pq}^{K}(\zeta-1)\varepsilon_4^J + \bar{A}_{pq}^{J}\varepsilon_3^{R^{1-\zeta}} + \tilde{A}_{pq}^{J}\varepsilon_3^{I^{1-\zeta}} + \hat{A}_{pq}^{J}\varepsilon_3^{K^{1-\zeta}}\Big)\Bigg\} > 0,\\
\lambda^K = \min\limits_{1\leq p\leq n}\Bigg\{c_p\zeta - \sum\limits_{q=1}^{n}\Big(\hat{A}_{pq}^{K}\Big((\zeta-1)\varepsilon_4^K + \varepsilon_4^{K^{1-\zeta}}\Big) + \hat{A}_{pq}^{R}\left((\zeta-1)\varepsilon_1^K\right) + \hat{A}_{pq}^{I}(\zeta-1)\varepsilon_2^K\\
\qquad + \hat{A}_{pq}^{J}(\zeta-1)\varepsilon_3^K + + \bar{A}_{pq}^{K}\varepsilon_4^{R^{1-\zeta}} + \tilde{A}_{pq}^{K}\varepsilon_4^{I^{1-\zeta}} + \check{A}_{pq}^{J}\varepsilon_4^{K^{1-\zeta}}\Big)\Bigg\} > 0.
\end{cases} \tag{7}$$

