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# Quadrature formulas for Fourier-Chebyshev coefficients*

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**Abstract:** The aim of this work is to construct a new quadrature formula for Fourier-Chebyshev coefficients based on the divided differences of the integrand at points-1, 1 and the zeros of the $n$th Chebyshev polynomial of the second kind. The interesting thing is that this quadrature rule is closely related to the well-known Gauss-Turán quadrature formula and similar to a recent result of Micchelli and Sharma, extending a particular case due to Micchelli and Rivlin.

**Key words:** Divided differences, Quadrature, Chebyshev polynomials, Fourier-Chebyshev coefficient
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## INTRODUCTION

Throughout this paper let $x_1, \cdots, x_n$ be zeros of the $n$th Chebyshev polynomial of the second kind $U_n(x)$ unless otherwise stated or implied. Let $N$ be the set of the natural numbers and $P_k$ the set of all polynomials of degree $\leqslant k$. Denoting that the points $\xi_1 > \cdots > \xi_k > \cdots > \xi_n$ are arbitrary, we adopt the customary notation $f[\xi_1^{j_1}, \cdots, \xi_k^{j_k}, \cdots, \xi_n^{j_n}, x]$ for the divided differences of $f$ at the points $\xi_1, \cdots, \xi_k, \cdots, \xi_n, x$ ($x$ may be identical to any one of $x_k$'s, $k = 1, \cdots, n$) with $\xi_k^{j_k}$ meaning that the point $\xi_k$ is repeated exactly $j_k$ times.

Bojanov (1996) gave a simple approach to the following quadrature established by Micchelli and Sharma (Micchelli, et al., 1983).

**Theorem 1.1** For every $n, s \in N$ and every polynomial $f \in P_{(2s+1)n-1}$, let $T_n(x)$ be the $n$ th Chebyshev polynomial of the first kind. Then we have

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} T_n(x) \mathrm{d}x = \frac{2^n \pi}{n} \sum_{j=1}^{s} \frac{1}{2(2j-1)} \binom{2j}{j} f[\xi_1^{2j-1}, \cdots, \xi_n^{2j-1}], \quad (1)$$

where $\xi_1, \cdots, \xi_n$ are the zeros of $T_n(x)$.

The quadrature rule Eq. (1) extends the following "remarkable" particular case due to Micchelli and Rivlin (1972).

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} T_n(x) \mathrm{d}x = \frac{\pi}{n 2^n} f'[\xi_1, \cdots, \xi_n], \quad (2)$$

which is exact for all polynomial $f \in P_{3n-1}$.

Recently, Gori and Micchelli (1996) considered the class $W_n$ of weight functions to consist of all nonnegative integrable functions $w$ on $[-1, 1]$ such that

$$w\sqrt{1-x^2} = \sum_{k=0}^{\infty}{}' \beta_k T_{2kn}(x), \quad (3)$$

where the prime on the summation indicates that the term corresponding to $k = 0$ is halved.

Accordingly, for every $w \in W_n$ and $f \in C[-1, 1]$ we have

$$\int_{-1}^{1} f(x) w(x) \mathrm{d}x = \frac{\pi}{2} \sum_{k=0}^{\infty}{}' \beta_k A_{2kn}(f), \quad (4)$$

where

$$A_n(f) = \frac{2}{\pi} \int_{-1}^{1} f(x) T_n(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}}. \quad (5)$$

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Thus formula Eq. (4) reduces to explicit expression for $A_{2kn}(f)$. But they used a rather complicated method (Gori et al., 1996) to obtain an implicit expression for Eq. (5) from a representation of the Fourier-Chebyshev coefficients $A_{kn}(f)$ in an infinite series along the divided differences of $f'$ at $\xi_1, \cdots, \xi_n$ with increasing multiplicities.

The goal here is to find explicit expression for

$$B_n(f) := \int_{-1}^{1} f(x) U_n(x) \sqrt{1-x^2}\,\mathrm{d}x. \tag{6}$$

Some related results are also derived. It is also interesting to mention that our result is similar to Eq. (1). We remark here that our results are closely related to a quadrature called Gauss-Turán type. For details of which interested readers, may refer to Gori et al. (1996) and Milovanović (1988; 2001) or Shi (1995; 1996; 1999) and references therein. This paper is organized as follows. In section 2, we state our main results. In section 3, we give some auxiliary lemmas which will be needed in proving the main results.

## MAIN RESULTS

The main purpose here is to obtain the following quadrature formulas.

