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Theoretical solution of a spherically isotropic hollow sphere for dynamic thermoelastic problems*

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Abstract: The separation of variables method was successfully used to resolve the spherically symmetric dynamic thermoelastic problem for a spherically isotropic elastic hollow sphere. Use of the integral transform can be avoided by means of this method, which is also appropriate for an arbitrary thickness hollow sphere subjected to arbitrary thermal and mechanical loads. Numerical results are presented to show the dynamic stress responses in the uniformly heated hollow spheres.

Key words: Separation of variables method, Spherically symmetric, Thermoelastic, Hollow sphere
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INTRODUCTION

Temperature field change will produce thermal stresses in structures. It is well known that different materials generate different stress responses, and that in some cases, thermal stresses are large enough to cause structure failure. So it is meaningful and important to study dynamic thermoelastic problems. Tusi and Kraus (1965) used Laplace transform to study the thermal stress-wave propagation due to thermal shock in an arbitrary thick-walled spherical shell. The dynamic thermal stress responses in a spherical shell subjected to arbitrary spherically symmetric temperature fields were investigated by Zaker (1968), whose technique was based on the integral theorem of hyperbolic initial value problem, together with the construction of image temperature fields in the regions outside the actual body. Hata (1991a, 1993) used the ray theory to obtain the dynamic thermal stress responses in a uniformly heated, isotropic hollow sphere, solid sphere, and transversely isotropic solid sphere. Wang (2000) applied finite Hankel transform and Laplace transform to analyze the thermal stress concentration in a spherically isotropic solid sphere.

Dynamic thermoelastic problems are usually solved by Laplace transform technique. But such method will involve the difficulty of inverse transform in some special cases. The ray theory is a good tool for Laplace inversion, but needs a large number of rays for a very thin spherical shell and hence becomes impractical (Pao et al., 1978). In this paper, a separation of variables method is applied to solve the spherically symmetric dynamic thermoelastic problem and avoid use of Laplace transform. First, a new dependent variable is introduced to rewrite the governing equation, the boundary conditions as well as the initial conditions. Then the thermal and mechanical loads are treated as inhomogeneous boundary conditions and a special function is introduced to transform the inhomogeneous boundary conditions to homogeneous ones. After which the orthogonal expansion technique is used to derive the time variables equation whose solution is easily obtained. The theoretical solution of the spherically isotropic hollow sphere for dynamic thermoelastic problem is finally obtained, and can be degenerated into that of the isotropic hollow sphere in a rather straightforward way. The dynamic thermal stress responses of spherically isotropic and isotropic hollow spheres are considered and relevant discussions are presented.

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BASIC EQUATIONS

The transient responses of a thermally shocked transversely isotropic solid sphere were considered by Hata (1993). If a spherical coordinate system (r, θ, φ) with the origin at the center of the sphere is used, then for the spherically symmetric problem, only the radial displacement $u_r = u_r(r, t)$ is nonzero. The strain-displacement relations are simplified

$$\gamma_{rr} = \frac{\partial u_r}{\partial r}, \quad \gamma_{\theta\theta} = \gamma_{\varphi\varphi} = \frac{u_r}{r}, \quad \gamma_{r\theta} = \gamma_{\theta\varphi} = \gamma_{\varphi r} = 0, \tag{1}$$

where $\gamma_{ij}(i, j = r, \theta, \varphi)$ are the strain components. The constitutive relations are

$$\begin{aligned} \gamma_{rr} &= \frac{1}{E_r}\sigma_{rr} - 2\frac{v_{r\theta}}{E_r}\sigma_{\theta\theta} + \alpha_r T(r, t), \\ \gamma_{\theta\theta} &= -\frac{v_{r\theta}}{E_r}\sigma_{rr} + \frac{1 - v_{\theta\varphi}}{E_\theta}\sigma_{\theta\theta} + \alpha_\theta T(r, t), \end{aligned} \tag{2}$$

where σ_{rr} and $\sigma_{\theta\theta}$ are stress components of r and θ direction, respectively. E_r and E_θ are Young's moduli of r and θ direction, respectively. $v_{r\theta}$ and $v_{\theta\varphi}$ are Poisson's ratios. α_r and α_θ are the coefficients of linear thermal expansion of r and θ direction, respectively. $T(r, t)$ is the temperature change. Eq.(2) can be rewritten as