$$
\begin{aligned}
D^{\alpha}V_1 \leq & \sum_{p=1}^{n}|e_p^R(t)|^{\zeta-1}\Bigg[(-c_p)|e_p^R(t)|^{\zeta}+\sum_{q=1}^{n}a_{pq}^R\Big(\vartheta_p^{RR}|e_q^R(t)|+\vartheta_p^{RI}|e_q^I(t)|+\vartheta_p^{RJ}|e_q^J(t)|+\vartheta_p^{RK}|e_q^K(t)|\Big)\\
&-\sum_{q=1}^{n}a_{pq}^I\Big(\vartheta_p^{IR}|e_q^R(t)|+\vartheta_p^{II}|e_q^I(t)|+\vartheta_p^{IJ}|e_q^J(t)|+\vartheta_p^{IK}|e_q^K(t)|\Big)-\sum_{q=1}^{n}a_{pq}^J\Big(\vartheta_p^{JR}|e_q^R(t)|+\vartheta_p^{JI}|e_q^I(t)|\\
&+\vartheta_p^{JJ}|e_q^J(t)|+\vartheta_p^{JK}|e_q^K(t)|\Big)-\sum_{q=1}^{n}a_{pq}^K\Big(\vartheta_p^{KR}|e_q^R(t)|+\vartheta_p^{KI}|e_q^I(t)|+\vartheta_p^{KJ}|e_q^J(t)|+\vartheta_p^{KK}|e_q^K(t)|\Big)\Bigg]\\
=&\sum_{p=1}^{n}(-c_p)|e_p^R(t)|^{\zeta}+\sum_{p=1}^{n}\sum_{q=1}^{n}a_{pq}^R|e_p^R(t)|^{\zeta-1}\Big(\vartheta_q^{RR}|e_q^R(t)|+\vartheta_q^{RI}|e_q^I(t)|+\vartheta_q^{RJ}|e_q^J(t)|+\vartheta_q^{RK}|e_q^K(t)|\Big)\\
&-\sum_{p=1}^{n}\sum_{q=1}^{n}a_{pq}^I|e_p^R(t)|^{\zeta-1}\Big(\vartheta_q^{IR}|e_q^R(t)|+\vartheta_q^{II}|e_q^I(t)|+\vartheta_q^{IJ}|e_q^J(t)|+\vartheta_q^{IK}|e_q^K(t)|\Big)\\
&-\sum_{p=1}^{n}\sum_{q=1}^{n}a_{pq}^J|e_p^R(t)|^{\zeta-1}\Big(\vartheta_q^{JR}|e_q^R(t)|+\vartheta_q^{JI}|e_q^I(t)|+\vartheta_q^{JJ}|e_q^J(t)|+\vartheta_q^{JK}|e_q^K(t)|\Big)\\
&-\sum_{p=1}^{n}\sum_{q=1}^{n}a_{pq}^K|e_p^R(t)|^{\zeta-1}\Big(\vartheta_q^{KR}|e_q^R(t)|+\vartheta_q^{KI}|e_q^I(t)|+\vartheta_q^{KJ}|e_q^J(t)|+\vartheta_q^{KK}|e_q^K(t)|\Big)\\
=&\sum_{p=1}^{n}(-c_p)|e_p^R(t)|^{\zeta}+\sum_{p=1}^{n}\sum_{q=1}^{n}\Big(a_{pq}^R\vartheta_q^{RR}|e_p^R(t)|^{\zeta-1}|e_q^R(t)|+a_{pq}^R\vartheta_q^{RI}|e_p^R(t)|^{\zeta-1}|e_q^I(t)|+a_{pq}^R\vartheta_q^{RJ}|e_p^R(t)|^{\zeta-1}\\
&\cdot|e_q^J(t)|+a_{pq}^R\vartheta_q^{RK}|e_p^R(t)|^{\zeta-1}|e_q^K(t)|\Big)-\sum_{p=1}^{n}\sum_{q=1}^{n}\Big(a_{pq}^I\vartheta_q^{IR}|e_p^R(t)|^{\zeta-1}|e_q^R(t)|+a_{pq}^I\vartheta_q^{II}|e_p^R(t)|^{\zeta-1}|e_q^I(t)|\\
&+a_{pq}^I\vartheta_q^{IJ}|e_p^R(t)|^{\zeta-1}|e_q^J(t)|+a_{pq}^I\vartheta_q^{IK}|e_p^R(t)|^{\zeta-1}|e_q^K(t)|\Big)-\sum_{p=1}^{n}\sum_{q=1}^{n}\Big(a_{pq}^J\vartheta_q^{JR}|e_p^R(t)|^{\zeta-1}|e_q^R(t)|\\
&+a_{pq}^J\vartheta_q^{JI}|e_p^R(t)|^{\zeta-1}|e_q^I(t)|+a_{pq}^J\vartheta_q^{JJ}|e_p^R(t)|^{\zeta-1}|e_q^J(t)|+a_{pq}^J\vartheta_q^{JK}|e_p^R(t)|^{\zeta-1}|e_q^K(t)|\Big)\\
&-\sum_{p=1}^{n}\sum_{q=1}^{n}\Big(a_{pq}^K\vartheta_q^{KR}|e_p^R(t)|^{\zeta-1}|e_q^R(t)|+a_{pq}^K\vartheta_q^{KI}|e_p^R(t)|^{\zeta-1}|e_q^I(t)|+a_{pq}^K\vartheta_q^{KJ}|e_p^R(t)|^{\zeta-1}|e_q^J(t)|\\
&+a_{pq}^K\vartheta_q^{KK}|e_p^R(t)|^{\zeta-1}|e_q^K(t)|\Big).
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
D^{\alpha}V_1 \leq & \sum_{q=1}^{n}\Bigg\{\Big(a_{pq}^R\vartheta_q^{RR}-a_{pq}^I\vartheta_q^{IR}-a_{pq}^J\vartheta_q^{JR}-a_{pq}^K\vartheta_q^{KR}\Big)(\zeta-1)\varepsilon_1^R+\Big(a_{pq}^R\vartheta_q^{RI}-a_{pq}^I\vartheta_q^{II}-a_{pq}^J\vartheta_q^{JI}-a_{pq}^K\vartheta_q^{KI}\Big)\\
&\cdot(\zeta-1)\varepsilon_2^R+\Big(a_{pq}^R\vartheta_q^{RJ}-a_{pq}^I\vartheta_q^{IJ}-a_{pq}^J\vartheta_q^{JJ}-a_{pq}^K\vartheta_q^{KJ}\Big)(\zeta-1)\varepsilon_3^R+\Big(a_{pq}^R\vartheta_q^{RK}-a_{pq}^I\vartheta_q^{IK}-a_{pq}^J\vartheta_q^{JK}\\
&-a_{pq}^K\vartheta_q^{KK}\Big)(\zeta-1)\varepsilon_4^R+\Big(a_{pq}^R\vartheta_q^{RR}-a_{pq}^I\vartheta_q^{IR}-a_{pq}^J\vartheta_q^{JR}-a_{pq}^K\vartheta_q^{KR}\Big)\varepsilon_1^{R^{1-\zeta}}-c_p\zeta\Bigg\}V_1\\
&+\sum_{q=1}^{n}\Bigg\{\Big(a_{pq}^R\vartheta_q^{RI}-a_{pq}^I\vartheta_q^{II}-a_{pq}^J\vartheta_q^{JI}-a_{pq}^K\vartheta_q^{KI}\Big)\varepsilon_2^{R^{1-\zeta}}\Bigg\}V_2+\sum_{q=1}^{n}\Bigg\{\Big(a_{pq}^R\vartheta_q^{RJ}-a_{pq}^I\vartheta_q^{IJ}-a_{pq}^J\vartheta_q^{JJ}\\
&-a_{pq}^K\vartheta_q^{KJ}\Big)\varepsilon_3^{R^{1-\zeta}}\Bigg\}V_3+\sum_{q=1}^{n}\Bigg\{\Big(a_{pq}^R\vartheta_q^{RK}-a_{pq}^I\vartheta_q^{IK}-a_{pq}^J\vartheta_q^{JK}-a_{pq}^K\vartheta_q^{KK}\Big)\varepsilon_4^{R^{1-\zeta}}\Bigg\}V_4.
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