**Theorem 2.1** For $n$, $s \in N$, let us denote by

$$-1 = x_{n+1} < x_n < \cdots < x_1 < x_0 = 1$$

the zeros of $(1-x^2)U_n(x)$. Let $f \in P_{(2s+1)n+2s-1}$. Then we have

$$B_n(f) = \frac{2^n \pi}{n+1} \sum_{j=1}^{s} \frac{(-1)^{j+1}}{4^{j(n+1)}} \binom{2j}{j} \sum_{k=0}^{n+1}{}'' f[x_0^{j-1}, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^{j-1}, x_k], \tag{7}$$

where the primes on the summation indicate the two terms corresponding to $k=0$ and $k=n+1$ are halved.

**Corollary 2.2** Let $f \in P_{(2s+1)n+2s-1}$. Then we have (with other symbols as in Theorem 2.1)

$$B_n(f) = \frac{2^n \pi}{n+1} \sum_{j=1}^{s} \frac{(-1)^{j+1}}{(2j-1)4^{j(n+1)}} \binom{2j}{j} \{f'[x_0^{j-1}, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^{j-1}] + \frac{1}{2}\{f[x_0^{j}, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^{j-1}] + f[x_0^{j-1}, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^{j}]\}\}. \tag{8}$$

In the next two results, let us introduce the notation

$$I_n(f) = \int_{-1}^{1} f(x) U_n(x) (1-x^2)^{-1/2}\,\mathrm{d}x.$$

**Theorem 2.3** For $n$, $s \in N$, let us denote by

$$-1 = x_{n+1} < x_n < \cdots < x_1 < x_0 = 1$$

the zeros of $(1-x^2)U_n(x)$. Let $f \in P_{(2s+1)n+2s+1}$. Other symbols are as in Theorem 2.1. Then we have

$$I_n(f) = \frac{\pi}{2}(f(1) + (-1)^n f(-1)) + \frac{2^n \pi}{n+1} \sum_{j=1}^{s} \frac{(-1)^{j}}{4^{j(n+1)}} \binom{2j}{j} \sum_{k=0}^{n+1}{}'' f[x_0^{j}, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^{j}, x_k], \tag{9}$$

**Corollary 2.4** Let $n$, $s \in N$, $f \in P_{(2s+1)n+2s+1}$. Other symbols are as in Theorem 2.1. Then we have

$$I_n(f) = \frac{\pi}{2}(f(1) + (-1)^n f(-1)) +$$
$$\frac{2^n\pi}{n+1}\sum_{j=1}^{s}\frac{(-1)^j}{(2j-1)4^{j(n+1)}}\binom{2j}{j}\{f[x_0^j, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^j] -$$
$$\frac{1}{2}\{f[x_0^{j+1}, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^j] + f[x_0^j, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^{j+1}]\}\}. \qquad (10)$$

It is also interesting that Eqs. (7) − (10) are closely related to the Gauss-Turán quadrature rule. For Gauss-Turán quadrature rule, see Gori et al. (1996); Bojanov (1996); Shi (1995; 1999) and references therein.

## AUXILIARY LEMMAS

Here we shall state some lemmas which will be needed in the proofs of our theorems.

**Lemma 3.1** (Yang et al., 2002) We have

$$\int_{-1}^{1} p_{n-1}(x)U_n(x)^{2m+1}(1-x^2)^{m+1/2}dx = 0, \quad p_{n-1}\in P_{n-1}, \ m \in N. \qquad (11)$$

**Lemma 3.2** (Yang et al., 2002) We have

$$\int_{-1}^{1} l_k(x)U_n(x)^{2m}(1-x^2)^{m+1/2}dx = \frac{\pi}{4^m(n+1)}\binom{2m}{m}(1-x_k^2), m \in N. \qquad (12)$$

**Lemma 3.3** (Yang et al., 2002) We have

$$\int_{-1}^{1}(1\pm x)U_n(x)^{2m}(1-x^2)^{m-1/2}dx = \frac{\pi}{4^m}\binom{2m}{m}, \quad m \in N. \qquad (13)$$

**Lemma 3.4** We have

$$\int_{-1}^{1}(1+x)U_n(x)(1-x^2)^{-1/2}dx = \pi,$$
$$\int_{-1}^{1}(1-x)U_n(x)(1-x^2)^{-1/2}dx = (-1)^n\pi.$$

**Proof** Since we have Gradshteyn et al. Formula 3.612. 1-3.612.2(1980).

$$\int_0^{\pi}\frac{\sin(n+1)\theta}{\sin\theta}d\theta = \frac{1+(-1)^n}{2}\pi,$$
$$\int_0^{\pi}\frac{\sin(n+1)\theta\cos\theta}{\sin\theta}d\theta = \frac{1+(-1)^{n+1}}{2}\pi,$$

hence it follows by making the substitution $\theta = \arccos x$ and some calculation.

