ISSN 1009 – 3095 *Journal of Zhejiang University SCIENCE* V.3, No.1, P.94 – 99, Jan. – Mar., 2002
http://www.periodicals.com.cn; http://www.zju.edu.cn/depart/zdxb
http://lib.zju.edu.cn/eindex.htm; http://www.zjupress.com; jzu_s@mail.hz.zj.cn

Decomposition in blocks at the level of wavelet coefficients and $T(1)$ theorem on Hardy space*

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Received June 18, 2000; revision accepted Apr.18, 2001
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Abstract: This paper deals with the establishment of $T(1)$ theorem on Hardy space H^1 under condition of weak regularity. An operator or a function is identified on the basis of their wavelet coefficients which are regrouped on some blocks. The actions of each block operator (pseudo-annular operator) on each block function (atom) are exactly analyzed to establish $T(1)$ theorem on Hardy space.

Key words: hardy space, wavelet coefficients, blocks
Document code: A **CLC number:** O177.8

INTRODUCTION

In this paper, $S(R^n)$ denotes fast decreasing function space, $S'(R^n)$ the dual space of $S(R^n)$ and $S'(R^n \times R^n)$ the dual space of $S'(R^{2n})$, BMO the bounded mean oscillation space.

Fourier transformation may be used to study a convolution operator's continuity, but cannot be easily used to analyse a non-convolution operator. So some other analyse methods must be founded to study it. A famous result is David Journe's $T(1)$ theorem (David et al., 1984). Let T be an operator which is continuous from $S(R^n)$ to $S'(R^n)$, then T is associated with a kernel distribution $K(x,y) \in S'(R^n \times R^n)$ in the sense that $\langle Tf, g\rangle = \langle K(x,y), g(x)f(y)\rangle$ for test functions f and g. Given $0 < \gamma \leqslant 1$, one assumes that $K(x,y)$ satisfies the pointwise conditions

$$|K(x,y)| \leqslant A|x-y|^{-n} \qquad (1)$$

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leqslant B\frac{|x-x'|^{\gamma}}{|x-y|^{n+\gamma}}, \ \forall |x-x'| \leqslant \frac{1}{2}|x-y|. \qquad (2)$$

$T(1)$ thoerem states that: If T satisfies conditions Eq.(1) and Eq.(2), then T can extend to a bounded operator on L^2, if and only if T satisfies

$$T(1) \in BMO, \ T^t(1) \in BMO\ . \qquad (3)$$

and the weak boundedness property:

$$|\langle Tf, g\rangle| \leqslant Ct^n(\|f\|_{\infty} + t\|\nabla f\|_{\infty}) \cdot (\|g\|_{\infty} + t\|\nabla g\|_{\infty}), \ \forall f, g \in C_0^{\infty}(R^n),$$
$$\text{and } \max\{\text{diamsupp } f, \text{diamsupp } g\} \leqslant t. \qquad (4)$$

To establish the $T(1)$ theorem in different function spaces, the pointwise conditions Eq.(1) and Eq.(2) are not necessary. After David and Journé have established their famous $T(1)$ theorem, many other researchers tried to establish a strong continuity under a more and more weak regularity. Meyer replaced Eq.(1) and Eq.(2) by the integral estimates:

$$\sup_{r>0}\int_{r \leqslant |x-y| \leqslant 2r}(|K(x,y)| + |K(y,x)|)\mathrm{d}y \leqslant C, \qquad (5)$$

$$\sup_{\substack{r>0\\ |u|+|v| \leqslant r}}\int_{2^k r \leqslant |x-y| \leqslant 2^{k+1} r}\{(|K(x+u,y+v) - K(x,y)| + |K(y+u,x+v) - K(y,x)|)\mathrm{d}y\} = B(k),$$ where $k = 1,2,3,\cdots$, and $B(k)$ satisfy: (6)

* Project supported by the National Natural Science Foundation of China (No.1000102) and Education Ministry of China.

$$\sum_{k=1}^{\infty} kB(k) < \infty . \quad (7)$$

Meyer (1985) proved that: If T satisfies Eqs.(4),(5),(6) and (7), and T satisfies also

$$T(1) = T^t(1) = 0. \quad (8)$$

then T continues on $\dot{B}_1^{0,1}$.