D^{\alpha}V_2 \leq & \sum_{q=1}^{n}\Bigg\{\Big(a_{pq}^{R}\vartheta_q^{IR}+a_{pq}^{I}\vartheta_q^{RR}+a_{pq}^{J}\vartheta_q^{KR}-a_{pq}^{K}\vartheta_q^{JR}\Big)(\zeta-1)\varepsilon_1^{I}+\Big(a_{pq}^{R}\vartheta_q^{II}+a_{pq}^{I}\vartheta_q^{RI}+a_{pq}^{J}\vartheta_q^{KI}-a_{pq}^{K}\vartheta_q^{JI}\Big) \\
& \cdot(\zeta-1)\varepsilon_2^{I}+\Big(a_{pq}^{R}\vartheta_q^{IJ}+a_{pq}^{I}\vartheta_q^{RJ}+a_{pq}^{J}\vartheta_q^{KJ}-a_{pq}^{K}\vartheta_q^{JJ}\Big)(\zeta-1)\varepsilon_3^{I}+\Big(a_{pq}^{R}\vartheta_q^{IK}+a_{pq}^{I}\vartheta_q^{RK}+a_{pq}^{J}\vartheta_q^{KK} \\
& -a_{pq}^{K}\vartheta_q^{JK}\Big)(\zeta-1)\varepsilon_4^{I}+\Big(a_{pq}^{R}\vartheta_q^{II}+a_{pq}^{I}\vartheta_q^{RI}+a_{pq}^{J}\vartheta_q^{KI}-a_{pq}^{K}\vartheta_q^{JI}\Big)\varepsilon_2^{I^{1-\zeta}}-c_p\zeta\Bigg\}V_2+\sum_{q=1}^{n}\Bigg\{\Big(a_{pq}^{R}\vartheta_q^{IR} \\
& +a_{pq}^{I}\vartheta_q^{RR}+a_{pq}^{J}\vartheta_q^{KR}-a_{pq}^{K}\vartheta_q^{JR}\Big)\varepsilon_1^{I^{1-\zeta}}\Bigg\}V_1+\sum_{q=1}^{n}\Bigg\{(a_{pq}^{R}\vartheta_q^{IJ}+a_{pq}^{I}\vartheta_q^{RJ}+a_{pq}^{J}\vartheta_q^{KJ}-a_{pq}^{K}\vartheta_q^{JJ})\varepsilon_3^{I^{1-\zeta}}\Bigg\}V_3 \\
& +\sum_{q=1}^{n}\Bigg\{(a_{pq}^{R}\vartheta_q^{IK}+a_{pq}^{I}\vartheta_q^{RK}+a_{pq}^{J}\vartheta_q^{KK}-a_{pq}^{K}\vartheta_q^{JK})\varepsilon_4^{R^{1-\zeta}}\Bigg\}V_4.
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
D^{\alpha}V_3 \leq & \sum_{q=1}^{n}\Bigg\{\Big(a_{pq}^{R}\vartheta_q^{JR}-a_{pq}^{I}\vartheta_q^{KR}+a_{pq}^{J}\vartheta_q^{RR}+a_{pq}^{K}\vartheta_q^{IR}\Big)(\zeta-1)\varepsilon_1^{J}+\Big(a_{pq}^{R}\vartheta_q^{JI}-a_{pq}^{I}\vartheta_q^{KI}+a_{pq}^{J}\vartheta_q^{RI}+a_{pq}^{K}\vartheta_q^{II}\Big) \\
& \cdot(\zeta-1)\varepsilon_2^{J}+\Big(a_{pq}^{R}\vartheta_q^{JJ}-a_{pq}^{I}\vartheta_q^{KJ}+a_{pq}^{J}\vartheta_q^{RJ}+a_{pq}^{J}\vartheta_q^{IJ}\Big)(\zeta-1)\varepsilon_3^{J}+\Big(a_{pq}^{R}\vartheta_q^{JK}-a_{pq}^{I}\vartheta_q^{KK}+a_{pq}^{J}\vartheta_q^{RK} \\
& +a_{pq}^{K}\vartheta_q^{IK}\Big)(\zeta-1)\varepsilon_4^{J}+\Big(a_{pq}^{R}\vartheta_q^{JJ}-a_{pq}^{I}\vartheta_q^{KJ}+a_{pq}^{J}\vartheta_q^{RJ}+a_{pq}^{K}\vartheta_q^{IJ}\Big)\varepsilon_3^{J^{1-\zeta}}-c_p\zeta\Bigg\}V_3+\sum_{q=1}^{n}\Bigg\{\Big(a_{pq}^{R}\vartheta_q^{JR} \\
& -a_{pq}^{I}\vartheta_q^{KR}+a_{pq}^{J}\vartheta_q^{RR}+a_{pq}^{K}\vartheta_q^{IR}\Big)\varepsilon_1^{J^{1-\zeta}}\Bigg\}V_1+\sum_{q=1}^{n}\Bigg\{\Big(a_{pq}^{R}\vartheta_q^{JI}-a_{pq}^{I}\vartheta_q^{KI}+a_{pq}^{J}\vartheta_q^{RI}+a_{pq}^{K}\vartheta_q^{II}\Big)\varepsilon_2^{J^{1-\zeta}}\Bigg\}V_2 \\
& +\sum_{q=1}^{n}\Bigg\{\Big(a_{pq}^{R}\vartheta_q^{JK}-a_{pq}^{I}\vartheta_q^{KK}+a_{pq}^{J}\vartheta_q^{RK}+a_{pq}^{K}\vartheta_q^{IK}\Big)\varepsilon_4^{J^{1-\zeta}}\Bigg\}V_4.
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
D^{\alpha}V_4 \leq & \sum_{q=1}^{n}\Bigg\{\Big(a_{pq}^{R}\vartheta_q^{KR}+a_{pq}^{I}\vartheta_q^{JR}-a_{pq}^{J}\vartheta_q^{IR}+a_{pq}^{K}\vartheta_q^{RR}\Big)(\zeta-1)\varepsilon_1^{K}+\Big(a_{pq}^{R}\vartheta_q^{KI}+a_{pq}^{I}\vartheta_q^{JI}-a_{pq}^{J}\vartheta_q^{II}+a_{pq}^{K}\vartheta_q^{RI}\Big) \\
& \cdot(\zeta-1)\varepsilon_2^{K}+\Big(a_{pq}^{R}\vartheta_q^{KJ}+a_{pq}^{I}\vartheta_q^{JJ}-a_{pq}^{J}\vartheta_q^{IJ}+a_{pq}^{K}\vartheta_q^{RJ}\Big)(\zeta-1)\varepsilon_3^{K}+\Big(a_{pq}^{R}\vartheta_q^{KK}+a_{pq}^{I}\vartheta_q^{JK}-a_{pq}^{J}\vartheta_q^{IK} \\
& +a_{pq}^{K}\vartheta_q^{RK}\Big)(\zeta-1)\varepsilon_4^{K}+(a_{pq}^{R}\vartheta_q^{KK}+a_{pq}^{I}\vartheta_q^{JK}-a_{pq}^{J}\vartheta_q^{IK}+a_{pq}^{K}\vartheta_q^{RK})\varepsilon_4^{K^{1-\zeta}}-c_p\zeta\Bigg\}V_4+\sum_{q=1}^{n}\Bigg\{\Big(a_{pq}^{R}\vartheta_q^{KR} \\
& +a_{pq}^{I}\vartheta_q^{JR}-a_{pq}^{J}\vartheta_q^{IR}+a_{pq}^{K}\vartheta_q^{RR}\Big)\varepsilon_1^{K^{1-\zeta}}\Bigg\}V_1+\sum_{q=1}^{n}\Bigg\{\Big(a_{pq}^{R}\vartheta_q^{KI}+a_{pq}^{I}\vartheta_q^{JI}-a_{pq}^{J}\vartheta_q^{II}+a_{pq}^{K}\vartheta_q^{RI}\Big)\varepsilon_2^{K^{1-\zeta}}\Bigg\}V_2 \\
& +\sum_{q=1}^{n}\Bigg\{\Big(a_{pq}^{R}\vartheta_q^{KJ}+a_{pq}^{I}\vartheta_q^{JJ}-a_{pq}^{J}\vartheta_q^{IJ}+a_{pq}^{K}\vartheta_q^{RJ}\Big)\varepsilon_3^{K^{1-\zeta}}\Bigg\}V_3.
\end{aligned}
\tag{13}
$$