**Lemma 3.5** (Yang et al., 2002) Suppose that $\xi_0, \xi_1, \cdots, \xi_n \in [-1, 1]$, are different. Let $n, m_k \in N$, $k = 0, 1, \cdots, n$, and $g$ be a sufficiently differentiable function in $[-1, 1]$, then we have

$$\sum_{k=0}^{n} m_k g[\xi_0^{m_0}, \xi_1^{m_1}, \cdots, \xi_n^{m_n}, \xi_k] = g'[\xi_0^{m_0}, \xi_1^{m_1}, \cdots, \xi_n^{m_n}]. \qquad (14)$$

**Lemma 3.6** (Yang et al., 2002) Let $-1 = x_{n+1} < x_n < \cdots < x_1 < x_0 = 1$ be the zeros of $(1-x^2)U_n(x)$ and $g \in C[-1, 1]$, then we have

$$\frac{1}{n+1}\sum_{k=1}^{n}(1-x_k^2)g[x_0, x_{n+1}, x_k] - \frac{1}{2}(g(x_0)+g(x_{n+1})) = -\frac{1}{n+1}\sum_{k=0}^{n+1}{}'' g(x_k). \ (15)$$

**Corollary 3.7** Let $s \in N$ and $f$ be a sufficiently differentiable function in $[-1, 1]$, let $-1 =$

$x_{n+1}<x_n<\cdots<x_1<x_0=1$ be the zeros of $(1-x^2)U_n(x)$, then we have

$$\frac{1}{n+1}\sum_{k=1}^{n}(1-x_k^2)f[x_0^j,x_1^{2j-1},\cdots,x_n^{2j-1},\ x_{n+1}^j,x_k]-\frac{1}{2}\{f[x_0^j,x_1^{2j-1},\cdots,x_n^{2j-1},\ x_{n+1}^{j-1}]+f[x_0^{j-1},x_1^{2j-1},\cdots,x_n^{2j-1},\ x_{n+1}^{j}]\}$$
$$=-\frac{1}{n+1}\sum_{k=0}^{n+1}{}''f[x_0^{j-1},x_1^{2j-1},\cdots,x_n^{2j-1},\ x_{n+1}^{j-1},x_k]. \tag{16}$$

**Proof** Set $g(x)=f[x_0^{j-1},x_1^{2j-1},\cdots x_n^{2j-1},x_{n+1}^{j-1},x]$. We derive Eq. (16) from Eq. (15).

**Lemma 3.8** let $-1=x_{n+1}<x_n<\cdots<x_1<x_0=1$ be the zeros of $(1-x^2)U_n(x)$ and $s\in N$. Let $f$ be a sufficiently differentiable function in $[-1,1]$, then we have

$$f(x)=\sum_{j=0}^{2s-1}\sum_{k=1}^{n}f[x_0^{[(j+1)/2]},x_1^j,\cdots,x_n^j,x_{n+1}^{[(j+1)/2]},x_k]l_k(x)(x^2-1)^{[(j+1)/2]}\omega_n(x)^j+$$
$$\frac{1}{2}\sum_{j=1}^{s}\{f[x_0^j,x_1^{2j-1},\cdots,x_n^{2j-1},\ x_{n+1}^{j-1}](1+x)+f[x_0^{j-1},x_1^{2j-1},\cdots,x_n^{2j-1},\ x_{n+1}^{j}]\cdot(1-x)\}(x^2-1)^{j-1}\omega_n(x)^{2j-1}+R_{n,s}(f;x), \tag{17}$$

where

$$R_{s,n}(f;x)=f[x_0^s,x_1^{2s},\cdots,x_n^{2s},x_{n+1}^s,x](x^2-1)^s\omega_n(x)^{2s},$$
$$\omega_n(x)=\prod_{k=1}^{n}(x-x_k),\ l_k(x)=\frac{\omega_n(x)}{(x-x_k)\omega'_n(x_k)},\ k=1,2,\cdots,n,\ n\in N. \tag{18}$$