Han and Hofman (1993) proved that: If T satisfies almost the same conditions as Meyer's, then $\forall 1 \leqslant p, q \leqslant \infty$, T continues on $\dot{B}_p^{0,q}$ and $\forall 1 < p, q < \infty$, T continues on $\dot{F}_p^{0,q}$.

But all these ideas cannot be applied to the study of continuity on Hardy space H^1. This work uses a decomposition in blocks at the level of coefficients for operators and for functions, establishes a strong continuity on H^1, and thus a strong continuity on $L^p (1 < p < \infty)$.

MAIN THEOREM

If T satisfies Eqs.(4),(6) and (8), then one has:

(i), If T satisfies the following condition

$$\sum_{k=1}^{\infty} k^{\frac{3}{2}} B(k) < \infty , \quad (9)$$

then T is continuous on H^1.

(ii) If T satisfies the following condition

$$\sum_{k=1}^{\infty} k^{\frac{1}{2}} B(k) < \infty , \quad (10)$$

then $\forall 1 < p < \infty$, T is continuous on L^p.

PRELIMINARIES AND SOME IMPORTANT RESULTS

In this paper, $B(x, R)$ denotes a ball with radius R and with center at x, and denotes also $E_n = \{0,1\}^n \setminus \{0\}$ and $\Lambda_n = \{\lambda = (\varepsilon, j, k), \varepsilon \in E_n, j \in Z, k \in Z^n\}$. Then $\forall \varepsilon \in E_n$, denotes $\Phi^{(\varepsilon)}(x)$ wavelets of Daubechies in $C_0^2(B(0, 2^M))$, and $\Phi_\lambda(x) = \Phi_{j,k}^{(\varepsilon)}(x) = 2^{\frac{n}{2}} \Phi^{(\varepsilon)}(2^j x - k)$, and $\{\Phi_\lambda(x)\}_{\lambda \in \Lambda_n}$ is an orthonormal basis for $L^2(R^n)$. According to Schwartz's kernel theorem, a distribution $K(x, y) \in S'(R^n \times R^n)$ is associated with a linear continuous mapping T: $S(R^n) \rightarrow S'(R^n)$

$$Tf(x) = \int t\, K(x, y) f(y) \mathrm{d}y$$

Then one can analyse an operator on wavelet bases in $2n$ dimension. One denotes $\Gamma = \{(j, k, l), j \in Z, k, l \in Z^n\}$, and $\Lambda = \Lambda_{2n} = \{\lambda = (\varepsilon, \varepsilon', j, k, l), (\varepsilon, \varepsilon') \in E_{2n}, (j, k, l) \in \Gamma\} = E_{2n} \times \Gamma$. If an operator T satisfies Eq.(4), then one can define

$$a_\lambda = a_{j,k,l}^{\varepsilon,\varepsilon'} = \langle K(x, y), \Phi_\lambda(x, y) \rangle, \quad \forall \lambda \in \Lambda. \quad (11)$$

and the following equality is true in the sense of distribution:

$$K(x, y) = \sum_{\lambda \in \Lambda} a_\lambda \Phi_\lambda(x, y). \quad (12)$$

The Beylkin-Coifman-Rokhlin algorithm (Beylkin et al., 1991) says that $\{a_\lambda\}_{\lambda \in \Lambda}$ becomes a new representation for an operator. $\forall \alpha \in \{0,1\}^n$, one denotes $\delta_\alpha = 1$, if $\alpha = 0$; $\delta_\alpha = 0$, if $\alpha \neq 0$. Furthermore, $\forall R > 0$, one denotes $A(R) = \sup_{\substack{\lambda \in \Lambda \\ 2^{R-1} \leqslant |k-l| < 2^R}} |a_\lambda|$. Then one has:

Theorem 1 If T satisfies Eq.(4) and Eq.(6), then one has:

