http://www.zju.edu.cn/jzus; E-mail: jzus@zju.edu.cn
ISSN 1009-3095 *Journal of Zhejiang University SCIENCE* V.4, No.4, P.379-387, July-Aug., 2003

Free vibration of piezoelectric annular plate*

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Received Oct. 10, 2002; revision accepted Jan. 5, 2003

Abstract: General solutions for coupled three dimensional equations of piezoelectric media were used in this work to obtain some analytical solutions for free vibration of piezoelectric annular plates. These solutions not only satisfy the governing equations at every point in the concerned region but also satisfy the prescribed boundary conditions at every point on the boundaries. Therefore, they are three-dimensional exact. Numerical results are finally tabulated.

Key words: Piezoelectric media, Sectorial annular plate, Free vibration
Document code: A **CLC number:** O343.8

INTRODUCTION

Due to their wide applications in engineering, piezoelectric crystal plates attracted much attention and had been extensively studied as reported. However, most works were based on various two-dimensional approximations (Dokmeci, 1980; Lawson, 1942; Mindlin, 1952; 1972; Lee *et al.*, 1987; Wang *et al.*, 2000).

If a solution satisfies the governing equations at every point in the studied region and the prescribed boundary conditions at every point on the boundaries, it is called three-dimensional exact solution, which is always pursued by researchers because it represents advances in mechanics theory and provides the benchmarks for assessing approximate methods. Three-dimensional exact solutions for the free vibrations of rectangular piezoelectric plates had been obtained by employing Fourier series expansions (Heyliger *et al.*, 1995; Chen *et al.*, 1998; Gao *et al.*, 1998; Ray *et al.*, 1998; Kapuria *et al.*, 1998) obtained three-dimensional axisymmetric piezothermoelastic solution of a finite transversely isotropic piezoelectric clamped circular plate. Ding *et al.* (1999) presented three-dimensional exact solutions for the free axisymmetric vibration of circular piezoelectric plates by utilizing the state space method as well as finite Hankel transform.

Ding *et al.* (1996) obtained the general solution for the coupled equations of transversely isotropic piezoelectricity by introducing two displacements functions. One satisfies a second-order partial differential equation while the other satisfies a sixth-order one. The form of this solution is very simple and keeps unchanged either in Cartesian coordinates or in cylindrical coordinates. Hence it can be a powerful tool to solve some static and dynamic problems in piezoelectricity. Ding *et al.* (2000) obtained three-dimensional exact solutions of piezoelectric circular plates which included the non-axisymmetric case through these general solutions (Ding *et al.*, 1996). This paper extends previous works (Ding *et al.*, 2000) to the case of annular plates. The three-dimensional exact solutions for the free vibrations of transversely isotropic piezoelectric circular annular plates were obtained under some boundary conditions. Both axisymmetric and non-axisymmetric cases can also be considered. Some numerical results are finally tabulated.

* Project (No. 10002016) supported by the National Natural Science Foundation of China

GENERAL SOLUTIONS

By introducing two displacement functions ψ and F, Ding *et al*. (1996) obtained a general solution for the coupled linear dynamic equations for piezoelectric media. In circular cylindrical coordinates (r, θ, z), the general solution can be written as

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} - \frac{\partial}{\partial r}A_1 F,\quad u_\theta = -\frac{\partial \psi}{\partial r} - \frac{1}{r}\frac{\partial}{\partial \theta}A_1 F,$$
$$w = A_2 F,\quad \phi = A_3 F, \tag{1}$$

where u_r, u_θ and w are three displacement components, ϕ is the electric potential, and the differential operators A_1, A_2 and A_3 are

$$A_1 = [(c_{13} + c_{44})\varepsilon_{33} + (e_{15} + e_{31})e_{33}]\frac{\partial^3}{\partial z^3} + [(c_{13} + c_{44})\varepsilon_{11} + (e_{15} + e_{31})e_{15}]\Lambda\frac{\partial}{\partial z} \tag{2}$$

$$A_2 = c_{44}\varepsilon_{33}\frac{\partial^4}{\partial z^4} + \left\{[c_{11}\varepsilon_{33} + c_{44}\varepsilon_{11} + (e_{15} + e_{31})^2]\Lambda - \rho\varepsilon_{33}\frac{\partial^2}{\partial t^2}\right\}\frac{\partial^2}{\partial z^2} + c_{11}\varepsilon_{11}\Lambda\Lambda - \rho\varepsilon_{11}\Lambda\frac{\partial^2}{\partial t^2} \tag{3}$$

$$A_3 = c_{44}\varepsilon_{33}\frac{\partial^4}{\partial z^4} + \left\{[c_{11}\varepsilon_{33} + c_{44}\varepsilon_{15} - (c_{13} + c_{44})(e_{15} + e_{31})]\Lambda - \rho\varepsilon_{33}\frac{\partial^2}{\partial t^2}\right\}\frac{\partial^2}{\partial z^2} + c_{11}e_{15}\Lambda\Lambda - \rho e_{15}\Lambda\frac{\partial^2}{\partial t^2}, \tag{4}$$

where $\Lambda = \partial^2/\partial r^2 + (1/r)\partial/\partial r + (1/r^2)\partial^2/\partial\theta^2$ is the two-dimensional Laplacian operator.