**Proof** Let $1=x_0>x_1>\cdots>x_n>x_{n+1}=-1$ be the zeros of $(1-x^2)\omega_n(x)$. We use the following strategy. First, we interpolate $f$ at the points $1=x_0>x_1>\cdots>x_n>x_{n+1}=-1$. Let $L_{n+1}(f;x)$ be the unique Lagrange interpolating polynomial for $f$ at the nodes $x_0,x_1,\cdots,x_n,x_{n+1}$. We have by Newton's divided difference formula that

$$f(x)=L_{n+1}(f;x)+f[x_0,x_1,\cdots,x_n,x_{n+1},x](x^2-1)\omega_n(x), \tag{19}$$

Second, we interpolate $f[x_0,x_1,\cdots,x_n,x_{n+1},x]$ at the points $x_1>\cdots>x_n$. Applying Newton's divided difference formula again and noting Eq. (19), we obtain

$$f(x)=\sum_{k=1}^{n}f(x_k)l_k(x)+\frac{1}{2}\{f[x_0,x_1,\cdots,x_n](1+x)+f[x_1,\cdots,x_n,x_{n+1}](1-x)\}\omega_n(x)+$$
$$\sum_{k=1}^{n}f[x_0,x_1,\cdots,x_{n+1},x_k]l_k(x)(x^2-1)\omega_n(x)+$$
$$f[x_0,x_1^2,\cdots,x_n^2,x_{n+1},x](x^2-1)\omega_n(x)^2$$

Next, we interpolate $f[x_0^{(j+1)/2]},x_1^j,\cdots,x_n^j,x_{n+1}^{[(j+1)/2]},x]$ at the points $x_0,x_1,\cdots,x_n,x_{n+1}$ if $j\in N$ is odd and at the points $x_1,\cdots,x_n$ otherwise. This process continues until $j=2s-1$. It is easy to check that we can finally arrive at the relation Eq. (17). The proof is done.

## PROOFS OF THEOREMS

We are now ready to prove our main results.

**Proof of Theorem 2.1**

**Proof** Integrating both sides of Eq. (17) against the weight function $(1-x^2)^{1/2}$ over the interval $[-1,1]$, recalling Eq. (18) and nothing that $\omega_n(x)=2^{-n}U_n(x)$, we have successively by using Eqs. (11)-(13) and Eq. (16),

$$
\begin{aligned}
B_n(f) = & \sum_{j=0}^{2s-1}\sum_{k=1}^{n} 2^{-jn} f[x_0^{[(j+1)/2]}, x_1^j, \cdots, x_n^j, x_{n+1}^{[(j+1)/2]}, x_k] \times \\
& \int_{-1}^{1} l_k(x)(x^2-1)^{[(j+1)/2]} U_n(x)^{j+1}(1-x^2)^{1/2}\mathrm{d}x + \\
& \frac{1}{2}\sum_{j=1}^{s} 2^{-(2j-1)n}\{f[x_0^j, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^j]\int_{-1}^{1}(1+x)(x^2-1)^{j-1}U_n(x)^{2j}(1-x^2)^{1/2}\mathrm{d}x + \\
& f[x_0^{j-1}, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^{j-1}]\int_{-1}^{1}(1-x)(x^2-1)^{j-1}U_n(x)^{2j}(1-x^2)^{1/2}\mathrm{d}x\} + E_{s,n}(f) = \\
& \sum_{j=1}^{s}\frac{2^n(-1)^j}{4^{jn}}\{\sum_{k=1}^{n} f[x_0^j, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^j, x_k]\int_{-1}^{1} l_k(x)(1-x^2)^{j+1/2}U_n(x)^{2j}\mathrm{d}x - \\
& \frac{1}{2}\{f[x_0^j, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^{j-1}]\int_{-1}^{1}(1+x)(1-x^2)^{j-1/2}U_n(x)^{2j}\mathrm{d}x + \\
& f[x_0^{j-1}, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^j]\int_{-1}^{1}(1-x)(1-x^2)^{j-1/2}U_n(x)^{2j}\mathrm{d}x\}\} + E_{s,n}(f) = \\
& 2^n\pi\sum_{j=1}^{s}\frac{(-1)^{j+1}}{4^{j(n+1)}}\binom{2j}{j}\{\frac{1}{n+1}\sum_{k=1}^{n}(1-x_k^2)f[x_0^j, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^j, x_k] - \\
& \frac{1}{2}\{f[x_0^j, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^{j-1}] + [x_0^{j-1}, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^j]\}\} + E_{s,n}(f) = \\
& \frac{2^n\pi}{n+1}\sum_{j=1}^{s}\frac{(-1)^{j+1}}{4^{j(n+1)}}\binom{2j}{j}\sum_{k=0}^{n+1}{}'' f[x_0^{j-1}, x_1^{2j}, \cdots, x_n^{2j}, x_{n+1}^{j-1}, x_k] + E_{s,n}(f),
\end{aligned}
$$

