ISSN 1009-3095 *Journal of Zhejiang University SCIENCE* V.4, No.3, P.258-263, May-June, 2003

http://www.zju.edu.cn/jzus; http://www.periodicals.com.cn; jzus@zju.edu.cn

An adaptive strategy for controlling chaotic system*

CAO Yi-jia(曹一家)†1, ZHANG Hong-xian(张红先)2

([1] *College of Electrical Engineering, Zhejiang University, Hangzhou 310027, China*)

([2] *Department of Electrical Engineering, Huazhong University of Science and Technology, Wuhan 430074, China*)

†E-mail: yijiacao@cee.zju.edu.cn

Received June 6, 2002; revision accepted Dec. 8, 2002

Abstract: This paper presents an adaptive strategy for controlling chaotic systems. By employing the phase space reconstruction technique in nonlinear dynamical systems theory, the proposed strategy transforms the nonlinear system into canonical form, and employs a nonlinear observer to estimate the uncertainties and disturbances of the nonlinear system, and then establishes a state-error-like feedback law. The developed control scheme allows chaos control in spite of modeling errors and parametric variations. The effectiveness of the proposed approach has been demonstrated through its applications to two well-known chaotic systems: Duffing oscillator and Rössler chaos.

Key Words: Chaos control, Nonlinear control, Adaptive control

Document code: A **CLC number:** TP273+.2

INTRODUCTION

Several chaotic systems have been developed and thoroughly analyzed in recent decades. A chaotic system is a nonlinear deterministic system having complex and unpredictable behaviour. The control of chaotic systems has received increasing interest in recent years (Ott *et al.*, 1990; Fradkov *et al.*, 1996; Femat *et al.*, 1999; Cao, 2000). In 1990, Ott, Grebogi and Yorke (OGY) suggested a method for controlling chaotic systems by stabilizing one of the many unstable periodic orbits embedded in a chaotic attractor, using small time-dependent perturbations in the form of feedback to an accessible system parameter; then its sensitive dependence on initial conditions was exploited to achieve control with minimum control effort (Ott *et al.*, 1990). Since then, many results have been reported in the literature. For instance, Lyapunov-based control methods, variable structure control, discrete-time control, adaptive control, and robust asymptotic linearization.

Recently, adaptive and robust control of chaos is an exciting problem under intensive research (Femat *et al.*, 1999; Cao, 2000). This is because chaotic systems are sensitive to parametric variations; and as some existing control procedures have feedback structures, they are not robust against uncertainties and lead only to local stability. More robust control strategies borrowed from conventional engineering methods of nonlinear control have been considered. These methods include linear control feedbacks, sliding mode control, extended differential geometric approach, adaptive control techniques, etc.

The concept of observer has found useful applications in the synchronization of chaotic systems, and can also be applied to control chaotic systems with output dependent nonlinearity. However, the observer design of a general nonlinear system is a difficult problem in control and estimation theory (Jiang, 2001a; Jiang *et al.*, 2001b). A variety of methods were developed in recent years for some nonlinear systems. Four approaches are generally available for construction of nonlinear observers.

In this work, we study the robust stabilization of chaotic systems and propose a new adaptive control strategy based on phase space recon-

* Project supported by National Natural Science Foundation of China (No. 60074040) and the Teaching and Research Award Program for Outstanding Young Teachers (TRAPOYT) in Higher Education Institutions of China

struction technique and nonlinear observers in the nonlinear dynamic system theory. The main idea is to lump the uncertainties and disturbances in a nonlinear function which can be interpreted as a new state in an externally observer system. Thus, the new state is estimated by means of a state observer, and plays a role in rejecting the uncertainties and disturbances in the control strategy.

THE ROBUST CONTROL STRATEGY

Consider the following nonlinear systems whose trajectories are contained in a chaotic attractor:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{u}(t) \tag{1}$$

$$y(t) = h(\boldsymbol{x}(t)) \tag{2}$$

where $\boldsymbol{x} \in \boldsymbol{R}^m$ is the state vector, y is the output variable, $\boldsymbol{f}$: $\boldsymbol{R}^m \to \boldsymbol{R}^m$ and h: $\boldsymbol{R}^m \to \boldsymbol{R}$ are smooth nonlinear functions, $\boldsymbol{x} \in \boldsymbol{R}^m$ is the control term to be designed.