The displacement functions ψ and F satisfy

$$\left(c_{66}\Lambda + c_{44}\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2}\right)\psi = 0,\quad L_0 F = 0, \tag{5}$$

where

$$L_0 = a_4\Lambda\Lambda\Lambda + \left(a_3\frac{\partial^2}{\partial z^2} + a_6\frac{\partial^2}{\partial t^2}\right)\Lambda\Lambda + \left(a_2\frac{\partial^4}{\partial z^4} + a_5\frac{\partial^4}{\partial t^4} + a_7\frac{\partial^4}{\partial z^2\partial t^2}\right)\Lambda + a_1\frac{\partial^6}{\partial z^6} + a_8\frac{\partial^6}{\partial z^4\partial t^2} + a_9\frac{\partial^6}{\partial z^2\partial t^4}. \tag{6}$$

Here a_n $(n = 1, 2, \cdots, 9)$ can be expressed in terms of elastic constants c_{ij}, dielectric constants ε_{ij} ane piezoelectric coefficients e_{ij} as follows:

$$\begin{aligned}
&a_1 = c_{44}(e_{33}^2 + c_{33}\varepsilon_{33}),\quad a_4 = c_{11}(e_{15}^2 + c_{44}\varepsilon_{11}),\\
&a_5 = \rho^2\varepsilon_{11},\\
&a_2 = c_{33}[c_{44}\varepsilon_{11} + (e_{15} + e_{31})^2] + \varepsilon_{33}[c_{11}c_{33} + c_{44}^2 - (c_{13} + c_{44})^2] + e_{33}[2c_{44}e_{15} + c_{11}e_{33} - 2(c_{13} + c_{44})(e_{15} + e_{31})],\\
&a_3 = c_{44}[c_{11}\varepsilon_{33} + (e_{15} + e_{31})^2] + \varepsilon_{11}[c_{11}c_{33} + c_{44}^2 - (c_{13} + c_{44})^2] + e_{15}[2c_{11}e_{33} + c_{44}e_{15} - 2(c_{13} + c_{44})(e_{15} + e_{31})],\\
&a_6 = -\rho[e_{15}^2 + (c_{11} + c_{44})\varepsilon_{11}],\\
&a_8 = -\rho[e_{33}^2 + (c_{44} + c_{33})\varepsilon_{33}],\ a_9 = \rho^2\varepsilon_{33},\\
&a_7 = -\rho[2e_{15}e_{33} + (c_{44} + c_{33})\varepsilon_{11} + (c_{11} + c_{44})\varepsilon_{33} + (e_{15} + e_{31})^2].
\end{aligned} \tag{7}$$

MECHANICAL AND ELECTRIC QUANTITIES

Consider a transversely isotropic piezoelectric annular plate of height h_0, outer radius r_0, inner radius r_1 and center angle θ_0 as shown in Fig.1. The center of the upper surface is taken as the origin of the circular cylindrical coordinates (r, θ, z) and the positive direction of z-axis points from the upper surface to the bottom surface. The elastic symmetric axis of the piezoelectric material coincides with the z-axis.

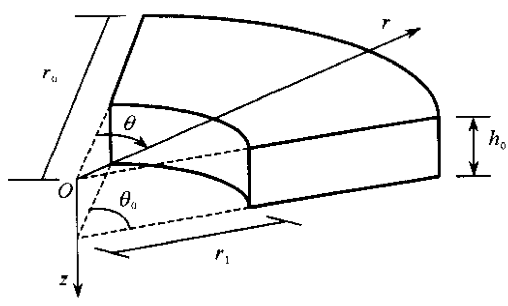

Fig.1 The geometry of the annular plate in the polar coordinates system

If two non-dimensional coordinates are defined as

$$\xi = r/r_0,\quad \zeta = z/h_0 \tag{8}$$

and the resonant frequency of the annular plate is denoted by ω, the displacement functions ψ and F can be written as

$$\begin{cases} F = \dfrac{h_0^5}{c_{11}\varepsilon_{33}} f(\zeta) H_\mu(k\zeta)\cos(\mu\theta)e^{i\omega t} \\ \psi = h_0^2 g(\zeta) H_\mu(\bar{k}\xi)\sin(\mu\theta)e^{i\omega t} \end{cases} \quad (9)$$

where $f(\zeta)$ and $g(\zeta)$ are unknown functions, k, $\bar{k}$ and μ are undetermined parameters, respectively, and

$$H_\mu(\bullet) = AJ_\mu(\bullet) + BY_\mu(\bullet) \quad (10)$$

where $J_\mu(\bullet)$ and $Y_\mu(\bullet)$ are the first and second Bessel functions of order μ, respectively. A and B are constants.

Substituting Eq.(9) into Eq.(5) yields the equations satisfied by $f(\zeta)$ and $g(\zeta)$

$$B_1 f^{(6)}(\zeta) + B_2 f^{(4)}(\zeta) + B_3 f^{(2)}(\zeta) + B_4 f(\zeta) = 0 \quad (11)$$

and

$$g^{(2)}(\zeta) + B_5 g(\zeta) = 0 \quad (12)$$

respectively. Where $f^{(n)}\zeta$ and $g^{(n)}(\zeta)$ denote the nth derivative of $f(\zeta)$ and $g(\zeta)$ with respect to ζ, respectively. The parameters B_n ($n = 1,2,3,4,5$) are

$$\begin{cases} B_1 = \bar{a}_1, \quad B_2 = -(\bar{a}_2 k^2 t_0^2 + \bar{a}_8 \Omega^2) \\ B_3 = \bar{a}_3 k^4 t_0^4 + \bar{a}_7 k^2 t_0^2 \Omega^2 + \Omega^4 \\ B_4 = -(\bar{a}_4 k^6 t_0^6 + \bar{a}_6 k^4 t_0^4 \Omega^2 + \bar{a}_5 k^2 t_0^2 \Omega^4) \\ B_5 = -(c_{66}\bar{k}^2 t_0^2 - c_{11}\Omega^2)/c_{44} \end{cases} \quad (13)$$

where $\Omega^2 = \rho\omega^2 h_0^2/c_{11}$, $t_0 = h_0/r_0$ and

$$\begin{cases} \bar{a}_n = a_n/(c_{11}^2\varepsilon_{33})(n = 1,2,3,4) \\ \bar{a}_n = a_n/(\rho c_{11}\varepsilon_{33})(n = 6,7,8) \\ \bar{a}_5 = a_5/(\rho^2\varepsilon_{33}) \end{cases} \quad (14)$$