$$\begin{aligned} \sigma_{rr} &= c_{11}\gamma_{rr} + 2c_{12}\gamma_{\theta\theta} - \beta_1 T(r, t), \\ \sigma_{\theta\theta} &= c_{12}\gamma_{rr} + (c_{22} + c_{23})\gamma_{\theta\theta} - \beta_2 T(r, t), \end{aligned} \tag{3}$$

where $c_{ij}(i, j = 1, 2, 3)$ are elastic constants and $\beta_i(i = 1, 2)$ are stress-temperature constants, respectively. The equation of motion is

$$\frac{\partial \sigma_{rr}}{\partial r} + 2\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho\frac{\partial^2 u_r}{\partial t^2}, \tag{4}$$

where ρ is the mass density. Substituting Eq.(1) into Eq.(3), we obtain

$$\begin{aligned} \sigma_{rr} &= c_{11}\frac{\partial u_r}{\partial r} + 2c_{12}\frac{u_r}{r} - \beta_1 T(r, t), \\ \sigma_{\theta\theta} &= c_{12}\frac{\partial u_r}{\partial r} + (c_{22} + c_{23})\frac{u_r}{r} - \beta_2 T(r, t). \end{aligned} \tag{5}$$

Then substituting Eq.(5) into Eq.(4), yields the following governing equation

$$\frac{\partial^2 u_r}{\partial r^2} + \frac{2}{r}\frac{\partial u_r}{\partial r} - \frac{\mu_1^2}{r^2}u_r = \frac{1}{c_L^2}\frac{\partial^2 u_r}{\partial t^2} + g(r, t), \tag{6}$$

where

$$\begin{aligned} \mu_1^2 &= 2\frac{c_{22} + c_{23} - c_{12}}{c_{11}}, \quad c_L = \sqrt{\frac{c_{11}}{\rho}}, \\ g(r, t) &= 2\frac{\beta_1 - \beta_2}{c_{11}}\frac{T(r, t)}{r} + \frac{\beta_1}{c_{11}}\frac{\partial T(r, t)}{\partial r}. \end{aligned} \tag{7}$$

The boundary conditions are

$$r = a, \quad \sigma_{rr}(a, t) = p_1(t), \tag{8a}$$

$$r = b, \quad \sigma_{rr}(b, t) = p_2(t), \tag{8b}$$

where a and b are the inner and outer radii of the hollow sphere, respectively. $p_1(t)$ and $p_2(t)$ are the prescribed pressures on the internal and external surfaces, respectively. The initial conditions $(t = 0)$ are

$$u_r(r, 0) = u_0(r), \quad \dot{u}_r(r, 0) = v_0(r), \tag{9}$$

where a dot over the quantity denotes its partial derivative with respect to t, and $u_0(r)$ and $v_0(r)$ are known functions.

THEORETICAL DEVELOPMENT AND THE SOLUTION

First, a new dependent variable $w(r, t)$ is introduced as

$$u_r = r^{-1/2}w(r, t) \tag{10}$$

Then Eqs.(6), (8) and (9) become

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{\mu^2}{r^2}w = \frac{1}{c_L^2}\frac{\partial^2 w}{\partial t^2} + g_1(r, t), \tag{11}$$

$$r = a, \quad \frac{\partial w}{\partial r} + h\frac{w}{r} = p_a(t), \tag{12a}$$

$$r = b, \quad \frac{\partial w}{\partial r} + h\frac{w}{r} = p_b(t), \tag{12b}$$

$$w(r, 0) = u_1(r), \quad \dot{w}(r, 0) = v_1(r), \tag{13}$$

where

$$\begin{aligned} \mu &= \sqrt{\mu_1^2 + 1/4}, \quad g_1(r, t) = r^{1/2}g(r, t), \\ h &= 2\frac{c_{12}}{c_{11}} - \frac{1}{2}, \\ p_a(t) &= \frac{\sqrt{a}}{c_{11}}[\beta_1 T(a, t) + p_1(t)], \\ p_b(t) &= \frac{\sqrt{b}}{c_{11}}[\beta_1 T(b, t) + p_2(t)], \\ u_1(r) &= r^{1/2}u_0(r), \quad v_1(r) = r^{1/2}v_0(r). \end{aligned} \tag{14}$$