1. Phase space reconstruction

When the dynamics of the system Eq.(1) is not known, phase space reconstruction technique is necessarily the first step to analyse a chaotic signal in terms of dynamical systems theory (Packard *et al.*, 1980). The reconstructed phase space can be used for qualitative analysis and quantitative statistical characterizations. A reconstruction viewpoint on communication systems via chaotic signals was developed by Itoh *et al.*, (1997). From the output signal $y(t)$ and its derivatives of successively higher order, we can reconstruct the following state:

$$\hat{\boldsymbol{X}}(t) = (y(t), \dot{y}(t), \cdots, y^{(n-1)}(t))^{\mathrm{T}} = (h(\boldsymbol{x}), L_f h(\boldsymbol{x}), \cdots, L_f^{(n-1)} h(\boldsymbol{x}))^{\mathrm{T}} = \boldsymbol{H}(\boldsymbol{x}) = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n)^{\mathrm{T}} \tag{3}$$

where L denotes the Lie derivative operator (Isidori, 1989)

$$L_f^j h(\boldsymbol{x}) = \sum_{i=1}^{n} \frac{\partial (L_f^{j-1} h)}{\partial x_i} f_i(\boldsymbol{x}).$$

It is easy to show that the reconstructed states $\hat{\boldsymbol{X}}$ satisfy the following differential equation:

$$\dot{\hat{\boldsymbol{X}}} = \boldsymbol{A}\hat{\boldsymbol{X}} = \boldsymbol{b}\phi(\hat{\boldsymbol{X}}) = \boldsymbol{b}\boldsymbol{u}'(t) \tag{4}$$

where

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n \times n} \tag{5}$$

$$\boldsymbol{b} = [0 \quad \cdots \quad 0 \quad 1]^{\mathrm{T}} \in \boldsymbol{R}^n$$

$$\phi(\hat{\boldsymbol{X}}) = L_f^m h(\boldsymbol{x}) = L_f^m h(\boldsymbol{H}^{-1}(\hat{\boldsymbol{X}}))$$

$$\boldsymbol{u}(t) = \left(\frac{\partial \boldsymbol{H}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right)^{+} \boldsymbol{b}\boldsymbol{u}'(t) = \boldsymbol{b}(\hat{\boldsymbol{X}})\boldsymbol{u}'(t) \tag{6}$$

where $\left(\frac{\partial \boldsymbol{H}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right)^{+}$ is the generalized inverse of $\frac{\partial \boldsymbol{H}(\boldsymbol{x})}{\partial \boldsymbol{x}}$ (Ben-Israel *et al.*, 1974), $\phi()$: $\boldsymbol{R}^n \to \boldsymbol{R}$ and $\boldsymbol{b}()$: $\boldsymbol{R}^n \to \boldsymbol{R}^m$, are smooth nonlinear functions, and $\boldsymbol{u}'(t)$ is a scalar control term. Takens (1980) proved that as long as n is sufficiently large (for example, $n > 2m$), generically, H is an embedding and $\partial H/\partial \boldsymbol{x}$ is of full rank.

The control problem is described as: for any initial conditions, given a target trajectory $y_r(t)$, design a control law $\boldsymbol{u}(\boldsymbol{t})$ such that the output of the system $y(t)$ tracks $y_r(t)$, i.e. $\lim_{t \to \infty} y(t) = 0$, where $y(t) = (y(t) - y_r(t))$.