If Eq.(11) has distinct eigenvalues λ_n ($n = 1, 2, \cdots, 6$), one has

$$f(\zeta) = \sum_{n=1}^{6} \beta_n e^{\lambda_n \zeta} \quad (15)$$

where β_n ($n = 1, 2, \cdots, 6$) are arbitrary constants. The solutions of Eq.(12) are

$$g(\zeta) = \begin{cases} \beta_7 e^{\lambda_7\zeta} + \beta_8 e^{-\lambda_7\zeta} & \lambda_7^2 > 0 \\ \beta_7\zeta + \beta_8 & \lambda_7^2 = 0 \\ \beta_7\cos(\bar{\lambda}_7\zeta) + \beta_8\sin(\bar{\lambda}_7\zeta) & \lambda_7^2 < 0 \end{cases} \quad (16)$$

where β_7 and β_8 are arbitrary constants and

$$\bar{\lambda}_7 = \sqrt{-\lambda_7^2}, \quad \lambda_7^2 = -B_5 \quad (17)$$

Substitution of Eq.(9) into Eq.(1) yields the displacements and electric potential

$$u_r = -h_0\left\{\left[\sum_{n=1}^{5} U_n f^{(n-1)}(\zeta)\right] k t_0 H'_\mu(k\xi) + \frac{\mu t_0}{\xi} g(\zeta) H_\mu(\bar{k}\xi)\right\}\cos(\mu\theta)e^{i\omega t} \quad (18a)$$

$$u_\theta = h_0\left\{\left[\sum_{n=1}^{5} U_n f^{(n-1)}(\zeta)\right](\mu t_0/\xi) H_\mu(k\xi) - \bar{k} t_0 g(\zeta) H'_\mu(\bar{k}\xi)\right\}\sin(\mu\theta)e^{i\omega t} \quad (18b)$$

$$w = h_0\left[\sum_{n=1}^{5} W_n f^{(n-1)}(\zeta)\right] H_\mu(k\xi)\cos(\mu\theta)e^{i\omega t} \quad (18c)$$

$$\phi = h_0\sqrt{c_{11}/\varepsilon_{33}}\left[\sum_{n=1}^{5} \Phi_n f^{(n-1)}(\zeta)\right] H_\mu(k\xi) \bullet \cos(\mu\theta)e^{i\omega t} \quad (18d)$$

where U_n, W_n and Φ_n ($n = 0, 1, \cdots, 5$) are

$$\begin{cases} U_1 = U_3 = U_5 = 0 \\ U_2 = -k^2 t_0^2[(c_{13} + c_{44})\varepsilon_{11} + (e_{15} + e_{31})e_{15}]/(c_{11}\varepsilon_{33}) \\ U_4 = [(c_{13} + c_{44})\varepsilon_{33} + (e_{15} + e_{31})e_{33}]/(c_{11}\varepsilon_{33}) \end{cases} \quad (19a)$$

$$\begin{cases} W_2 = W_4 = 0, W_5 = c_{44}/c_{11} \\ W_1 = (k^4 t_0^4 - k^2 t_0^2 \Omega^2)\varepsilon_{11}/\varepsilon_{33}, \\ W_3 = -k^2 t_0^2\left[c_{11}\varepsilon_{33} + c_{44}\varepsilon_{11} + (e_{15} + e_{31})^2\right]/(c_{11}\varepsilon_{33}) + \Omega^2 \end{cases} \quad (19b)$$

$$\begin{cases} \Phi_2 = \Phi_4 = 0, \\ \Phi_1 = (k^4 t_0^4 - k^2 t_0^2\Omega^2)e_{15}/\sqrt{c_{11}\varepsilon_{33}}, \\ \Phi_5 = c_{44}e_{33}/(c_{11}\sqrt{c_{11}\varepsilon_{33}}) \\ \Phi_3 = \{\Omega^2 e_{33}c_{11} - k^2 t_0^2[c_{11}e_{33} + c_{44}e_{15} - (c_{13} + c_{44})(e_{15} + e_{31})]\}/(c_{11}\sqrt{c_{11}\varepsilon_{33}}) \end{cases} \quad (19c)$$

The constitutive relations of transversely isotropic pizeoelectric materials can be written as (Ding *et al.*, 1996)

$$\begin{matrix} \{\boldsymbol{\sigma}\} = [\boldsymbol{C}]\{\boldsymbol{\gamma}\} - [\boldsymbol{e}]\{\boldsymbol{E}\} \\ \{\boldsymbol{D}\} = [\boldsymbol{e}]^T\{\boldsymbol{\gamma}\} + [\boldsymbol{\varepsilon}]\{\boldsymbol{E}\} \end{matrix} \quad (20)$$

where

$$\begin{matrix} \{\boldsymbol{\sigma}\} = [\sigma_r \quad \sigma_\theta \quad \sigma_z \quad \tau_{\theta z} \quad \tau_{rz} \quad \tau_{r\theta}]^T, \\ \{\boldsymbol{D}\} = [D_r \quad D_\theta \quad D_z]^T \end{matrix} \quad (21)$$

$$\{\boldsymbol{\gamma}\} = \left[\frac{\partial u_r}{\partial r} \quad \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \quad \frac{\partial w}{\partial z} \quad \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta} \quad \frac{\partial w}{\partial r} + \frac{\partial u_r}{\partial z} \quad \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right]^{\mathrm{T}} \tag{22}$$

$$\{\boldsymbol{E}\} = -\left[\frac{\partial \phi}{\partial r} \quad \frac{1}{r}\frac{\partial \phi}{\partial \theta} \quad \frac{\partial \phi}{\partial z}\right]^{\mathrm{T}} \tag{23}$$

$$[\boldsymbol{C}] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (c_{11}-c_{12})/2 \end{bmatrix} \tag{24}$$

$$[\boldsymbol{e}] = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix}, \quad [\boldsymbol{\varepsilon}] = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix} \tag{25}$$