Then we transform the inhomogeneous boundary conditions into the homogeneous ones by assuming

$$w(r,t) = w_1(r,t) + w_2(r,t), \tag{15}$$

where $w_2(r,t)$ satisfies the inhomogeneous boundary conditions, which can be taken as

$$w_2(r,t) = d_1(r-a)^m p_b(t) + d_2(r-b)^m p_a(t), \tag{16}$$

where

$$d_1 = \frac{b^{1-m}}{m(1-s)^{m-1} + h(1-s)^m}, \quad d_2 = \frac{b^{1-m}}{m(s-1)^{m-1} + h(s-1)^m/s}, \quad s = \frac{a}{b}, \tag{17}$$

in which $m \geqslant 2$ is an arbitrary integer that satisfies

$$[m(1-s)^{m-1} + h(1-s)^m][m(s-1)^{m-1} + h(s-1)^m/s] \neq 0. \tag{18}$$

Substituting Eq.(15) into Eqs.(11)-(13), gives

$$\frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r}\frac{\partial w_1}{\partial r} - \frac{\mu^2}{r^2} w_1 = \frac{1}{c_L^2}\frac{\partial^2 w_1}{\partial t^2} + g_2(r,t), \tag{19}$$

$$r = a, \quad \frac{\partial w_1}{\partial r} + h\frac{w_1}{r} = 0, \tag{20a}$$

$$r = b, \quad \frac{\partial w_1}{\partial r} + h\frac{w_1}{r} = 0, \tag{20b}$$

$$w_1(r,0) = u_2(r), \quad \dot{w}_1(r,0) = v_2(r), \tag{21}$$

where

$$g_1(r,t) = g_1(r,t) + \frac{1}{c_L^2}\frac{\partial w_2(r,t)}{\partial t^2} + \frac{\mu^2}{r^2} w_2(r,t) - \frac{1}{r}\frac{\partial w_2(r,t)}{\partial r} - \frac{\partial^2 w_2(r,t)}{\partial r^2}, \tag{22}$$

$$u_2(r) = u_1(r) - w_2(r,0), \quad v_2(r) = v_1(r) - \dot{w}_2(r,0)$$

By using the separation of variables method, the solution of Eq.(19) can be obtained in the following form

$$w_1(r,t) = \sum_i R_i(r) F_i(t), \tag{23}$$

where $F_i(t)$ are unknown functions about t, and $R_i(r)$ are given by

$$R_i(r) = J_\mu(k_i r) Y(\mu, k_i, a) - Y_\mu(k_i r) J(\mu, k_i, a), \tag{24}$$

in which $J_\mu(k_i r)$ and $Y_\mu(k_i r)$ are Bessel functions of the first and second kinds, respectively, and k_i, arranged in an ascending order, are a series of positive roots of the following eigenequation

$$J(\mu, k_i, a) Y(\mu, k_i, b) - J(\mu, k_i, b) Y(\mu, k_i, a) = 0, \tag{25}$$

where

$$J(\mu, k_i, r) = \frac{\mathrm{d}J_\mu(k_i r)}{\mathrm{d}r} + h\frac{J_\mu(k_i r)}{r}, \quad Y(\mu, k_i, r) = \frac{\mathrm{d}Y_\mu(k_i r)}{\mathrm{d}r} + h\frac{Y_\mu(k_i r)}{r}, \tag{26}$$

It can be shown that $w_1(r,t)$ given in Eq.(23) satisfies the homogeneous boundary conditions in Eq.(20). Substituting Eq.(23) into Eq.(19), gives

$$-c_L^2 \sum_i k_i^2 F_i(t) R_i(r) = \sum_i R_i(r) \frac{\mathrm{d}^2 F_i(t)}{\mathrm{d}t^2} + c_L^2 g_2(r,t). \tag{27}$$

By virtue of the orthogonal property of Bessel functions, it is easy to verify the following equation

$$\int_a^b r R_i(r) R_j(r) \mathrm{d}r = N_i \delta_{ij}, \tag{28}$$

where δ_{ij} is the Kronecker delta, and

$$N_i = \frac{1}{2k_i^2}\left\{ b^2\left[\frac{\mathrm{d}R_i(b)}{\mathrm{d}r}\right]^2 - a^2\left[\frac{\mathrm{d}R_i(a)}{\mathrm{d}r}\right]^2 + k_i^2[b^2 R_i^2(b) - a^2 R_i^2(a)] - \mu^2[R_i^2(b) - R_i^2(a)]\right\}. \tag{29}$$

In the above equation, we denote $\mathrm{d}R_i(a)/\mathrm{d}r = \mathrm{d}R_i(r)/\mathrm{d}r\,|_{r=a}$ and $\mathrm{d}R_i(b)/\mathrm{d}r = \mathrm{d}R_i(r)/\mathrm{d}r\,|_{r=b}$. Utilizing Eq.(28), we can derive the following equation from Eq.(27)

$$\frac{d^2 F_i(t)}{dt^2} + \omega_i^2 F_i(t) = q_i(t), \tag{30}$$

where

$$\omega_i = k_i c_L, \quad q_i(t) = -\frac{c_L^2}{N_i}\int_a^b r g_2(r,t) R_i(r) \mathrm{d}r. \tag{31}$$