2. Nonlinear observer

The nonlinear system Eq.(4) can be rewritten in the following form:

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 \\ \dot{\hat{x}}_2 &= \hat{x}_3 \\ &\vdots \\ \dot{\hat{x}}_n &= \phi(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n) + u'(t) \end{aligned} \tag{7}$$

where $\hat{x}_2 = \dot{\hat{x}}_1, \cdots, \hat{x}_n = \hat{x}_1^{(n-1)}$. First we assume that the nonlinear function $\phi()$ is confined by a global Lipschitz condition, i.e.,

$$\phi(\boldsymbol{x}) - \phi(\hat{\boldsymbol{x}}) \leqslant L \| \boldsymbol{x} - \hat{\boldsymbol{x}} \|, \forall \boldsymbol{x}, \hat{\boldsymbol{x}} \in \boldsymbol{R}^n \tag{8}$$

where L is a global Lipschitz constant. Let the state variables of the nonlinear observer be z_1, z_2, $\cdots$, z_{n+1}. Since $\hat{x}_1$ is available and under the global Lipschitz condition Eq.(8), we can construct the following nonlinear observer system:

$$\dot{z}_1 = z_2 - g_1[z_1 - \hat{x}_1(t)]$$

$$\dot{z}_2 = z_3 - g_2[z_1 - \hat{x}_1(t)]$$
$$\vdots$$
$$\dot{z}_n = z_{n+1} - g_n[z_1 - \hat{x}_1(t)] + u'(t)$$
$$\dot{z}_{n+1} = -g_{n+1}[z_1 - \hat{x}_1(t)], \quad (9)$$

where g_1, g_2, $\cdots$, g_{n+1} are nonlinear functions chosen so that the state variables of system Eq. (9) can track the state variables of nonlinear system Eq. (7), i.e.

$z_1(t) \to \hat{x}_1(t)$, $z_2(t) \to \hat{x}_2(t)$, $\cdots$, $z_{n+1} \to a(t)$,

where $a(t) = \phi(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n)$.

There are many studies on choosing appropriate nonlinear functions g_1, g_2, $\cdots$, g_{n+1} to achieve the convergence of the nonlinear observer Eq. (9) (Levant, 1993; 1998; Zhang *et al.*, 1998; Han, 1995; Jiang, 2001a). In this work, we consider the existing approaches (Levant, 1993; 1998; Zhang *et al.*, 1998; Han, 1995) and choose the nonlinear functions g_i ($i = 1, 2, \cdots, n$), as $\beta_i x^{\frac{1}{2}}$ sign(x) and g_{n+1} as β_{n+1} sign (x) to provide effective tracking (the convergence of the nonlinear observer can be referred to in Levant (1993; 1998)). To illustrate this, the following example is given.

Example: Given a chaotic Van der Pol's equation, which has the same form as Eq. (4):

$$\ddot{x} = f(x, \dot{x}, t) + f_e(t) \quad (10)$$

where $f(x, \dot{x}, t) = -x - 5.0(x^2 - 1)\dot{x}$, $f_e(t) = 5.0\cos(2.463t)$; initial state is the origin.

Obviously, the system Eq. (10) takes the form Eq. (7). The nonlinear observer Eq. (9) is chosen as follows:

$$\dot{z}_1 = z_2 - \beta_1 |z_1 - x(t)|^{1/2}\text{sign}(z_1 - x(t))$$
$$\dot{z}_2 = z_3 - \beta_2 |z_1 - x(t)|^{1/2}\text{sign}(z_1 - x(t))$$
$$\dot{z}_3 = -\beta_3\text{sign}(z_1 - x(t)) \quad (11)$$

The performance of the nonlinear observer Eq. (11) is given in Fig. 1, where x_1, x_2 are denoted by solid line, while z_1, z_2 are denoted by dotted line. It is shown that the nonlinear observer can effectively trace both state variables of the system Eq. (7).