Substitution of the displacements and electric potential into the constitutive relations of piezoelectric material yields the stresses and electric displacements

$$\sigma_r = \Big\{(c_{12} - c_{11})\Big[\sum_{n=1}^{5} U_n f^{(n-1)}(\zeta)\Big] k^2 t_0^2 H''_\mu(k\xi) + \Big[c_{12} k^2 t_0^2 \sum_{n=1}^{5} U_n f^{(n-1)}(\zeta) + c_{13} \sum_{n=1}^{5} W_n f^{(n)}(\zeta) + e_{31}\sqrt{\frac{c_{11}}{\varepsilon_{33}}} \sum_{n=1}^{5} \Phi_n f^{(n)}(\zeta)\Big] H_\mu(k\xi) + 2c_{66}\Big[\frac{\bar{k}\mu t_0}{\xi} H'_\mu(\bar{k}\xi) - \frac{\mu t_0}{\xi^2} H_\mu(\bar{k}\xi)\Big] g(\zeta)\Big\}\cos(\mu\theta)\mathrm{e}^{i\omega t} \tag{26}$$

$$\sigma_\theta = \Big\{(c_{11} - c_{12})\Big[\sum_{n=1}^{5} U_n f^{(n-1)}(\zeta)\Big] k^2 t_0^2 H''_\mu(k\xi) + \Big[c_{11} k^2 t_0^2 \sum_{n=1}^{5} U_n f^{(n-1)}(\zeta) + c_{13} \sum_{n=1}^{5} W_n f^{(n)}(\zeta) + e_{31}\sqrt{\frac{c_{11}}{\varepsilon_{33}}} \sum_{n=1}^{5} \Phi_n f^{(n)}(\zeta)\Big] H_\mu(k\xi) - 2c_{66}\Big[\frac{\bar{k}\mu t_0}{\xi} H'_\mu(\bar{k}\xi) - \frac{\mu t_0}{\xi^2} H_\mu(\bar{k}\xi)\Big] g(\zeta)\Big\}\cos(\mu\theta)\mathrm{e}^{i\omega t} \tag{27}$$

$$\sigma_z = \Big[c_{13} k^2 t_0^2 \sum_{n=1}^{5} U_n f^{(n-1)}(\zeta) + c_{33} \sum_{n=1}^{5} W_n f^{(n)}(\zeta) + e_{33}\sqrt{\frac{c_{11}}{\varepsilon_{33}}} \sum_{n=1}^{5} \Phi_n f^{(n)}(\zeta)\Big] H_\mu(k\xi)\cos(\mu\phi)\mathrm{e}^{i\omega t} \tag{28}$$

$$\tau_{\theta z} = \Big\{\Big[c_{44} \sum_{n=1}^{5} U_n f^{(n)}(\zeta) - c_{44} \sum_{n=1}^{5} W_n f^{(n-1)}(\zeta) - e_{15}\sqrt{\frac{c_{11}}{\varepsilon_{33}}} \sum_{n=1}^{5} \Phi_n f^{(n-1)}(\zeta)\Big](\mu t_0/\xi) H_\mu(k\xi) - c_{44}\bar{k} t_0 g'(\zeta) H'_\mu(\bar{k}\xi)\Big\}\sin(\mu\theta)\mathrm{e}^{i\omega t} \tag{29}$$

$$\tau_{rz} = \Big\{\Big[c_{44} \sum_{n=1}^{5} U_n f^{(n)}(\zeta) - c_{44} \sum_{n=1}^{5} W_n f^{(n-1)}(\zeta) - e_{15}\sqrt{\frac{c_{11}}{\varepsilon_{33}}} \sum_{n=1}^{5} \Phi_n f^{(n-1)}(\zeta)\Big] k t_0 H'_\mu(k\xi) + c_{44}\bar{k} t_0 g'(\zeta) H_\mu(\bar{k}\xi)\Big\}\cos(\mu\phi)\mathrm{e}^{i\omega t} \tag{30}$$

$$\tau_{r\theta} = \Big\{2c_{66}\Big[\sum_{n=1}^{5} U_n f^{(n-1)}(\zeta)\Big]\Big[(k\mu t_0^2/\varepsilon) H'_\mu(k\xi) - (\mu t_0^2/\xi^2) H_\mu(k\xi)\Big] - c_{66} k^2 t_0^2 \big[2H''_\mu(\bar{k}\xi) - H_\mu(\bar{k}\xi)\big] g(\zeta)\Big\}\sin(\mu\theta)\mathrm{e}^{i\omega t} \tag{31}$$