(i) $\forall t \geqslant 0$, if T satisfies the following condition:

$$\sum_{k=1}^{\infty} k^t B(k) < \infty , \quad (13)$$

then T satisfies

$$\sum_{R=1}^{\infty} R^t A(e) < \infty , \quad (14)$$

(ii) If T satisfies Eq.(14) with $t = 0$ and Eq.(8), then one has:

$$\sum_{m \in Z^n} a_{j,k+m\delta_\varepsilon, l+\delta_{\varepsilon'}}^{\varepsilon,\varepsilon'} = 0, \forall (\varepsilon, \varepsilon') \in E_{2n}, |\varepsilon||\varepsilon'| = 0 \text{ and } (j, k, l) \in \Gamma. \quad (15)$$

In Deng et al., (1998), such a result is proved for $n = 1$ and $t = 1$; here, one can repeat almost word for word the same proof, and finishes the proof of this theorem.

Let $U = \{u = (\varepsilon, \varepsilon', R), (\varepsilon, \varepsilon') \in E_{2n}, R \in Z, R \geqslant 0\}$. If T satisfies Eqs.(4), (6) and (13) with $t = 0$, applying Eq.(11), one can decompose an operator into a group of compact operator $\{T_u\}_{u \in U}$ where $T_u = T_R^{\varepsilon,\varepsilon'}$ is defined by its kernel-distribution $K_R^{\varepsilon,\varepsilon'}(x, y) = \sum_{\lambda \in \Gamma} a_R^{\varepsilon,\varepsilon'}(\lambda) \Phi_{(\varepsilon,\varepsilon',\lambda)}(x, y)$ (where a compact operator means an operator maps a function with compact support to a function with compact support modulate a sufficiently regular function). In fact, $\forall \lambda = (j, k, l) \in \Gamma$, one defines $a_R^{\varepsilon, \varepsilon'}(\lambda)$ as follows:

If $R = 0$, one defines:

$$a_0^{\varepsilon, \varepsilon'}(\lambda) = \begin{cases} a_{j,k,k}^{(\varepsilon,\varepsilon')} - (\delta_\varepsilon + \delta_{\varepsilon'}) \sum_{m \neq 0} a_{j, k+m\delta_\varepsilon, k+m\delta_{\varepsilon'}}^{(\varepsilon,\varepsilon')}, & \forall \lambda = (j,k,k); \\ 0, & \text{otherwise}. \end{cases}$$

If $R > 0$, one defines:

$$a_R^{\varepsilon, \varepsilon'}(\lambda) = \begin{cases} a_{(\varepsilon, \varepsilon', \lambda)}, & 2^{R-1} \leqslant |k - l| < 2^R; \\ -(\delta_\varepsilon + \delta_{\varepsilon'}) \sum_{m: 2^{R-1} \leqslant |m - k\delta_\varepsilon - l\delta_{\varepsilon'}| < 2^R} a_{j, k+m\delta_\varepsilon, l+m\delta_{\varepsilon'}}^{(\varepsilon,\varepsilon')}, & \forall \lambda = (j,k,k); \\ 0, & \text{otherwise}. \end{cases} \quad (16)$$

One sees that each $T_R^{\varepsilon, \varepsilon'}$ is a compact operator, and here one calls it a pseudo-annular operator. One has:

Theorem 2 If T satisfies Eqs. (4), (6) and (14) with $t = 0$, then one has:

(i) $T = \sum_{(\varepsilon, \varepsilon', R) \in U} T_R^{\varepsilon, \varepsilon'}$.

(ii) If $R > 0$, then $\| T_R^{\varepsilon, \varepsilon'} \|_{L^2 \to L^2} \leqslant CR^{\frac{(\delta_\varepsilon + \delta_{\varepsilon'})}{2}} A(R)$, and these estimations are sharps.

In Deng et al., (1998), such a conclusion has been proved in dimension 1 with Haar's bases, and almost the same ideas can be used to prove this theorem. End.

Now one presents two kinds of function blocks.

Definition 1 For $1 < p \leqslant \infty$, one calls $a(x)$ a p-atom with module C in H^1, if there is a cube Q such that (i), Supp $a(x) \subset Q$; (ii), $\| a(x) \|_{L^p} \leqslant C |Q|^{\frac{1}{p} - 1}$; and (iii), $\int a(x) dx = 0$. If $C = 1$, one calls also $a(x)$ a p-atom; furthermore, if $p = 2$, then one calls $a(x)$ an atom.