constant $\xi > 0$ such that $\min(\lambda^R, \lambda^I, \lambda^J, \lambda^K) \geq \xi > 0$. This implies that

$$D^{\alpha}V(t, e(t)) \leq -\xi V(t, e(t)), \text{ for } t \neq t_k. \tag{15}$$

When $t = t_k$ $(k = 1, 2 \ldots, m)$, we obtain inequality (16) (on the next page).

Therefore, as a consequence of Lemma 4 and inequalities (15) and (16), it can be concluded that the impulsive fractional-order QVNN as QVNNs (1) is Mittag-Leffler stable.

$$\begin{aligned}
D^{\alpha}V(t,e(t)) =& \sum_{q=1}^{n}\bigg\{-c_p\zeta+\bar{A}_{pq}^{R}\Big((\zeta-1)\varepsilon_1^{R}+\varepsilon_1^{R^{1-\zeta}}\Big)+\bar{A}_{pq}^{I}(\zeta-1)\varepsilon_2^{R}+\bar{A}_{pq}^{J}(\zeta-1)\varepsilon_3^{R}+\bar{A}_{pq}^{K}(\zeta-1)\varepsilon_4^{R}\\
&+\tilde{A}_{pq}^{R}\varepsilon_1^{I^{1-\zeta}}+\check{A}_{pq}^{R}\varepsilon_1^{J^{1-\zeta}}+\hat{A}_{pq}^{R}\varepsilon_1^{K^{1-\zeta}}\bigg\}V_1+\sum_{q=1}^{n}\bigg\{-c_p\zeta+\tilde{A}_{pq}^{R}\Big((\zeta-1)\varepsilon_1^{I}\Big)+\tilde{A}_{pq}^{I}\Big((\zeta-1)\varepsilon_2^{I}\\
&+\varepsilon_2^{I^{1-\zeta}}\Big)+\tilde{A}_{pq}^{J}(\zeta-1)\varepsilon_3^{I}+\tilde{A}_{pq}^{K}(\zeta-1)\varepsilon_4^{I}+\bar{A}_{pq}^{I}\varepsilon_2^{R^{1-\zeta}}+\check{A}_{pq}^{I}\varepsilon_2^{J^{1-\zeta}}+\hat{A}_{pq}^{I}\varepsilon_2^{K^{1-\zeta}}\bigg\}V_2(t)\\
&+\sum_{q=1}^{n}\bigg\{-c_p\zeta+\check{A}_{pq}^{R}\Big((\zeta-1)\varepsilon_1^{J}\Big)+\check{A}_{pq}^{I}(\zeta-1)\varepsilon_2^{J}+\check{A}_{pq}^{J}\Big((\zeta-1)\varepsilon_3^{J}+\varepsilon_3^{J^{1-\zeta}}\Big)+\check{A}_{pq}^{K}(\zeta-1)\varepsilon_4^{J}\\
&+\bar{A}_{pq}^{J}\varepsilon_3^{R^{1-\zeta}}+\tilde{A}_{pq}^{J}\varepsilon_3^{I^{1-\zeta}}+\hat{A}_{pq}^{J}\varepsilon_3^{K^{1-\zeta}}\bigg\}V_3+\sum_{q=1}^{n}\bigg\{-c_p\zeta+\hat{A}_{pq}^{R}\Big((\zeta-1)\varepsilon_1^{K}\Big)+\hat{A}_{pq}^{I}(\zeta-1)\varepsilon_2^{K}\\
&+\hat{A}_{pq}^{J}(\zeta-1)\varepsilon_3^{K}+\hat{A}_{pq}^{K}\Big((\zeta-1)\varepsilon_4^{K}+\varepsilon_4^{K^{1-\zeta}}\Big)+\bar{A}_{pq}^{K}\varepsilon_4^{R^{1-\zeta}}+\tilde{A}_{pq}^{K}\varepsilon_4^{I^{1-\zeta}}+\check{A}_{pq}^{J}\varepsilon_4^{K^{1-\zeta}}\bigg\}V_4\\
\le& -\lambda^{R}V_1-\lambda^{I}V_2-\lambda^{J}V_3-\lambda^{K}V_4.
\end{aligned}\tag{14}$$

$$\begin{aligned}
V(t_k^{+},e(t_k^{+})) =& \sum_{p=1}^{n}\zeta^{-1}\Big|e_p^{R}(t_k)+\eta_{kp}^{R}(e_p^{R}(t_k))\Big|^{\zeta}+\sum_{p=1}^{n}\zeta^{-1}\Big|e_p^{I}(t_k)+\eta_{kp}^{I}(e_p^{I}(t_k))\Big|^{\zeta}\\
&+\sum_{p=1}^{n}\zeta^{-1}\Big|e_p^{J}(t_k)+\eta_{kp}^{J}(e_p^{J}(t_k))\Big|^{\zeta}+\sum_{p=1}^{n}\zeta^{-1}\Big|e_p^{K}(t_k)+\eta_{kp}^{K}(e_p^{K}(t_k))\Big|^{\zeta}\\
=& \sum_{p=1}^{n}\zeta^{-1}\Big|h_p^{R}(t_k)-h^{*}-\delta_{kp}^{R}\Big(h_p(t_k)-h^{*}\Big)\Big|^{\zeta}+\sum_{p=1}^{n}\zeta^{-1}\Big|h_p^{I}(t_k)-h^{*}-\delta_{kp}^{I}\Big(h_p(t_k)-h^{*}\Big)\Big|^{\zeta}\\
&+\sum_{p=1}^{n}\zeta^{-1}\Big|h_p^{J}(t_k)-h^{*}-\delta_{kp}^{J}\Big(h_p(t_k)-h^{*}\Big)\Big|^{\zeta}+\sum_{p=1}^{n}\zeta^{-1}\Big|h_p^{K}(t_k)-h^{*}-\delta_{kp}^{K}\Big(h_p(t_k)-h^{*}\Big)\Big|^{\zeta}\\
=& \sum_{p=1}^{n}\zeta^{-1}\Big|1-\delta_{kp}^{R}\Big|^{\zeta}\Big|h_p^{R}(t_k)-h^{*}\Big|^{\zeta}+\sum_{p=1}^{n}\zeta^{-1}\Big|1-\delta_{kp}^{I}\Big|^{\zeta}\Big|h_p^{I}(t_k)-h^{*}\Big|^{\zeta}\\
&+\sum_{p=1}^{n}\zeta^{-1}\Big|1-\delta_{kp}^{J}\Big|^{\zeta}\Big|h_p^{J}(t_k)-h^{*}\Big|^{\zeta}+\sum_{p=1}^{n}\zeta^{-1}\Big|1-\delta_{kp}^{K}\Big|^{\zeta}\Big|h_p^{K}(t_k)-h^{*}\Big|^{\zeta}\\
\le& V(t_k,e(t_k)).
\end{aligned}\tag{16}$$

Remark 2 Theorem 1 gives the Mittag-Leffler stability conditions for the impulsive fractional-order QVNNs (1). This condition depends on the parameters of NNs and the activation functions.