$$D_r = \Big\{\Big[-e_{15} \sum_{n=1}^{5} U_n f^{(n)}(\zeta) + e_{15} \sum_{n=1}^{5} W_n f^{(n-1)}(\zeta) - \varepsilon_{11}\sqrt{\frac{c_{11}}{\varepsilon_{33}}} \sum_{n=1}^{5} \Phi_n f^{(n-1)}(\zeta)\Big] k t_0 H'_\mu(k\xi) + e_{15}\frac{\mu t_0}{\xi} g'(\zeta) H_\mu(\bar{k}\xi c)\Big\}\cos(\mu\theta)\mathrm{e}^{i\omega t} \tag{32}$$

$$D_\theta = \Big\{\Big[e_{15} \sum_{n=1}^{5} U_n f^{(n)}(\zeta) - e_{15} \sum_{n=1}^{5} W_n f^{(n-1)}(\zeta) + \varepsilon_{11}\sqrt{\frac{c_{11}}{\varepsilon_{33}}} \sum_{n=1}^{5} \Phi_n f^{(n-1)}(\zeta)\Big](\mu t_0/\xi) H_\mu(k\xi) - e_{15}\bar{k} t_0 g'(\zeta) H'_\mu(\bar{k}\xi)\Big\}\sin(\mu\theta)\mathrm{e}^{i\omega t} \tag{33}$$

$$D_z = \Big[e_{31} k^2 t_0^2 \sum_{n=1}^{5} U_n f^{(n-1)}(\zeta) + e_{33} \sum_{n=1}^{5} W_n f^{(n)}(\zeta) - \sqrt{c_{11}\varepsilon_{33}} \sum_{n=1}^{5} \Phi_n f^{(n)}(\zeta)\Big] H_\mu(k\xi)\cos(\mu\theta)\mathrm{e}^{i\omega t} \tag{34}$$

BOUNDARY CONDITIONS

If k and $\bar{k}$ satisfy

$$H_\mu(k) = H'_\mu(\bar{k}) = 0 \quad \text{or}$$
$$H_\mu(ks) = H'_\mu(\bar{k}s) = 0 \tag{35}$$

where $s = r_1/r_0$, on the boundary of $r = r_0$, i.e., $\xi = 1$ or $r = r_1$, i.e., $\xi = s$ one has

$$u_z = 0,\ u_\theta = 0,\ \phi = 0 \quad \text{and}$$
$$(c_{11} - c_{12})u_r + r\sigma_{rr} = 0 \tag{36}$$

If k and $\bar{k}$ satisfy

$$H'_\mu(k) = H_\mu(\bar{k}) = 0 \ \text{or}$$
$$H'_\mu(ks) = H_\mu(\bar{k}s) = 0 \tag{37}$$

on the boundary of $r = r_0$, i.e., $\xi = 1$ or $r = r_1$, i.e., $\xi = s$ one has

$$u_r = 0,\ \tau_{rz} = 0,\ D_r = 0 \ \text{and}$$
$$(c_{11} - c_{12})u_\theta + r\tau_{r\theta} = 0 \tag{38}$$

Hence, four boundary conditions can be defined on the outer and inner circumferential boundaries: (1) boundary condition satisfies Eq.(36) on the boundaries of $\xi = 1$ and $\xi = s$; (2) boundary condition satisfies Eq.(38) on the boundaries of $\xi = 1$ and $\xi = s$; (3) boundary condition on the boundary of $\xi = 1$ satisfies Eq.(36) and on the boundary of $\xi = s$ satisfies Eq.(38); (4) boundary condition on the boundary of $\xi = 1$ satisfies Eq. (38) and on the boundary of $\xi = s$ satisfies Eq.(36).

In a similar manner, if $\mu = n\pi/\theta_0 (n = 1, 2, \cdots)$ on the boundaries of $\theta = 0$ and $\theta = \theta_0$, one has

$$U_\theta = 0,\ T_{r\theta} = T_{\theta z} = 0,\ D_\theta = 0 \tag{39}$$

If $\mu = (2n + 1)\pi/(2\theta_0)\ (n = 0, 1, 2, \cdots)$, on the boundary of $\theta = 0$ one has Eq.(39) and on the boundary of $\theta = \theta_0$, one has

$$u_r = w = 0,\ \phi = 0,\ \sigma_\theta = 0 \tag{40}$$

If $\cos(\mu\theta)$ and $\sin(\mu\theta)$ in the two displacement functions are interchanged and $\mu = n\pi/\theta_0 (n = 1, 2, \cdots)$, boundary conditions on the boundaries of $\theta = 0$ and $\theta = \theta_0$ both satisfy Eq.(40). Consequently, three boundary conditions can be defined on the other two boundaries ($\theta = 0$ and $\theta = \theta_0$) of the annular plates: (I) boundary conditions on the boundaries of $\theta = 0$ and $\theta = \theta_0$ both satisfy Eq.(39); (II) boundary conditions on the boundaries of $\theta = 0$ and $\theta = \theta_0$ both satisfy Eq.(40); (III) boundary condition on the boundary of $\theta = 0$ satisfies Eq.(39) and on the boundary of $\theta = \theta_0$ satisfies Eq. (40).

If the center angle $\theta_0 = 2\pi$ and μ is an integer, which implies $\sin(\mu\theta) = \sin(\mu\theta + 2\mu\pi)$ and $\cos(\mu\theta) = \cos(\mu\theta + 2\mu\pi)$, the sectorial annular plate degenerate to circular annular plate. If $\mu = 0$, all physical quantities are independent on θ and lead to axisymmetric case.

FREQUENCY EQUATION

For the case of free vibration, at the upper and bottom surfaces of the annular plate, the mechanical conditions are

$$\sigma_z = \tau_{rz} = \tau_{\theta z} = 0 \tag{41a}$$

and the electric conditions are

$$D_z = 0 \ \text{or} \ \phi = 0 \qquad \text{(41b or 41c)}$$

For the sake of simplify, Case (1) is corresponded to Eq.(41b) while Case (2) to Eq. (41c).