Where

$$
\begin{aligned}
E_{s,n}(f) := & \int_{-1}^{1} R_{s,n}(f;x)(1-x^2)^{1/2}\mathrm{d}x = \\
& \frac{1}{4^{sn}}\int_{-1}^{1} f[x_0^s, x_1^{2s}, \cdots, x_n^{2s}, x_n^{2s}, x_{n+1}^s, x](x^2-1)^s U_n(x)^{2s+1}(1-x^2)^{1/2}\mathrm{d}x.
\end{aligned}
$$

If $f \in P_{(2s+1)n+2s-1}$, then $f[x_0^s, x_1^{2s}, \cdots, x_n^{2s}, x_n^{2s}, x_{n+1}^s, x] \in P_{n-1}$. It follows from Eq.(11) that $E_{s,n}(f)=0$.

**Proof of Corollary 2.2**

**Proof** For $1 \leqslant j \leqslant s$, $s \in N$, it is easy to check that

$$
\begin{aligned}
\sum_{k=0}^{n+1}{}'' f[x_0^{j-1}, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^{j-1}, x_k] = & \frac{1}{2j-1}\sum_{k=0}^{n+1} m_{k,j} f[x_0^{j-1}, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^{j-1}, x_k] + \\
\frac{1}{2}\{f[x_0^j, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^{j-1}] + f[x_0^{j-1}, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^j], \quad & (20)
\end{aligned}
$$

where

$$
m_{k,j} = \begin{cases} j-1, & \text{if } k = 0 \text{ or } n+1, \\ 2j-1, & \text{if } k = 1, \cdots, n. \end{cases}
$$

It follows from Lemma 3.5 that

$$
\sum_{k=0}^{n+1} m_{k,j} f[x_0^{j-1}, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^{j-1}, x_k] = \sum_{k=0}^{n+1} f'[x_0^{j-1}, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^{j-1}]. \quad (21)
$$

Now Corollary 2.2 follows from Theorem 2.1 and Eqs.(20)-(21).

**Proof of Theorem 2.3**

**Proof** We have by Newton's divided fifference formula that

$$
f(x) = \frac{1}{2}(f(x_0)(1+x) + f(x_{n+1})(1-x)) + f[x_0, x_{n+1}, x](x^2-1).
$$

Multiplying both sides of the above equation by $U_n(x)$ and integrating with respect to weight $(1-x^2)^{-1/2}$ and noting Eq.(14), we obtain

$$I_n(f) = \frac{1}{2}\{f(x_0)\int_{-1}^{1}(1+x)U_n(x)(1-x^2)-1/2\mathrm{d}x + f(x_{n+1})\int_{-1}^{1}(1-x)U_n(x)$$
$$(1-x^2)^{-1/2}\mathrm{d}x\} - \int_{-1}^{1} f[x_0, x_{n+1}, x]U_n(x)(1-x^2)^{1/2}\mathrm{d}x =$$
$$\frac{\pi}{2}(f(x_0) + (-1)^n f(x_{n+1})) - \int_{-1}^{1} f[x_0, x_{n+1}, x]U_n(x)(1-x^2)^{1/2}\mathrm{d}x. \qquad (22)$$

Now replacing $f(x)$ in Theorem 2.1 by $f[x_0, x_{n+1}, x]$ and a straightforward calculation by using Eq.(22), we see that Theorem 2.3 follows from Theorem 2.1 and Lemma 3.1 since $f[x_0, x_{n+1}, x] \in P_{(2s+1)n+2s-1}$ if $f \in P_{(2s+1)n+2s+1}$.

**Proof of Corollary 2.4**

**Proof** The proof of Corollary 2.4 is similar to that of Corollary 2.2.

$$\sum_{k=0}^{n+1}{}'' f[x_0^j, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^j, x_k] = \frac{1}{2j-1}\{\sum_{j=1}^{s} m_{k,j} f'[x_0^j, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^j] -$$
$$\frac{1}{2}\{f[x_0^{j+1}, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^j] + f[x_0^j, x_1^{2j-1}, \cdots, x_n^{2j-1}, x_{n+1}^{j+1}\}\},$$

where

$$m_{k,j} = \begin{cases} j, & \text{if } k = 0 \text{ or } n+1, \\ 2j-1 & \text{if } k = 1, \cdots, n. \end{cases}$$

Now Corollary 2.4 follows from Theorem 2.3 and Lemma 3.5.


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