If $a(x)$ is a p-atom, then there exists a constant C_p such that $\| a(x) \|_{H^1} \leqslant C_p$. Furthermore, according to wavelet theory, for each function $f(x) \in H^1$, there is a unique sequence of numbers $\{a_\lambda\}_{\lambda \in \Lambda_n}$ such that $f(x) = \sum_{\lambda \in \Lambda_n} a_\lambda \Phi_\lambda(x)$. $\forall j \in R$, $k \in R^n$, one notes $Q_{j,k} = \prod_{i=1}^{n} [2^{-j} k_i, 2^{-j}(k_i + 1)]$. Then one introduces the second kind of function blocks:

Definition 2 One calls $\{a_\lambda\}_{\lambda \in \Lambda_n}$ an atom at the level of coefficients, if there is a cube $Q_{s,p}$, such that $\{a_\lambda\}_{\lambda \in \Lambda_n}$ satisfies: (i), if $Q_{j,k}$ is not contained in $Q_{s,p}$, then $a_\lambda = 0$; (ii), $\sum |a_\lambda|^2 \leqslant 2^{ns}$.

Since $a_\lambda(x) = \sum_{\lambda \in \Lambda} a_\lambda \Phi_\lambda(x) \in H^1$, and $a_\lambda(x)$ is a 2-atom with constant C, hence one calls also $a_\lambda(x)$ an atom at the level of coefficients.

About function blocks, one has the following result:

Theorem 3 The following four conditions are equivalent:

(i) $f(x) \in H^1$.

(ii) There exists a sequence of numbers $\{\lambda_m\}_{m \in Z} \in l^1$ and a sequence of atoms $\{a_m(x)\}_{m \in Z}$ such that: $f(x) = \sum_{m \in Z} \lambda_m a_m(x)$.

(iii) There exists a sequence of numbers $\{\lambda_m\}_{m \in Z} \in l^1$ and a sequence of ∞-atoms $\{a_m(x)\}_{m \in Z}$ such that: $f(x) = \sum_{m \in Z} \lambda_m a_m(x)$.

(iv) There exists a series of numbers $\{\lambda_m\}_{m \in Z} \in l^1$ and a series of atoms at the level of coefficients $\{\{a_\lambda^m\}_{\lambda \in \Lambda_n}\}_{m \in Z}$ such that: $f(x) = \sum_{m \in Z} \lambda_m \sum_{\lambda \in \Lambda} a_\lambda^m \Phi_\lambda(x)$.

In chapter 5 of (Meyer, 1990 – 1991), it is proved that (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iv); in Coifman and Weiss (1977), one proved that (i)$\Leftrightarrow$(iii). Hence, one gets the conclusion that i, ii, iii and iv are equivalent in this theorem. End.

Let $\phi(x) \in C_0^1(B(0, 2^M))$, let $R > 0$, $\{a_k\}_{k \in Z^n} \in l^1$ such that $a_k = 0$, $\forall |k| > 2^R$ and $\sum_{k \in Z^n} a(k) = 0$, then there is a constant C which is not dependent on R, and one has the following conclusion which will be used in the following section:

Theorem 4 $\| \sum_{k \in Z^n} a(k) \phi(x - k) \|_{H^1} \leqslant CR \| \{a_k\}_{k \in Z^n} \|_{l^1}$

Its proof can be founded in Deng et al.

ACTION OF $T_R^{\varepsilon,\varepsilon'}$ ON ATOMS

In this section, one considers the action of $T_R^{\varepsilon,\varepsilon'}$ on an atom at the level of coefficient $a_{s,p}(x)$ where $a_{s,p}(x) = \sum_{\lambda=(\varepsilon,j,k)\in\Lambda, Q_{j,k}\subset Q_{s,p}} a_{s,p}(\lambda)\Phi_\lambda(x)$. One decomposes $T_R^{\varepsilon,\varepsilon'} a_{s,p}(x)$ into two parts: $I_R^{\varepsilon,\varepsilon'}(h,s,p)(x)$ and $I_R^{\varepsilon,\varepsilon'}(l,s,p)(x)$. For each $R>0$, fixed, one considers two cases for ε'.