Substitution of Eqs.(28) – (30) into Eq. (41a) yields

$$g'(0) = g'(1) = 0 \tag{42}$$

and

$$\begin{cases} c_{13}k^2 t_0^2 \sum_{n=1}^{5} U_n f^{(n-1)}(0) + c_{33}\sum_{n=1}^{5} W_n f^{(n)}(0) + e_{33}\sqrt{\dfrac{c_{11}}{\varepsilon_{33}}}\sum_{n=1}^{5}\Phi_n f^{(n)}(0) = 0 \\ c_{13}k^2 t_0^2 \sum_{n=1}^{5} U_n f^{(n-1)}(1) + c_{33}\sum_{n=1}^{5} W_n f^{(n)}(1) + e_{33}\sqrt{\dfrac{c_{11}}{\varepsilon_{33}}}\sum_{n=1}^{5}\Phi_n f^{(n)}(1) = 0 \\ c_{44}\sum_{n=1}^{5} U_n f^{(n)}(0) - c_{44}\sum_{n=1}^{5} W_n f^{(n-1)}(0) - e_{15}\sqrt{\dfrac{c_{11}}{\varepsilon_{33}}}\sum_{n=1}^{5}\Phi_n f^{(n-1)}(0) = 0 \\ c_{44}\sum_{n=1}^{5} U_n f^{(n)}(1) - c_{44}\sum_{n=1}^{5} W_n f^{(n-1)}(1) - e_{15}\sqrt{\dfrac{c_{11}}{\varepsilon_{33}}}\sum_{n=1}^{5}\Phi_n f^{(n-1)}(1) = 0 \end{cases} \tag{43a}$$

Substitution of Eq.(34) or (18d) into Eq.(41b) or (41c), respectively, yields

$$\begin{cases} e_{31}k^2t_0^2\sum_{n=1}^{5}U_nf^{(n-1)}(0)+e_{33}\sum_{n=1}^{5}W_nf^{(n)}(0)- \\ \quad\sqrt{c_{11}\varepsilon_{33}}\sum_{n=1}^{5}\Phi_nf^{(n)}(0)=0 \\ e_{31}k^2t_0^2\sum_{n=1}^{5}U_nf^{(n-1)}(1)+e_{33}\sum_{n=1}^{5}W_nf^{(n)}(1)- \\ \quad\sqrt{c_{11}\varepsilon_{33}}\sum_{n=1}^{5}\Phi_nf^{(n)}(1)=0 \end{cases} \tag{43b}$$

or

$$\begin{cases} \sum_{n=1}^{5}\Phi_nf^{(n-1)}(0)=0 \\ \sum_{n=1}^{5}\Phi_nf^{(n-1)}(1)=0 \end{cases} \tag{43c}$$

Substituting Eq.(16) into Eq.(42) yields homogeneous equations with respect to parameters β_7 and β_8. For the non-trivial solutions of the homogeneous equations, one has

$$\lambda_7^2=0 \text{ or } \bar{\lambda}_7=m\pi \quad (m=1,2,\cdots) \tag{44}$$

From Eq.(17) and the last one of Eq.(13), one has

$$\Omega^2=[c_{66}\bar{k}^2t_0^2+c_{44}(m\pi)^2]/c_{11} \quad (m=0,1,2,\cdots) \tag{45}$$

which is a characteristic frequency equation for the free vibration of a piezoelectric annular plates. Substituting Eq.(15) into Eqs.(41a) and (41b) or (41c) also yields homogeneous equations with respect to parameters β_n ($n=1,2,\cdots,6$)

$$[T_{mn}]\{\beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4 \quad \beta_5 \quad \beta_6\}^{\mathrm{T}}=0 \tag{46}$$

where

$$\begin{cases} T_{1m}=c_{13}k^2t_0^2\sum_{n=1}^{5}U_n\lambda_m^{n-1}+c_{33}\sum_{n=1}^{5}W_n\lambda_m^n+ \\ \quad e_{33}\sqrt{\frac{c_{11}}{\varepsilon_{33}}}\sum_{n=1}^{5}\Phi_n\lambda_m^n, \ T_{2m}=T_{1m}\mathrm{e}^{\lambda_m} \\ T_{3m}=c_{44}\sum_{n=1}^{5}U_n\lambda_m^n-c_{44}\sum_{n=1}^{5}W_n\lambda_m^{n-1}- \\ \quad e_{15}\sqrt{\frac{c_{11}}{\varepsilon_{33}}}\sum_{n=1}^{5}\Phi_n\lambda_m^{n-1}, \ T_{4m}=T_{3m}\mathrm{e}^{\lambda_m} \\ T_{5m}=e_{31}k^2t_0^2\sum_{n=1}^{5}U_n\lambda_m^{n-1}+e_{33}\sum_{n=1}^{5}W_n\lambda_m^n- \\ \quad\sqrt{c_{11}\varepsilon_{33}}\sum_{n=1}^{5}\Phi_n\lambda_m^n, \ T_{6m}=T_{5m}\mathrm{e}^{\lambda_m} \end{cases}$$

$$(m=1,2,\cdots,6) \tag{47a}$$

and

$$T_{5m}=e_{31}k^2t_0^2\sum_{n=1}^{5}U_n\lambda_m^{n-1}+e_{33}\sum_{n=1}^{5}W_n\lambda_m^n-\sqrt{c_{11}\varepsilon_{33}}\sum_{n=1}^{5}\Phi_n\lambda_m^n, \ T_{6m}=T_{5m}\mathrm{e}^{\lambda_m} \quad (m=1,2,\cdots,6) \tag{47b}$$