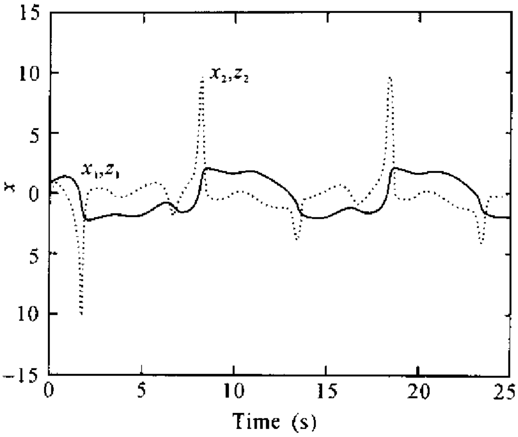

Fig. 1 Performance of the nonlinear observer Eq. (11) ($\beta_1 = 10.0$, $\beta_2 = 100.0$, $\beta_3 = 10.0$)

3. Nonlinear feedback control law

Let v_0 and v_i ($i = 1, \cdots, n$) be the reference input and its ith derivative of the system Eq. (7), $\varepsilon_1 = v_0 - z_1$, $\varepsilon_2 = v_1 - z_2$, $\cdots$, $\varepsilon_n = v_{n-1} - z_n$; then ε_1, ε_2, $\cdots$, and ε_n are the errors between the reference input and observed system output, and their differentials, respectively. Introduce a complementary control input, $u_0(t)$, described as follows:

$$u_0(t) = u'(t) + a(t) \quad (12)$$

then system Eq. (7) becomes:

$$\hat{x}^{(n-1)} = \hat{x}^{(n)} = u_0(t). \quad (13)$$

To guarantee the stability of the origin of the system Eq. (13), $u_0(t)$ is designed to take the following feedback form:

$$u_0 = k_1\varepsilon_1 + k_2\varepsilon_2 + \cdots + k_n\varepsilon_n + v_n, \quad (14)$$

where k_1, k_2, $\cdots$, k_n are constants to be chosen. Putting Eq. (14) into Eq. (13), we have:

$$\hat{x}^{(n)} - v_n - k_1(v_0 - z_1) - k_2(v_1 - z_2) - \cdots - k_n(v_{n-1} - z_n) = 0. \quad (15)$$

From the above nonlinear observer design, we have $z_1(t) \to \hat{x}_1(t)$, $z_2(t) \to \hat{x}_2(t)$, $\cdots$, $z_n(t) \to \hat{x}_n(t)$, as $t \to \infty$. Let $\tilde{y}(t) = x(t) - v_0(t)$, then Eq. (15) becomes:

$$\tilde{y}^{(n)}(t) + k_n\tilde{y}^{(n-1)} + \cdots + k_1\tilde{y}(t) = 0, \quad (16)$$

If the stability of the origin of the system Eq. (16) is guaranteed, then $\tilde{y}(t) \to 0$, as $t \to \infty$. So the control objective has been achieved. To make the closed-loop system Eq. (16) exponentially stable, k_i are chosen so that $s^{n+1} + k_n s^n + \cdots + k_2 s + k_1$ is a Hurwitz polynomial (Isidori, 1989). It should be noted here that system Eq.

(13) is controlled by $u_0(t)$ obtained from the nonlinear combination of system state-errors.

In the proposed control law Eq.(12), where given the reference signal, the nonlinear observer achieves estimation of state variables of the controlled system: e.g. $z_1(t)$, $z_2(t)$, $\cdots$, $z_n(t)$ and the estimation of $\phi(\hat{x}_1(t), \hat{x}_2(t), \cdots, \hat{x}_n(t))$, (i.e. $z_{n+1}(t)$). The control input can be divided into two parts:

$$u'(t) = u_0(t) - z_{n+1}(t) \quad (17)$$

where $-z_{n+1}(t)$ and $u_0(t)$ jointly play a role in rejecting disturbances and controlling the system.

Using the phase space reconstruction technique and the states of the nonlinear observers Eq.(9), the control law for system Eq.(1) becomes:

$$\boldsymbol{u}(t) = \boldsymbol{b}(z)(u_0(t) - z_{n+1}(t)) \quad (18)$$

The above analysis shows that, in theory, control law Eq.(18) would guarantee the control objective when n and parameters β_i, k_i are suitably chosen. However, in practice, this approach may have a disadvantage, i.e., even though H is an embedding when n is larger enough, the larger the n, the higher the computational complexity. However, with the rapid development of computer facilities, this disadvantage can be easily overcome. In fact, in the following simulation study, it is not a serious problem.