If $\varepsilon'\neq 0$, decompose $a_{s,p}(x)$ in two parts $a_{s,p}^h(x) = \sum_{\lambda\in\Lambda, j-s\geq R, Q_{j,k}\subset Q_{s,p}} a_{s,p}(\lambda)\Phi_\lambda(x)$ and $a_{s,p}^l(x) = \sum_{u=0}^{R-1} a_{s,p}^{(u)}(x)$ where $a_{s,p}^{(u)}(x) = \sum_{\lambda\in\Lambda, j-s=u, Q_{j,k}\subset Q_{s,p}} a_{s,p}(\lambda)\Phi_\lambda(x)$. Then one has:

$T_R^{\varepsilon,\varepsilon'} a_{s,p}(x) = T_R^{\varepsilon,\varepsilon'} a_{s,p}^h(x) + T_R^{\varepsilon,\varepsilon'} a_{s,p}^l(x) = I_R^{\varepsilon,\varepsilon'}(h,s,p)(x) + I_R^{\varepsilon,\varepsilon'}(l,s,p)(x)$.

If $\varepsilon' = 0$, then $I_R^{\varepsilon,\varepsilon'}(h,s,p)(x) = \int \sum_{\lambda\in\Gamma, j-s\geq R} a_R^{\varepsilon,\varepsilon'}(\lambda)\Phi_{(\varepsilon,\varepsilon',\lambda)}(x,y)\, a_{s,p}(y)\mathrm{d}y$ and $I_R^{\varepsilon,\varepsilon'}(l,s,p)(x) = \int \sum_{\lambda\in\Gamma, 0\leq j-s\leq R} a_R^{\varepsilon,\varepsilon'}(\lambda)\Phi_{(\varepsilon,\varepsilon',\lambda)}(x,y)\, a_{s,p}(y)\mathrm{d}y$.

One has the following conclusion:

Theorem 5 (i) $I_R^{\varepsilon,\varepsilon'}(h,s,p)(x)$ is a 2-atom with module $CR^{\frac{\delta_\varepsilon+\delta_{\varepsilon'}}{2}} A(R)$.

(ii) $\| I_R^{\varepsilon,\varepsilon'}(l,s,p)(x)\|_{H^1} \leq CR^{\frac{1+2\delta_\varepsilon+\delta_{\varepsilon'}}{2}} A(R)$, and these estimations are sharps.

Proof.

(i) Let $\widetilde{Q}_{s,p} = \prod_{i=1}^n [2^{-s}(p_i - 2^{2M}), 2^{-s}(p_i - 2^{2M}))$, then

$$\mathrm{Supp} I_R^{\varepsilon,\varepsilon'}(h,s,p)(x) \subset \widetilde{Q}_{s,p}$$

According to (ii) of Theorem 2, one has:

$$| I_R^{\varepsilon,\varepsilon'}(h,s,p)(x)\|_{L^2} \leq CR^{\frac{\delta_\varepsilon+\delta_{\varepsilon'}}{2}} A(R)\, \| a_{s,p}(x)\|_{L^2}.$$

Furthermore, one has:

$$\int I_R^{\varepsilon,\varepsilon'}(h,s,p)(x)\mathrm{d}x = 0.$$

Then $I_R^{\varepsilon,\varepsilon'}(h,s,p)(x)$ is a 2-atom with constant $CR^{\frac{\delta_\varepsilon+\delta_{\varepsilon'}}{2}} A(R)$.

(ii) Now one calculates $\| I_R^{\varepsilon,\varepsilon'}(l,s,p)(x)\|_{H^1}$, and considers three cases.