corresponds to Eq.(41b) or

$$T_{5m}=\sum_{n=1}^{5}\Phi_n\lambda_m^{n-1}, \ T_{6m}=T_{5m}\mathrm{e}^{\lambda_m} \quad (m=1,2,\cdots,6) \tag{47c}$$

corresponds to Eq.(41c). For nontrivial solutions of β_m ($m=1,2,\cdots,6$), the determinant of coefficients of the homogeneous Eq.(46) should vanish, i.e.,

$$|T_{mn}|=0 \tag{48}$$

which gives another characteristic frequency equation for the free vibration of piezoelectric annular plates, so the free vibration of annular plates can be categorized into two kinds with frequency equations corresponding to Eq.(45) and Eq.(48), respectively. In fact, frequency Eq.(45) denotes the in-plane vibration of an annular plate. If a frequency Ω is obtained by Eq.(45), it usually does not satisfy Eq.(48). So Eq.(46) has only trivial solution, namely, $\beta_n=0$ ($n=1,2,\cdots,6$). As a result, one has $f(\zeta)=0$ and $w=0$. It means the frequency Ωs obtained by Eq.(45) corresponds to in-plane vibration of annular plate.

In Eq.(45) and (48), the parameters μ, k and $\bar{k}$ are determined by the boundary conditions. Obviously, infinite frequencies can be obtained from Eq.(45) for each $\bar{k}$ as well as from Eq.(48) for each pair of μ and k since Eq.(48) is a transcendental equation with respect to non-dimensional frequency Ω.

NUMERICAL EXAMPLES

It is noteworthy that the boundary conditions on the boundary $\theta=0$ and $\theta=\theta_0$, wave number n in the circumferential direction and the center angle θ_0 of the sectorial annular plate are all related to the parameter μ, namely, $\mu = n\pi/\theta_0$ corresponds to boundary conditions (I) or (II) and $\mu = (2n+1)\pi/2\theta_0$ to boundary conditions (III). This means that different boundary condition on the boundary $\theta = 0$ and $\theta = \theta_0$, wave number n and value of center angle θ_0 may probably have the same value of the parameter μ. For instances, $n = 1$, $\theta_0 = \pi/0.9$, and boundary conditions (I) lead to $\mu = 0.9$ while $n = 1$, $\theta_0 = \pi/0.6$ and boundary condition (III) also lead to $\mu = 0.9$. Consequently, only the value of the parameter μ is given in the following numerical examples.

Example 1. Consider a circular annular plate with material constants $c_{11} = 13.9 \times 10^{10}$ Pa, $c_{12} = 7.78 \times 10^{10}$ Pa, $c_{13} = 7.43 \times 10^{10}$ Pa, $c_{33} = 11.5 \times 10^{10}$ Pa, $c_{44} = 2.56 \times 10^{10}$ Pa, $e_{15} = 12.7$ C/m^2, $e_{31} = -5.2$ C/m^2, $e_{33} = 15.1$ C/m^2, $\varepsilon_{11} = 6.46 \times 10^{-9}$ F/m, $\varepsilon_{33} = 5.62 \times 10^{-9}$ F/m. Its ratio of inner radius to outer radius $s = 0.5$. Tables 1 – 4 show the first non-dimensional frequencies under four boundary conditions when $\mu = 1, 2, 3$ and $t_0 = 0.1 - 0.5$. Tables 5 – 8 show the first five non-dimensional frequencies under four boundary conditions for $t_0 = 0.1$.

Table 1 The first frequency of circular annular plate under boundary condition (1)

t_0	$\mu = 1$		$\mu = 2$		$\mu = 3$	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
0.1	0.1034	0.1023	0.1166	0.1152	0.1380	0.1362
0.2	0.3604	0.3517	0.4011	0.3910	0.4655	0.4530
0.3	0.6925	0.6718	0.7626	0.7396	0.8718	0.8452
0.4	1.0562	1.0239	1.1545	1.1193	1.3059	1.2667
0.5	1.4320	1.3896	1.5566	1.5112	1.7474	1.6975

Table 2 The first frequency of circular annular plate under boundary condition (2)

t_0	$\mu = 1$		$\mu = 2$		$\mu = 3$	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
0.1	0.0049	0.0049	0.0191	0.0191	0.0411	0.0409
0.2	0.0195	0.0195	0.0740	0.0734	0.1541	0.1519
0.3	0.0433	0.0430	0.1587	0.1564	0.3176	0.3105
0.4	0.0755	0.0749	0.2662	0.2608	0.5126	0.4984
0.5	0.1153	0.1140	0.3902	0.3805	0.7262	0.7044

Table 3 The first frequency of circular annular plate under boundary condition (3)

t_0	$\mu = 1$		$\mu = 2$		$\mu = 3$	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
0.1	0.0403	0.0401	0.0588	0.0585	0.0884	0.0876
0.2	0.1511	0.1490	0.2156	0.2118	0.3129	0.3059
0.3	0.3120	0.3051	0.4337	0.4224	0.6094	0.5917
0.4	0.5041	0.4903	0.6852	0.6648	0.9388	0.9100
0.5	0.7149	0.6935	0.9541	0.9249	1.2821	1.2436

Table 4 The first frequency of circular annular plate under boundary condition (4)

t_0	$\mu = 1$		$\mu = 2$		$\mu = 3$	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
0.1	0.0234	0.0233	0.0345	0.0344	0.0524	0.0521
0.2	0.0901	0.0893	0.1305	0.1289	0.1937	0.1905
0.3	0.1915	0.1884	0.2720	0.2664	0.3929	0.3831
0.4	0.3184	0.3112	0.4433	0.4317	0.6251	0.6069
0.5	0.4628	0.4505	0.6332	0.6147	0.8753	0.8485