SIMULATIONS

To demonstrate the performance of the control strategy, simulations with the Duffing oscillator and Rössler chaotic system are provided. For Duffing oscillator $\phi(x, \dot{x}, t) = -p_1 x - \mu x^3 - p_2 \dot{x}$ and $f_e(t) = q\cos(\Omega t)$. This system can be used to model various physical phenomena such as the buckling of an elastic beam in a magnetic field. Here, the simulation parameter values are $p_1 = 0.4$, $p_2 = -1.1$, $\mu = 1.0$, $\Omega = 1.8$, and $q = 1.8$; the Duffing oscillator displays chaotic behavior. The origin (0, 0) is an equilibrium point of the Duffing oscillator for $f_e(t) + u = 0$. However, such an equilibrium is a saddle point. Let us choose $y_r(t) = 0$ as control objective. The control parameters are taken as $k_1 = 12.0$, $k_2 = 36.0$, $\beta_1 = 10.0$, $\beta_2 = 100.0$ and $\beta_3 = 10.0$. Fig.2 shows the position and velocity trajectories before and after the control is activated at time $t = 20$. The trajectories are stabilized in the equilibrium point in spite of uncertainties in the model. Fig.2 also shows the control signal $u(t)$, represented by the solid line. Stabilization of the origin is achieved when the control action $u(t)$ counteracts the external force $f_e(t)$, represented by the dotted line (i.e. $u(t) = -f_e(t)$). Let us now consider the case of tracking of trajectories. Our objective is to control the chaotic Duffing oscillator to the periodic solution that it exhibits when $q = 0.62$. It is a (finite-time) section of the chaotic trajectory. Fig.3 displays the dynamical behavior of the

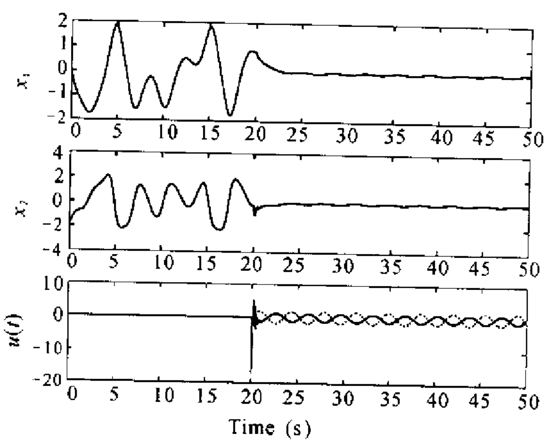

Fig.2 Control performance for the case of stabilization of the origin

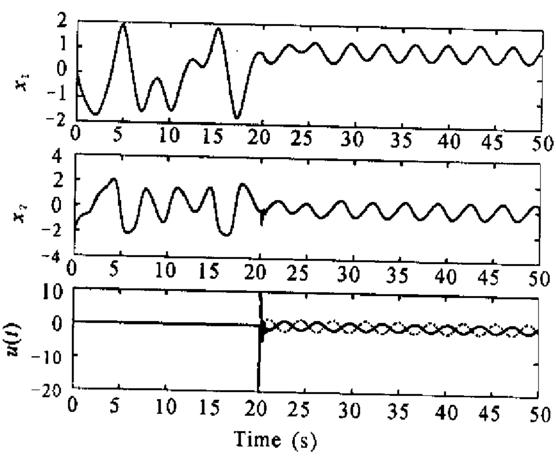

Fig.3 Control performance for the case of stabilization of the periodic motion ($q = 0.62$)

controlled position $x_1(t)$, and velocity $x_2(t)$, which after a transient, attain the periodic behavior. In this case, the controller is also activated at $t=20$. For both cases, the controller can achieve the control objectives very rapidly.