(1) If $\varepsilon = 0$, one has:

$$\| T_R^{0,\varepsilon'} a_{s,p}^l(x)\|_{H^1} \leq C\sum_{u=0}^{R-1} \| T_R^{0,\varepsilon'} a_{s,p}^{(u)}(x)\|_{H^1} \leq C\sum_{u=0}^{R-1} \| \sum_{k,l} a_R^{0,\varepsilon'}(u+s,k,l)\, a_{u+s,l}^{\varepsilon'} \Phi_{u+s,k}^{(0)}(x)\|_{H^1}$$

Then one applies Theorem 4, and gets:

$$\| T_R^{0,\varepsilon'} a_{s,p}^l(x)\|_{H^1} \leq CR\sum_{u=0}^{R-1} \| \sum_{k,l} a_R^{0,\varepsilon'}(u+s,k,l)\, a_{u+s,l}^{\varepsilon'} \Phi_{u+s,k}^{(0)}(x)\|_{L^1}$$

Therefore, one has:

$$\| T_R^{0,\varepsilon'} a_{s,p}^l(x)\|_{H^1} \leq CRA(R)\sum_{u=0}^{R-1} \sum_{Q_{s+u,l}\subset Q_{s,p}} 2^{-\frac{u+s}{2}} | a_{s+u,l}^{\varepsilon'}| \leq CRA(R)2^{\frac{-s}{2}} \sum_{u=0}^{R-1} \Big(\sum_{Q_{s+u,l}\subset Q_{s,p}} | a_{s+u,l}^{\varepsilon'}|^2\Big)^{\frac{1}{2}} \leq CR^{\frac{3}{2}} A(R)2^{\frac{-s}{2}} \Big(\sum_{u=0}^{R-1}\sum_{Q_{s+u,l}\subset Q_{s,p}} | a_{s+u,l}^{\varepsilon'}|^2\Big)^{\frac{1}{2}} \leq CR^{\frac{3}{2}} A(R).$$

(2) If $\varepsilon' = 0$, then one has:

$$\| I_R^{\varepsilon,0}(l,s,p)\|_{H^1} = \| \int \sum_{\lambda\in\Gamma, 0\leq j-s\leq R} a_R^{\varepsilon,0}(\lambda)\Phi_{(\varepsilon,0,\lambda)}(x,y)a_{s,p}(y).\mathrm{d}y\|_{H^1} \leq \| \int \sum_{\lambda\in\Gamma, 0\leq j-s\leq R} a_R^{\varepsilon,0}(\lambda)\Phi_{(\varepsilon,0,\lambda)}(x,y)a_{s,p}(y).\mathrm{d}y\|_{\dot B_1^{0,1}} \leq$$

$$\sum_{0\leq j-s\leq R}\sum_{k\in Z^n} | \int \sum_{l\in Z^n} a_R^{\varepsilon,0}(j,k,l)\Phi^{(0)}(2^j y - l).\, a_{s,p}(y)\mathrm{d}y| \leq \sum_{0\leq j-s\leq R} \sup \sum_{k\in Z^n} | a_R^{\varepsilon,0}(j,k,l)| \sup \sum_{l\in Z^n} |\Phi^{(0)}(2^j y - l)| \int | a_{s,p}(y)|\mathrm{d}y \leq CRA(R).$$

(3) If $|\varepsilon\|\varepsilon'|\neq 0$, then one has:

$$\| I_R^{(\varepsilon,\varepsilon')}(l,s,p)(x)\|_{H^1} = \| \sum_{0\leq j-s\leq R-1}\sum_{k\in Z^n} a_R^{\varepsilon,\varepsilon'}(j,k,l)a_{s,p}(\varepsilon',j,l).\, \Phi_{j,k}^{(\varepsilon)}(x)\|_{H^1} \leq$$

$$CA(R)\sum_{u=0}^{R-1}\sum_{Q_{s+u,l}\subset Q_{s,p}} 2^{-\frac{u+s}{2}} | a_{s+u,l}^{\varepsilon'}| \leq CA(R)2^{\frac{-s}{2}} \sum_{u=0}^{R-1}\Big(\sum_{Q_{s+u,l}\subset Q_{s,p}} | a_{s+u,l}^{\varepsilon'}|^2\Big)^{\frac{1}{2}} \leq CR^{\frac{1}{2}} A(R)2^{\frac{-s}{2}}\Big(\sum_{u=0}^{R-1}\sum_{Q_{s+u,l}\subset Q_{s,p}} | a_{s+u,l}^{\varepsilon'}|^2\Big)^{\frac{1}{2}} \leq$$

$CR^{\frac{1}{2}}A(R)$.