Table 5 The first five frequencies of circular annular plate under boundary condition (1)

rank	$\mu = 1$		$\mu = 2$		$\mu = 3$	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
1	0.1034	0.1023	0.1166	0.1152	0.1380	0.1362
2	0.3527	0.3443	0.3637	0.3549	0.3818	0.3724
3	0.6765	0.6564	0.6851	0.6647	0.6994	0.6785
4	1.0324	1.0008	1.0393	1.0074	1.0507	1.0185
5	1.4009	1.3593	1.4065	1.3648	1.4159	1.3739

Table 6 The first five frequencies of circular annular plate under boundary condition (2)

rank	$\mu = 1$		$\mu = 2$		$\mu = 3$	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
1	0.0049	0.0049	0.0191	0.0191	0.0411	0.0409
2	0.1087	0.1075	0.1247	0.1232	0.1514	0.1493
3	0.3566	0.3481	0.3682	0.3593	0.3874	0.3778
4	0.6794	0.6593	0.6883	0.6678	0.7029	0.6819
5	1.0347	1.0030	1.0417	1.0098	1.0532	1.0210

Table 7 The first five frequencies of circular annular plate under boundary condition (3)

rank	$\mu = 1$		$\mu = 2$		$\mu = 3$	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
1	0.0403	0.0401	0.0588	0.0585	0.0884	0.0876
2	0.2234	0.2193	0.2369	0.2324	0.2592	0.2540
3	0.5153	0.5010	0.5254	0.5107	0.5420	0.5268
4	0.8572	0.8310	0.8650	0.8385	0.8779	0.8511
5	1.2200	1.1830	1.2262	1.1891	1.2366	1.1992

Table 8 The first five frequencies of circular annular plate under boundary condition (4)

rank	$\mu=1$		$\mu=2$		$\mu=3$	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
1	0.0234	0.0233	0.0345	0.0344	0.0524	0.0521
2	0.2104	0.2067	0.2227	0.2186	0.2428	0.2381
3	0.5053	0.4914	0.5151	0.5008	0.5312	0.5163
4	0.8494	0.8235	0.8571	0.8309	0.8698	0.8432
5	1.2137	1.1770	1.2199	1.1830	1.2302	1.1930

Example 2. Consider a sectorial annular plate with the same material as Example 1. The ratio of inner radius to outer radius $s=0.5$. The frequencies under four boundary conditions are tabulated in Tables 9 – 12.

Table 9 The first frequency of sectorial annular plate under boundary condition (1)

t_0	$\mu=1.8$		$\mu=0.9$		$\mu=0.6$	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
0.1	0.1133	0.1120	0.1026	0.1015	0.1005	0.0995
0.2	0.3909	0.3812	0.3577	0.3492	0.3515	0.3431
0.3	0.7452	0.7227	0.6879	0.6674	0.6770	0.6569
0.4	1.1302	1.0957	1.0498	1.0177	1.0344	1.0028
0.5	1.5259	1.4811	1.4239	1.3817	1.4043	1.3626

Table 10 The first frequency of sectorial annular plate under boundary condition (2)

t_0	$\mu=1.8$		$\mu=0.9$		$\mu=0.6$	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
0.1	0.0156	0.0156	0.0040	0.0040	0.0018	0.0018
0.2	0.0607	0.0603	0.0158	0.0158	0.0071	0.0071
0.3	0.1312	0.1295	0.0352	0.0351	0.0159	0.0159
0.4	0.2219	0.2179	0.0617	0.0613	0.0281	0.0280
0.5	0.3279	0.3204	0.0946	0.0937	0.0434	0.0432

Table 11 The first frequency of sectorial annular plate under boundary condition (3)

t_0	$\mu=1.8$		$\mu=0.9$		$\mu=0.6$	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
0.1	0.0542	0.0539	0.0391	0.0389	0.0362	0.0360
0.2	0.1998	0.1964	0.1469	0.1449	0.1367	0.1349
0.3	0.4042	0.3940	0.3038	0.2971	0.2840	0.2780
0.4	0.6418	0.6230	0.4917	0.4783	0.4616	0.4493
0.5	0.8973	0.8698	0.6982	0.6774	0.6579	0.6385

Table 12 The first frequency of sectorial annular plate under boundary condition (4)

t_0	$\mu=1.8$		$\mu=0.9$		$\mu=0.6$	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
0.1	0.0317	0.0316	0.0227	0.0226	0.0210	0.0210
0.2	0.1205	0.1191	0.0875	0.0867	0.0812	0.0805
0.3	0.2523	0.2473	0.1862	0.1832	0.1734	0.1708
0.4	0.4130	0.4025	0.3100	0.3031	0.2897	0.2835
0.5	0.5922	0.5752	0.4511	0.4392	0.4231	0.4122

CONCLUSIONS

1. Application of the general solution for coupled equations for piezoelectric media yielded three-dimensional exact solutions for the free vibration of a piezoelectric sectorial annular plate under several boundary conditions. When the circular center angle is equal to 2π and μ is an integer, the proposed solutions are simplified to that of a circular annular plate.

2. Tables 1 – 12 show that the non-dimensional frequency Ω for the boundary condition (1) is the biggest, while that for the boundary condition (ii) is the least when the other conditions are the same. These numerical results also show that the non-dimensional frequency Ω for the Case (1) is high than that for the Case (2), although the differences between the two cases are few.

3. The proposed method can be extended to analyze the static and dynamic behaviors of annular plates.

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