Remark 3 In the past few decades, NNs have been studied extensively for their wide applications. Mittag-Leffler stability analysis of RVNNs and CVNNs has attracted the attention of many researchers. Liu P et al. (2018) introduced a class of integer-order recurrent NNs with unbounded time-varying delays. Using the geometrical properties of non-monotonic activation functions, they showed that the addressed system has exactly $(2K_i+1)^n$ equilibrium points, of which (K_i+1) were locally asymptotically stable while others were unstable. Tyagi et al. (2016) provided sufficient conditions for Mittag-Leffler stability of the equilibrium points for CVNNs. In this study, sufficient conditions are analyzed to achieve the coexistence and Mittag-Leffler stability of equilibrium points. The number of equilibrium points of QVNNs is larger than that of CVNNs.

Theorem 2 Assume that the conditions of Lemmas 2 and 3 hold, and $\beta_k^{\nu}(h^{\nu}(t_k)) = -\delta(h^{\nu}(t_k)-h^{*})$

$(k=1,2,\ldots,m)$, where h^* is the steady state of the impulsive NNs (1). If $|1-\delta_{kp}^{\nu}|\leq 1$ and there exist

$$\begin{aligned}\psi^R=&\min_{1\leq p\leq n}\left[c_p-\sum_{q=1}^{n}\left(|a_{pq}^R|+|a_{pq}^I|+|a_{pq}^J|+|a_{pq}^K|\right)\right.\\&\left.\cdot\left(\vartheta_q^{RR}+\vartheta_q^{IR}+\vartheta_q^{JR}+\vartheta_q^{KR}\right)\right]>0,\\\psi^I=&\min_{1\leq p\leq n}\left[c_p-\sum_{q=1}^{n}\left(|a_{pq}^R|+|a_{pq}^I|+|a_{pq}^J|+|a_{pq}^K|\right)\right.\\&\left.\cdot\left(\vartheta_q^{RI}+\vartheta_q^{II}+\vartheta_q^{JI}+\vartheta_q^{KI}\right)\right]>0,\\\psi^J=&\min_{1\leq p\leq n}\left[c_p-\sum_{q=1}^{n}\left(|a_{pq}^R|+|a_{pq}^I|+|a_{pq}^J|+|a_{pq}^K|\right)\right.\\&\left.\cdot\left(\vartheta_q^{RJ}+\vartheta_q^{IJ}+\vartheta_q^{JJ}+\vartheta_q^{KJ}\right)\right]>0,\\\psi^K=&\min_{1\leq p\leq n}\left[c_p-\sum_{q=1}^{n}\left(|a_{pq}^R|+|a_{pq}^I|+|a_{pq}^J|+|a_{pq}^K|\right)\right.\\&\left.\cdot\left(\vartheta_q^{RK}+\vartheta_q^{IK}+\vartheta_q^{JK}+\vartheta_q^{KK}\right)\right]>0,\end{aligned}\tag{17}$$

then the impulsive fractional-order QVNN (1) is Mittag-Leffler stable.

Proof Consider the following Lyapunov function candidate:

$$V(t,e(t))=V_1+V_2+V_3+V_4,\tag{18}$$

where

$$\begin{aligned}V_1=&\sum_{p=1}^{n}|e_p^R(t)|,\ V_2=\sum_{p=1}^{n}|e_p^I(t)|,\\V_3=&\sum_{p=1}^{n}|e_p^J(t)|,\ V_4=\sum_{p=1}^{n}|e_p^K(t)|.\end{aligned}$$

When $t\neq t_k$ $(k=1,2,\ldots,m)$, calculating the fractional-order derivative of $V(t,e(t))$ along the trajectories of Eq. (3), we obtain inequality (19) (on the next page).

Similarly, we can obtain inequalities (20)–(22) (on the next page). Adding inequalities (19)–(22), we obtain inequality (23) (on the next page).

From inequality (17), we can select a positive constant $\xi>0$ such that $\min(\psi^R,\psi^I,\psi^J,\psi^K)\geq\xi>0$. This implies that

$$D^{\alpha}V(t,e(t))\leq-\xi V(t,e(t)),\ t\neq t_k.\tag{24}$$

When $t=t_k$ $(k=1,2,\ldots,m)$, we have

$$\begin{aligned}V(t_k^+,e(t_k^+))=&\sum_{p=1}^{n}\left|e_p^R(t_k)+\eta_{kp}^R(e_p^R(t_k))\right|\\&+\sum_{p=1}^{n}\left|e_p^I(t_k)+\eta_{kp}^I(e_p^I(t_k))\right|\\&+\sum_{p=1}^{n}\left|e_p^J(t_k)+\eta_{kp}^J(e_p^J(t_k))\right|\\&+\sum_{p=1}^{n}\left|e_p^K(t_k)+\eta_{kp}^K(e_p^K(t_k))\right|\\=&\sum_{p=1}^{n}\left|h_p^R(t_k)-h^*-\delta_{kp}^R(h_p(t_k)-h^*)\right|\\&+\sum_{p=1}^{n}\left|h_p^I(t_k)-h^*-\delta_{kp}^I(h_p(t_k)-h^*)\right|\\&+\sum_{p=1}^{n}\left|h_p^J(t_k)-h^*-\delta_{kp}^J(h_p(t_k)-h^*)\right|\\&+\sum_{p=1}^{n}\left|h_p^K(t_k)-h^*-\delta_{kp}^K(h_p(t_k)-h^*)\right|\\=&\sum_{p=1}^{n}\left|1-\delta_{kp}^R\right|\cdot\left|h_p^R(t_k)-h^*\right|\\&+\sum_{p=1}^{n}\left|1-\delta_{kp}^I\right|\cdot\left|h_p^I(t_k)-h^*\right|\\&+\sum_{p=1}^{n}\left|1-\delta_{kp}^J\right|\cdot\left|h_p^J(t_k)-h^*\right|\\&+\sum_{p=1}^{n}\left|1-\delta_{kp}^K\right|\cdot\left|h_p^K(t_k)-h^*\right|\\\leq&V(t_k,e(t_k)).\end{aligned}\tag{25}$$

Therefore, from Lemma 4 and inequalities (24) and (25), it can be concluded that the impulsive fractional-order QVNN as QVNNs (1) is Mittag-Leffler stable.