Rössler chaos is widely used to study the control and synchronization of chaos. The circuit equations can be written in dimensionless form as follows:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 \\ x_1 + ax_2 \\ c + x_3(x_1 - b) \end{pmatrix} + \boldsymbol{u} \qquad (19)$$

$$y = x_2 \qquad (20)$$

where we take $a=0.2$, $b=5.7$, $c=0.2$, so that chaotic behaviour exists in this system, and a typical chaotic behaviour is shown in Fig. 4. By calculation, we have

$$\boldsymbol{H}(\boldsymbol{x}) = \begin{pmatrix} x_2 \\ x_1 + ax_2 \\ ax_1 + (a^2-1)x_2 - x_3 \end{pmatrix}$$

$$\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{x}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & a & 0 \\ a & a^2-1 & -1 \end{pmatrix}$$

$$\phi(\boldsymbol{x}) = -x_1x_3 + (a^2-1)x_1 + (a^3-2a)x_2 + (b-a)x_3 - c$$

The nonlinear observer Eq. (9) becomes:

$$\begin{aligned} \dot{z}_1 &= z_2 - g_1(z_1 - y) \\ \dot{z}_2 &= z_3 - g_2(z_1 - y) \\ \dot{z}_3 &= z_4 - g_3(z_1 - y) \\ \dot{z}_4 &= -g_4(z_1 - y) \end{aligned} \qquad (21)$$

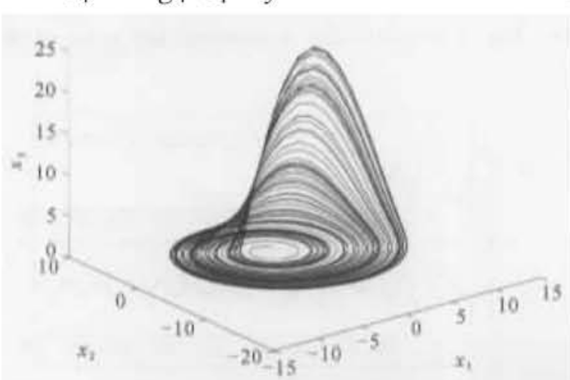

Fig.4 Chaotic orbits of the Rössler system

where g_i is a chosen function described above. The control for Rössler chaos is:

$$\boldsymbol{u}(t) = \left(\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{x}}\right)^{+} \boldsymbol{b}u'(t) = [0\ 0\ -1]^{\mathrm{T}}(u_0(t) - z_4(t)) \qquad (22)$$

where $u_0(t)$ is designed according to Eq. (14). Figs. 5 and 6 show the stabilization of the Rössler system at the origin and a perodic orbit $y_r(t) = \sin(t)$. The initial conditions for the system (19) were $(x_1(0), x_2(0), x_3(0)) = (-1, 0, -0.2)$ and for the nonlinear observer (9), $(z_1(0), z_2(0), z_3(0), z_4(0)) = (0, 0, 0, 0)$. The control feedback gain $k_1 = 1.0$, $k_2 = 3.0$, $k_3 = 3.0$, and the high-gain estimation parameters values $\beta_1 = 10.0$, $\beta_2 = 100.0$, $\beta_3 = 100.0$, $\beta_4 = 10.0$, were chosen. For both cases, the control input is activated at $t=20$. It can be seen from Figs. 5 and 6 that the control strategy can achieve the control objective very rapidly.