The solution of Eq.(30) is

$$F_i(t) = G_{1i}\cos\omega_i t + G_{2i}\sin\omega_i t + \frac{1}{\omega_i}\int_0^t q_i(\tau)\sin\omega_i(t-\tau)\mathrm{d}\tau \tag{32}$$

where

$$G_{1i} = \frac{1}{N_i}\int_a^b r u_2(r)R_i(r)\mathrm{d}r,$$
$$G_{2i} = \frac{1}{N_i\omega_i}\int_a^b r v_2(r)R_i(r)\mathrm{d}r, \tag{33}$$

Finally, the displacement solution can be obtained as follows

$$u_r(r,t) = r^{-1/2}[w_1(r,t) + w_2(r,t)]. \tag{34}$$

SOME PARTICULAR CASES

1. Isotropic material

If

$$c_{11} = c_{22} = E(1-v)/k, c_{12} = c_{23} = Ev/k,$$
$$k = (1+v)(1-2v), \alpha_\theta = \alpha_r = \alpha \tag{35}$$

where E and v are Young's modulus and Poisson's ratio, respectively, the solution obtained above then degenerates into that of an isotropic hollow sphere for the dynamic thermoelastic problem. If a solid sphere ($a=0$) is considered, we can just set $p_1(t)=0$, $p_a(t)=0$, $J(\mu, k_i, \alpha)$ and $Y(\mu, k_i, a)=1$ in relevant formulations to obtain the solution of an isotropic solid sphere for the dynamic thermoelastic problem.

2. Elastodynamic solution

If the temperature change $T(r,t)=0$, the solution becomes that of a spherically isotropic hollow sphere for the elastodynamic problem.

3. Fixed boundary conditions

For a hollow sphere fixed at the internal surface, instead of Eq.(8a), we have

$$r = a, \quad u_r(a,t) = 0. \tag{8a'}$$

Consequently, we have the following equations instead of Eqs.(12a) and (20a),

$$w(a,t)=0, \quad w_1(a,t)=0. \quad (12a'), (20a')$$

Then, we can just set $p_1(t)=0$, $p_a(t)=0$, $J(\mu, k_i, a) = J_\mu(k_i a)$ and $Y(\mu, k_i, a) = Y_\mu(k_i a)$ in relevant formulations to obtain the solution of the dynamic thermoelastic problem for a spherically isotropic hollow sphere with fixed internal surface.

NUMERICAL RESULTS AND DISCUSSIONS

In order to examine the present solution, both spherically isotropic and isotropic hollow spheres are considered to analyze the dynamic thermal stress responses. Suppose the spherical shells are subjected to the following loads

$$p_1(t) = p_2(t) = 0.0, \quad T(r,t) = T_0 H(t), \tag{36}$$

where $H_{(0)}$ denotes the Heaviside step function and T_0 is a prescribed temperature change, respectively. For the spherically isotropic hollow sphere, the constants of the material are list in the following

$$E_r = 100GPa, \quad E_\theta = 300GPa,$$
$$v_{r\theta} = v_{\theta\varphi} = 0.3, \quad \alpha_\theta = 2\alpha_r = 2.0\times 10^{-6}/K, \tag{37}$$

and for the isotropic hollow sphere, we take

$$E = 200\text{GPa}, \quad v = 0.3, \quad \alpha = 1.0\times 10^{-6}/K. \tag{38}$$

The time, coordinate and stresses are normalized as follows

$$t^* = \frac{c_L}{b-a}t, \quad \xi = \frac{r-a}{b-a}, \quad \sigma_i^* = \frac{\sigma_{ii}}{\sigma_0} \ (i = r, \theta), \tag{39}$$

where

$$\sigma_0 = \frac{\alpha E T_0}{1-2v} \tag{40}$$

In the following calculation, we take $m=2$, $s=0.2$, $T_0 = 100\text{K}$, $u_0(r)=0$, $v_0(r)=0$.