Remark 4 In reviewing the existing works (Wang F et al., 2015; Song et al., 2016a; Wang LM et al., 2017), fractional-order CVNNs (Wang F et al., 2015; Wang LM et al., 2017) and integer-order systems (Song et al., 2016a, 2016b) have been studied extensively over the last few decades. Recently, the Mittag-Leffler stability analysis of fractional-order QVNNs was investigated by Yang et al. (2018). However, the results on

$$\begin{aligned}D^\alpha V_1 \leq &\sum_{p=1}^{n} \operatorname{sgn}(e_p^R(t)) D^\alpha e_p^R(t)\\ =&\sum_{p=1}^{n}\Bigg[\Bigg(-c_p+\sum_{q=1}^{n}\Big(|a_{pq}^R|\vartheta_p^{RR}+|a_{pq}^I|\vartheta_p^{IR}+|a_{pq}^J|\vartheta_p^{JR}+|a_{pq}^K|\vartheta_p^{KR}\Big)\Bigg)|e_q^R(t)|+\sum_{q=1}^{n}\Big(|a_{pq}^R|\vartheta_p^{RI}+|a_{pq}^I|\vartheta_p^{II}\\ &+|a_{pq}^J|\vartheta_p^{JI}+|a_{pq}^K|\vartheta_p^{KI}\Big)|e_q^I(t)|+\sum_{q=1}^{n}\Big(|a_{pq}^R|\vartheta_p^{RJ}+|a_{pq}^I|\vartheta_p^{IJ}+|a_{pq}^J|\vartheta_p^{JJ}+|a_{pq}^K|\vartheta_p^{KJ}\Big)|e_q^J(t)|\\ &+\sum_{q=1}^{n}\Big(|a_{pq}^R|\vartheta_p^{RK}+|a_{pq}^I|\vartheta_p^{IK}+|a_{pq}^J|\vartheta_p^{JK}+|a_{pq}^K|\vartheta_p^{KK}\Big)|e_q^K(t)|\Bigg]. \end{aligned} \tag{19}$$

$$\begin{aligned}D^\alpha V_2 \leq &\sum_{p=1}^{n}\Bigg[\sum_{q=1}^{n}\Big(|a_{pq}^R|\vartheta_p^{IR}+|a_{pq}^I|\vartheta_p^{RR}+|a_{pq}^J|\vartheta_p^{KR}+|a_{pq}^K|\vartheta_p^{JR}\Big)|e_q^R(t)|+\sum_{q=1}^{n}\Big(-c_p+|a_{pq}^R|\vartheta_p^{II}+|a_{pq}^I|\vartheta_p^{RI}\\ &+|a_{pq}^J|\vartheta_p^{KI}+|a_{pq}^K|\vartheta_p^{JI}\Big)|e_q^I(t)|+\sum_{q=1}^{n}\Big(|a_{pq}^R|\vartheta_p^{IJ}+|a_{pq}^I|\vartheta_p^{RJ}+|a_{pq}^J|\vartheta_p^{KJ}+|a_{pq}^K|\vartheta_p^{JJ}\Big)|e_q^J(t)|\\ &+\sum_{q=1}^{n}\Big(|a_{pq}^R|\vartheta_p^{IK}+|a_{pq}^I|\vartheta_p^{RK}+|a_{pq}^J|\vartheta_p^{KK}+|a_{pq}^K|\vartheta_p^{JK}\Big)|e_q^K(t)|\Bigg]. \end{aligned} \tag{20}$$

$$\begin{aligned}D^\alpha V_3 \leq &\sum_{p=1}^{n}\Bigg[\sum_{q=1}^{n}\Big(|a_{pq}^R|\vartheta_p^{JR}+|a_{pq}^I|\vartheta_p^{KR}+|a_{pq}^J|\vartheta_p^{RR}+|a_{pq}^K|\vartheta_p^{IR}\Big)|e_q^R(t)|+\sum_{q=1}^{n}\Big(|a_{pq}^R|\vartheta_p^{JI}+|a_{pq}^I|\vartheta_p^{KI}\\ &+|a_{pq}^J|\vartheta_p^{RI}+|a_{pq}^K|\vartheta_p^{II}\Big)|e_q^I(t)|+\sum_{q=1}^{n}\Big(-c_p+|a_{pq}^R|\vartheta_p^{JJ}+|a_{pq}^I|\vartheta_p^{KJ}+|a_{pq}^J|\vartheta_p^{RJ}+|a_{pq}^K|\vartheta_p^{IJ}\Big)|e_q^J(t)|\\ &+\sum_{q=1}^{n}\Big(|a_{pq}^R|\vartheta_p^{JK}+|a_{pq}^I|\vartheta_p^{KK}+|a_{pq}^J|\vartheta_p^{RK}+|a_{pq}^K|\vartheta_p^{IK}\Big)|e_q^K(t)|\Bigg]. \end{aligned} \tag{21}$$