(iii) Finally, one proves that these estimations are sharps. For simplicity of notations, the verification is restricted to dimension 1. One considers three cases.

(1) If $\varepsilon = 0$, one chooses

$$K_R^{\varepsilon,\varepsilon'}(x,y) = \sum_{j=M+3}^{R-M}\sum_{k\in Z^n}\left(\Phi^{(0)}_{j,k\frac{5}{4}2^{R-1}}(x) - \Phi^{(0)}_{j,k\frac{7}{4}2^{R-1}}(x)\right)\Phi^{(1)}_{j,k}(y)$$

and chooses

$$a(x) = (R-2M-2)^{\frac{-1}{2}}\sum_{j=M+2}^{R-M}\sum_{k=0}^{2^j-1}\Phi^{(1)}(2^jx-k).$$

(2) If $\varepsilon' = 0$, one chooses

$$K_R^{\varepsilon,0}(x,y) = \sum_{j=4M+3}^{R-4M}\sum_{k=2^{4M}-2^{R-1}}^{2^{j-1}-2^{4M}-2^{R-1}}\Phi^{(1)}_{j,k}(x)(\Phi^{(0)}_{j,k}(y) - \Phi^{(0)}_{j,k+2^{R-1}}(y))$$

and chooses

$$a(x) = \Phi^{(1)}(x).$$

(3) If $|\varepsilon\|\varepsilon'|\neq 0$, one chooses

$$K_R^{\varepsilon,\varepsilon'}(x,y) = \sum_{j\in Z}\sum_{2^{R-1}\leqslant k-l<2^R}2^{jn}\Phi^{(1)}(2^jx-k).\Phi^{(1)}(2^jy-l)$$

and chooses

$$a(x) = R^{\frac{-1}{2}}\sum_{j=0}^{R-1}\sum_{k=0}^{2^j-1}\Phi^{(1)}(2^jx-k).$$

Then, one can use the above three examples to prove that all the estimations in (ii) are sharps.

PROOF OF MAIN THEOREM AND TWO REMARKS

First, one proves (i) of Main Theorem. For an arbitrary atom $a_{s,p}(x)$ at the level of coefficients, according to Theorem 5, one has: $\forall R>0$, $\|T_R^{\varepsilon,\varepsilon'}a_{s,p}(x)\|_{H^1}\leqslant CR^{\frac{1+2\delta_\varepsilon+\delta_{\varepsilon'}}{2}}A(R)$. If $R=0$ and $|\varepsilon\|\varepsilon'|\neq 0$, it is evi dent that $\|T_R^{\varepsilon,\varepsilon'}a_{s,p}(x)\|_{H^1}\leqslant C$. Further more, if $T(1) = T^t(1) = 0$, one applies (ii) of Theorem 1, if $R=0$ and $|\varepsilon\|\varepsilon'| = 0$, then $T_R^{\varepsilon,\varepsilon'} = 0$. One applies (i) of Theorem 1, and gets: if T satisfies Eqs.(4), (6), (8) and (9), then T is continuous on H^1.

Then, one proves (ii) of main theorem. Using Theorem 2, $\forall R>0$, one has: $\|T_R^{\varepsilon,\varepsilon'}\|_{L^2\to L^2}\leqslant CR^{\frac{\delta_\varepsilon+\delta_{\varepsilon'}}{2}}A(R)$. Then one has:

$$\forall f(x)\in L^2,\ \|Tf(x)\|_{L^2}\leqslant \|f(x)\|_{L^2}. \tag{17}$$

Hence, for an arbitrary ∞-atom $a_{s,p}(x)$, one has: $\|Ta_{s,p}(x)\|_{L^2}\leqslant C2^{\frac{ns}{2}}$. Furthermore, if T satisfies Eq.(13) with $t=0$, then one has