Figs. 1 and 2 depict the dynamic stress responses of a uniformly heated spherically isotropic hollow sphere and Figs. 3 and 4 show the dynamic stress responses of a uniformly heated isotropic hollow sphere for $s=0.2$. The dynamic stress responses of the uniformly heated isotropic hollow sphere had also been studied by Hata (1991b). Our results (Figs. 3 and 4) agree well with Hata's results (Figs. 2 and 5 therein). Thus, the validity of the method presented in this paper is verified. Comparing Fig. 1 with Fig. 3 and comparing Fig. 2 with Fig. 4, we find that the peak values of stress responses in the spherically isotropic hollow sphere are larger than those in the isotropic one for the same inner to outer radius ratio.

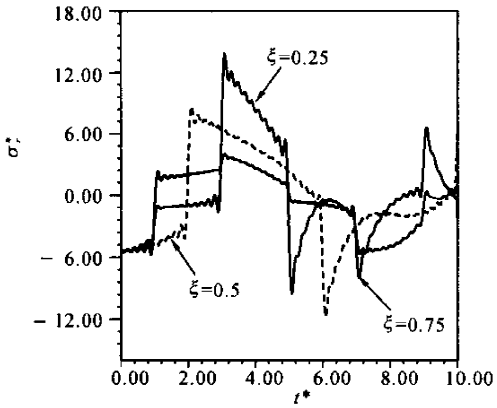

Fig.1 History of dynamic stress σ_r^* for spherically isotropic hollow sphere ($s = 0.2$)

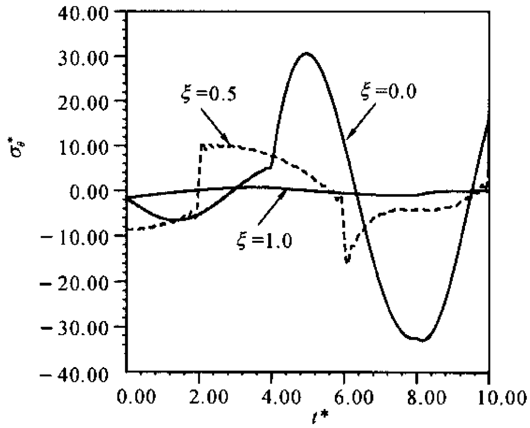

Fig.2 History of dynamic stress σ_θ^* for spherically isotropic hollow sphere ($s = 0.2$)

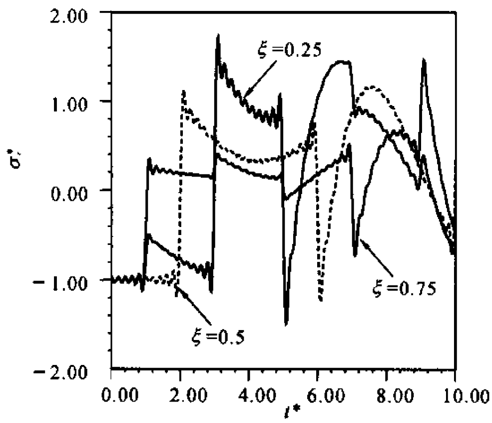

Fig.3 History of dynamic stress σ_r^* for isotropic hollow sphere ($s = 0.2$)

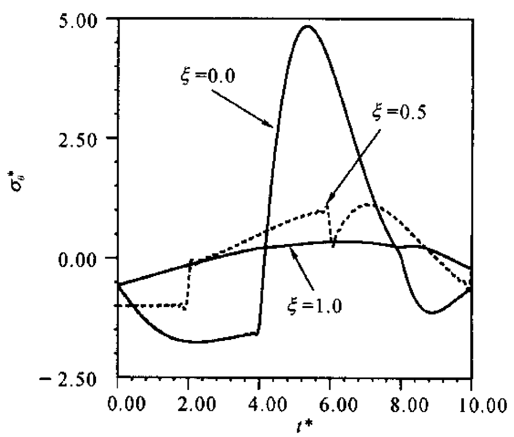

Fig.4 History of dynamic stress σ_θ^* for isotropic hollow sphere ($s = 0.2$)

It is noted here that the constitutive relations Eq.(2) for spherically isotropic material are valid only for hollow sphere. For a solid sphere, the center is actually a special point, where it is impossible to separate the r, θ and φ- directions. That is to say, the constitutive relations of Eq.(2) are invalid at the center. The dynamic stress responses in a solid sphere with spherical isotropy had also been studied both by Hata (1993) and Wang (2000), and the particular discussion are concentrated on the dynamic stress responses at the center. Apparently, it is meaningless to discuss the dynamic stress responses at the center by virtue of the theoretical solution, which is derived on the basis of Eq.(2).

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