$$\begin{aligned}lD^\alpha V_4 \leq &\sum_{p=1}^{n}\Bigg[\sum_{q=1}^{n}\Big(|a_{pq}^R|\vartheta_p^{KR}+|a_{pq}^I|\vartheta_p^{JR}+|a_{pq}^J|\vartheta_p^{IR}+|a_{pq}^K|\vartheta_p^{RR}\Big)|e_q^R(t)|+\sum_{q=1}^{n}\Big(|a_{pq}^R|\vartheta_p^{KI}+|a_{pq}^I|\vartheta_p^{JI}\\ &+|a_{pq}^J|\vartheta_p^{II}+|a_{pq}^K|\vartheta_p^{RI}\Big)|e_q^I(t)|+\sum_{q=1}^{n}\Big(|a_{pq}^R|\vartheta_p^{KJ}+|a_{pq}^I|\vartheta_p^{JJ}+|a_{pq}^J|\vartheta_p^{IJ}+|a_{pq}^K|\vartheta_p^{RJ}\Big)|e_q^J(t)|\\ &+\sum_{q=1}^{n}\Big(|a_{pq}^R|\vartheta_p^{KK}+|a_{pq}^I|\vartheta_p^{JK}+|a_{pq}^J|\vartheta_p^{IK}+|a_{pq}^K|\vartheta_p^{rK}\Big)|e_q^K(t)|\Bigg]. \end{aligned} \tag{22}$$

$$\begin{aligned}D^\alpha V(t,e(t)) \leq &-\min_{1\leq p\leq n}\Bigg[c_p-\sum_{q=1}^{n}\Big(|a_{pq}^R|+|a_{pq}^I|+|a_{pq}^J|+|a_{pq}^K|\Big)\Big(\vartheta_q^{RR}+\vartheta_q^{IR}+\vartheta_q^{JR}+\vartheta_q^{KR}\Big)\Bigg]V_1\\ &-\min_{1\leq p\leq n}\Bigg[c_p-\sum_{q=1}^{n}\Big(|a_{pq}^R|+|a_{pq}^I|+|a_{pq}^J|+|a_{pq}^K|\Big)\Big(\vartheta_q^{RI}+\vartheta_q^{II}+\vartheta_q^{JI}+\vartheta_q^{KI}\Big)\Bigg]V_2\\ &-\min_{1\leq p\leq n}\Bigg[c_p-\sum_{q=1}^{n}\Big(|a_{pq}^R|+|a_{pq}^I|+|a_{pq}^J|+|a_{pq}^K|\Big)\Big(\vartheta_q^{RJ}+\vartheta_q^{IJ}+\vartheta_q^{JJ}+\vartheta_q^{KJ}\Big)\Bigg]V_3\\ &-\min_{1\leq p\leq n}\Bigg[c_p-\sum_{q=1}^{n}\Big(|a_{pq}^R|+|a_{pq}^I|+|a_{pq}^J|+|a_{pq}^K|\Big)\Big(\vartheta_q^{RK}+\vartheta_q^{IK}+\vartheta_q^{JK}+\vartheta_q^{KK}\Big)\Bigg]V_4. \end{aligned} \tag{23}$$

impulsive fractional-order QVNNs have not been discussed to date. In light of this, we have analyzed the Mittag-Leffler stability of multiple equilibrium points for the impulsive fractional-order QVNNs using the Lyapunov direct method, which is much more complicated than that of CVNNs.

4 Numerical simulations

In this section, simulation results are given to illustrate the efficiency of the impulsive fractional-order QVNNs.

Example 1 Consider the following impulsive fractional-order QVNN:

$$\begin{cases} D^{0.9}h_1(t) = -c_1h_1(t) + a_{11}f_1(h_1(t)) \\ \qquad\qquad + a_{12}f_2(h_2(t)) + \mathcal{R}_1, \\ \Delta h_p(t_k) = \delta_{k1}(h_1(t_k) - h^*), \\ D^{0.9}h_2(t) = -c_2h_2(t) + a_{21}f_1(h_1(t)) \\ \qquad\qquad + a_{22}f_2(h_2(t)) + \mathcal{R}_2, \\ \Delta h_p(t_k) = \delta_{k2}(h_2(t_k) - h^*), \end{cases} \tag{26}$$

where $k = 1, 2, \ldots, m$, $p = 1, 2, \ldots, n$, $f_1(h_1(t)) = f_2(h_2(t)) = \tanh(t)$, $a_{11} = 8.6 + 0.12\mathrm{i} - 0.5\mathrm{j} + 1.2\mathrm{k}$, $a_{12} = 0.4 + 0.36\mathrm{i} + 0.5\mathrm{j} + 0.04\mathrm{k}$, $a_{21} = 0.3 - 0.1\mathrm{i} + 0.25\mathrm{j} + 0.42\mathrm{k}, a_{22} = 6 + 0.25\mathrm{i} + 0.1\mathrm{j} + 0.2\mathrm{k}$, $\mathcal{R}_1 = 0.5 + 0.4\mathrm{i} - 0.6\mathrm{j} + 0.6\mathrm{k}$, $\mathcal{R}_2 = 0.7 - 1.1\mathrm{i} + 0.4\mathrm{j} - 1.4\mathrm{k}$, $c_1 = c_2 = 4.5$, $\vartheta_1^{\nu_1\nu_2} = \vartheta_2^{\nu_1\nu_2} = 0.01$, and $\nu_1, \nu_2 \in \{R, I, J, K\}$. Calculation shows that inequality (7) in Theorem 1 is satisfied with $p = 1, 2$ and $q = 1, 2$. Consequently, QVNN (26) is Mittag-Leffler stable. Fig. 1 shows the trajectories of QVNN (26) with different initial values.