$$\int_{|x-y|\geqslant 2|y-y'|}|K(x,y)-K(x,y')|\mathrm{d}x<\infty. \tag{18}$$

$\forall Q_{s,p}$, one denotes $\phi_{s,p}(x)\in C_0^1(B(2^{-s}p, 2^{n+5-s}))$ such that $\phi_{s,p}(x) = 1$, if $x\in B(2^{-s}p, 2^{n+3-s})$. Then one has: $Ta_{s,p}(x) = \phi_{s,p}(x)Ta_{s,p}(x) + (1-\phi_{s,p}(x))Ta_{s,p}(x) = \phi_{s,p}(x)Ta_{s,p}(x) + (1-\phi_{s,p}(x))\int(K(x,y) - K(x,2^{-s}p)a_{s,p}(y)\mathrm{d}y = I_1(x)+I_2(x)$. One applies Eq.(17), $I_1(x)\in L^1$; one applies Eq.(18), $I_2(x)\in L^1$; then $Ta_{s,p}(x)\in L^1$.

Hence one has:

$$T \text{ is continuous from } H^1 \text{ into } L^1. \tag{19}$$

Using the interpolation theorem, one has:

$$T \text{ is continuous from } L^p \text{ into } L^p,\ \forall 1<p\leqslant 2. \tag{20}$$

Since $L^{p'}(2<p'<\infty,\ p' = \frac{p}{p-1})$ is the dual space of $L^p(1<p<2)$, one gets (ii). End.

Finally, one gives two remarks.

Remark 1 In Han et al., (1993) and Meyer, (1985), it is proved that: if T satisfies Eqs.(1), (4), (5), (6) and (8), and if T satisfies Eqs.(13) with $t=1$ in some sense, then: (i), T is continuous on $\dot{B}_p^{0,q}$, $\forall 1\leqslant p, q\leqslant\infty$; and (ii), T is continuous on $\dot{F}_p^{0,q}$, $\forall 1<p, q<\infty$. In our Main Theorem, one proved that if T satisfies Eqs.(1), (4), (5), (6) and (8), then one has: (i) If T satisfies Eq.(13) with $t=\frac{3}{2}$, then T is continuous on H^1; (ii) If T satisfies Eq.(13) with $t=\frac{1}{2}$, then T is continuous on L^p, $\forall 1<p<\infty$. That is to say, to establish a strong continuity for an operator on different function spaces, there exists three cases for the index t in Eq.(13): $t = \frac{1}{2}, 1, \frac{3}{2}$. The natural problem is that: for each p and q, what is the smallest index t in Eq.(13) such that if T satisfies Eqs.(1), (4), (5), (6), (8) and Eq.(13), then T is continous on $B_p^{0,q}$ or T is continous on $\dot{F}_p^{0,q}$?

Remark 2 Given two operators T as below:

A continuous Hilbert transformation, whose kernel distribution $K(x,y) = \frac{1}{x-y}$ or a diverse Hilbert transformation, whose matrix element on diagonal are zero, otherwise $\frac{1}{k-l}$.

Given ε, a sufficiently small positive real number, a continuous operator T, whose Kernel-distribution $K(x,y) = \sum a_{j,k,l}^{\varepsilon,\varepsilon'} \Phi_{j,k}^{\varepsilon}(x) \Phi_{j,l}^{\varepsilon'}(y)$, where

$$a_{j,k,l}^{\varepsilon,\varepsilon'} = \frac{1}{(2+|k-l|)^n \log^{\frac{5}{2}+\varepsilon}(2+|k-l|)}$$

Applying the three following steps, we see that the above operators are continuous from H^1 to H^1:

(i) Calculate the coefficients $a_{j,k,l}^{\varepsilon,\varepsilon'} = \langle T\Phi_{j,l}^{\varepsilon'}, \Phi_{j,k}^{\varepsilon} \rangle$ under the B-C-R algorithm.

(ii) Calculate $A(R) = \sup \sum_{l: 2^{R-1} \leq |k-l| < 2^R} \{|a_{j,k,l}^{\varepsilon,\varepsilon'}| + |a_{j,l,k}^{\varepsilon,\varepsilon'}|\}$, and verify that $\sum R^{\frac{3}{2}} A(R) < +\infty$.

(iii) Verify that $u_{j,k}^{\varepsilon} = \sum_{l \in Z^n} a_{j,l,k}^{0,\varepsilon'} = 0$.

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