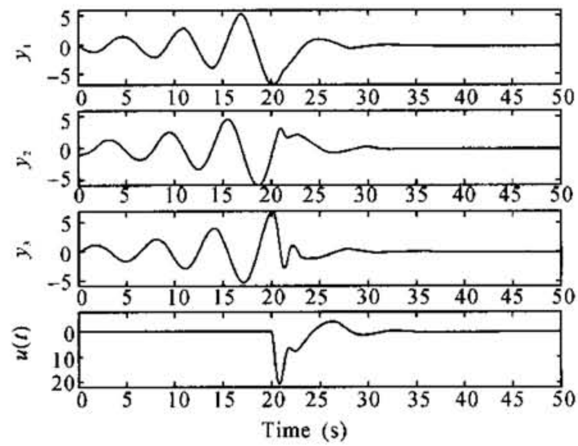

Fig.5 Control performance for the case of stabilization of the origin

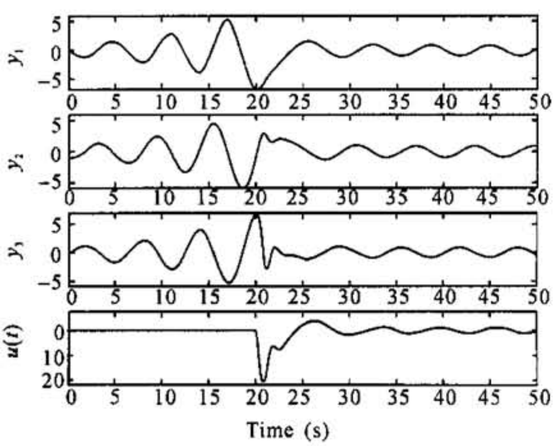

Fig.6 Control performance for the case of stabilization of the periodic motion ($y_r(t) = \sin(t)$)

CONCLUSIONS

An adaptive strategy to control nonlinear chaotic systems has been proposed. The control scheme does not depend on the system states and model, but employs the phase space reconstruction technique to transforms the system into canonical form and then employs a nonlinear observer to track the states and disturbances of the system. The effectiveness of the proposed approach has been demonstrated through its applications to two well-known chaotic systems: Duffing oscillator and Rössler chaos.

References

Ben-Israel, A. and Greville, T. N. E., 1974. Generalized Inverse: Theory and Applications. John Wiley, New York.

Cao, Y. J., 2000. A nonlinear adaptive approach to controlling chaotic oscillators. *Phys. Lett. A*, **270**(3/4): 171 – 175.

Femat, R., Alvarez-Ramirez, J., Castillo-Toledo, B. and Gonzalez, J., 1999. On robust chaos suppression in a class of nondriven oscillators: application to the chua's circuit. *IEEE Trans. Circuits. and Syst. I*, **46**(9): 1150 – 1152.

Fradkov, A. L. and Yu, A., 1996. Speed gradient control of chaotic continuous-time systems. *IEEE Trans. Circuits and Syst. I*, **43**: 907 – 913.

Han, J. Q., 1995. An extended nonlinear observer. *J. Control and Decision*, **10**: 85 – 89.

Isidori, A., 1989. Nonlinear Control Systems, 2nd ed. Springer-Verlag, New York.

Itoh, M., Wu, C. W. and Chua, L. O., 1997. Communication systems via chaotic signals from a reconstruction viewpoint. *Int. J. of Bifurcation and Chaos*, **7**: 275 – 286.

Jiang, L., 2001a. Nonlinear Adaptive Control and Applications in Power Systems. PhD Thesis, University of Liverpool, UK.

Jiang, L., Wu, Q. H., Zhang, C. and Zhou, X. X., 2001b. Observer-based nonlinear control of synchronous generator with perturbation estimation. *International Journal of Electrical Power and Energy Systems*, (6): 22 – 28.

Levant, A., 1993. Sliding order and sliding accuracy in sliding mode control. *Int. J. of Control*, **58**(6): 1247 – 1263.

Levant, A., 1998. Robust exact differentiation via sliding mode technique. *Automatica*, **34**(3): 379 – 384.

Ott, E., Grebogi, C. and Yorke, J. Y., 1990. Controlling chaos. *Phys. Rev. Lett.*, **64**: 1196 – 1199.

Packard, N. H., Crutchfield, J. P., Farmer, J. D. and Shaw, R. S., 1980. Geometry from a time series. *Phys. Rev. Lett.*, **45**: 712 – 715.

Takens, F., 1980. Dynamical Systems and Turbulence, Lecture Notes in Maths. Springer-Verlag, New York.

Zhang, C., Zhou, X. X., Cao Y. J. and Wu, Q. H., 1998. A disturbance auto-rejection control of TCSC. *In*: Proc. IEE Control 98, **2**: 1017 – 1023.