Example 2 Consider the following impulsive fractional-order QVNN:

$$\begin{cases} D^{0.7}h_1(t) = -c_1h_1(t) + a_{11}f_1(h_1(t)) \\ \qquad\qquad + a_{12}f_2(h_2(t)) + \mathcal{R}_1, \\ \Delta h_p(t_k) = \delta_{k1}(h_1(t_k) - h^*), \\ D^{0.7}h_2(t) = -c_2h_2(t) + a_{21}f_1(h_1(t)) \\ \qquad\qquad + a_{22}f_2(h_2(t)) + \mathcal{R}_2, \\ \Delta h_p(t_k) = \delta_{k2}(h_2(t_k) - h^*), \end{cases} \tag{27}$$

where $k = 1, 2, \ldots, m$, $p = 1, 2, \ldots, n$, $f_1(h_1(t)) = f_2(h_2(t)) = \exp(-t^2)$, $a_{11} = 6.3 + 0.1\mathrm{i} - 0.2\mathrm{j} + 1.4\mathrm{k}$, $a_{12} = 0.8 + 0.6\mathrm{i} + 0.1\mathrm{j} + 0.001\mathrm{k}$, $a_{21} = 0.3 + 0.3\mathrm{i} - 0.12\mathrm{j} + 0.4\mathrm{k}$, $a_{22} = 4.9 - 0.32\mathrm{i} + 0\mathrm{j} + 0.25\mathrm{k}$, $\mathcal{R}_1 = 0.9 + 0.8\mathrm{i} + 0.5\mathrm{j} + 0.4\mathrm{k}$, $\mathcal{R}_2 = 1.7 - 2.1\mathrm{i} +$

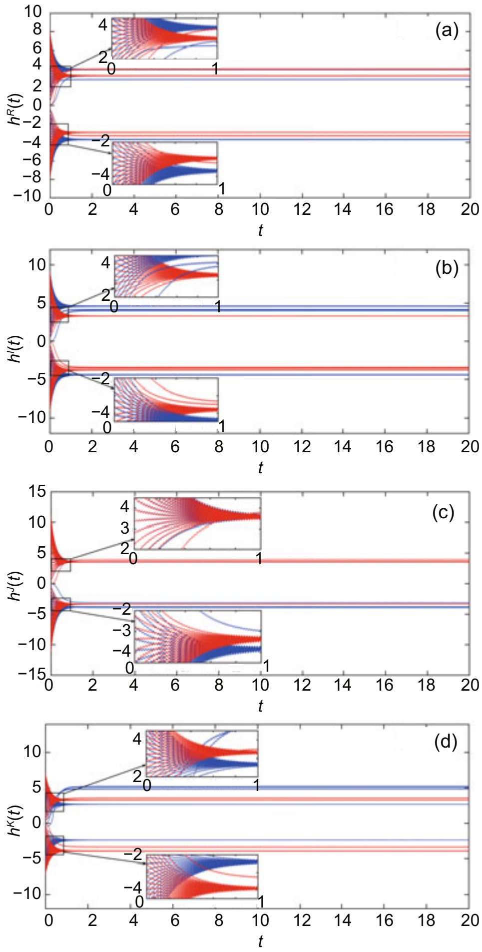

Fig. 1 Trajectories of state variables $h^R(t)$ (a), $h^I(t)$ (b), $h^J(t)$ (c), and $h^K(t)$ (d) in Example 1 with $\alpha = 0.9$ and $t_k = 0.02\mathrm{k}$

$0.4\mathrm{j} + 1.4\mathrm{k}, c_1 = c_2 = 5.5$, $\vartheta_1^{\nu_1\nu_2} = \vartheta_2^{\nu_1\nu_2} = 0.01$ and $\nu_1, \nu_2 \in \{R, I, J, K\}$. Calculation shows that inequality (7) in Theorem 1 is satisfied with $p = 1, 2$ and $q = 1, 2$. Consequently, QVNN (27) is Mittag-Leffler stable. Fig. 2 shows that the trajectories of QVNN (27) are Mittag-Leffler stable with different initial values.

5 Conclusions

In this study, we investigated the Mittag-Leffler stability analysis of multiple equilibrium points for fractional-order QVNNs with an impulse term. By employing the non-commutative property of quaternion multiplication, QVNNs were converted into four RVNNs. According to the definition of the activation functions, the existence of equilibrium points was also analyzed. Sufficient conditions were

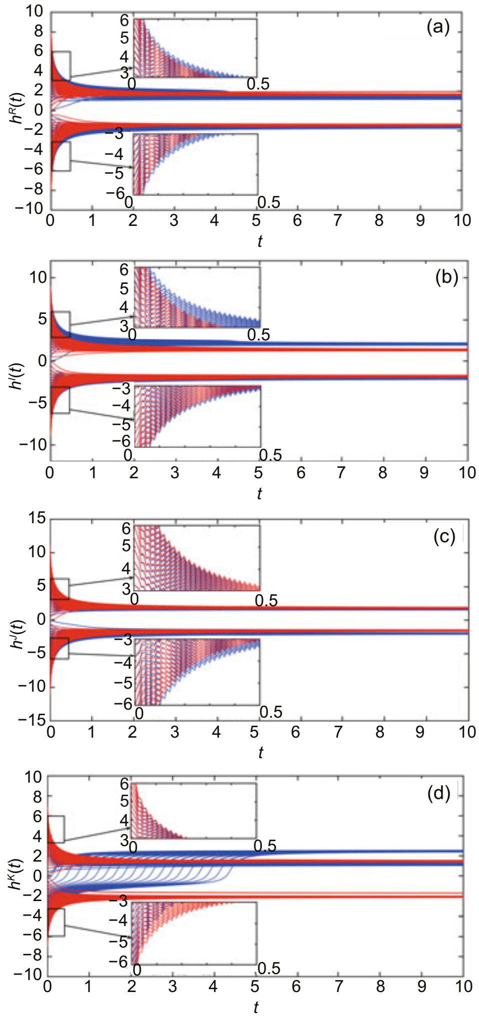

Fig. 2 Trajectories of state variables $h^R(t)$ (a), $h^I(t)$ (b), $h^J(t)$ (c), and $h^K(t)$ (d) in Example 2 with $\alpha = 0.7$ and $t_k = 0.02$k

derived to assure the Mittag-Leffler stability of multiple equilibrium points for QVNNs. Simulation results validated our theoretical solutions.

Contributors

K. UDHAYAKUMAR and R. RAKKIYAPPAN designed the research. K. UDHAYAKUMAR drafted the manuscript. Jin-de CAO helped organize the manuscript and gave some suggestions to improve the results. Jin-de CAO and Xue-gang TAN revised and finalized the paper.

Compliance with ethics guidelines

K. UDHAYAKUMAR, R. RAKKIYAPPAN, Jin-de CAO, and Xue-gang TAN declare that they have no conflict of interest.

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