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On the p-norm joint spectral radius

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Abstract: The p-norm joint spectral radius is defined by a bounded collection of square matrices with complex entries and of the same size. In the present paper the author investigates the p-norm joint spectral radius for integers. The method introduced in this paper yields some basic formulas for these spectral radii. The approach used in this paper provides a simple proof of Berger-Wang's relation concerning the ∞-norm joint spectral radius.

Key words: Joint spectral radius, Kronecker product, Matrix, Wavelets
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INTRODUCTION

Suppose Σ is a bounded collection of square matrices with complex entries and of the same size. Let $\mathscr{L}_n = \mathscr{L}_n(\Sigma)$ be the set of all products of length n of the elements of Σ and $\mathscr{L}_0$ the identity matrix $\boldsymbol{I}$. The p-norm joint spectral radius through the limit of normalized norms of products of Σ is given by Rota and Strang (1960) for $p = \infty$, and Jia (1995), Zhou (1998) for other p.

$$\rho_p(\Sigma) := \begin{cases} \limsup\limits_{n\to\infty} \sup\limits_{A\in\mathscr{L}_n} \|\boldsymbol{A}\|^{\frac{1}{n}}, & p = \infty; \\ \limsup\limits_{n\to\infty} \{\sum\limits_{A\in\mathscr{L}_n} \|\boldsymbol{A}\|^{p}\}^{\frac{1}{pn}}, & 1 \leqslant p < \infty, \end{cases} \quad (1)$$

where $\|\cdot\|$ is an arbitrary matrix norm.

Perhaps the most important application of the joint spectral radius concept is its use in cascade algorithms for wavelets of compact support (see e.g. Daubechies and Lagarias (1992a)) or subdivision algorithms for computer aided design (see e.g. Micchelli and Prautzsch (1989)). Wavelets of compact support can be constructed from linear combinations of integer translates of scaling functions (Cohen, 1990). These functions are solutions of the two-scale dilation equation. One typical example of a two-scale dilation equation in one dimension is the following:

$$\varphi(x) = c_0\varphi(2x) + c_1\varphi(2x-1) + \cdots + c_m\varphi(2x-m), \quad (2)$$

with $\sum_{j=0}^{m} c_j = 2$ and c_j, $j = 0, \cdots, m$, are given real coefficients. If φ determines a multiresolution, then the associated wavelet is given by Cohen (1990)

$$\psi(x) = \sum_{j=-m+1}^{1} (-1)^j c_{1-j}\varphi(2x-j).$$

It is known that the existence and the smoothness of the L_p solution φ can be characterized by using $\rho_p(\Sigma)$ with an appropriate Σ (Daubechies and Lagarias, 1992a; Jia, 1995; Colella and Heil, 1992; Heil and Colella, 1994). How can we efficiently calculate $\rho_p(\Sigma)$ for given p and Σ? One can easily see that the direct computation of this quantity has an exponentially increasing cost. We note that in case $p = \infty$, Berger and Wang (1992) proved:

$$\rho_\infty(\Sigma) = \limsup_{n\to\infty} \sup_{A\in\mathscr{L}_n} \rho(\boldsymbol{A})^{\frac{1}{n}}, \quad (3)$$

where $\rho(\cdot)$ on the right hand side is the matrix spectral radius. Thus, they settled a conjecture raised by Daubechies and Lagarias (1992a; 1992b). An elmentary proof of this significant result is given in Elsner (1995). The finiteness conjecture (Daubechies and Lagarias, 1992a; Lagarias and Wang, 1995) for any given finite set of matrices Σ asserts that there exists a finite k and a product $\boldsymbol{A} \in \mathscr{L}_k(\Sigma)$ such that

$$\rho_\infty(\Sigma) = \rho(\boldsymbol{A})^{\frac{1}{k}}.$$

This conjecture was disproved by Bousch and Mairesse (2002). On the other hand, if the matrices in $\sum$ can simultaneously be upper-triangularized or simultaneously Hermitianized, then $\rho_\infty(\sum) = \max_{A\in\sum}\{\rho(A)\}$ (Colella and Heil, 1992). In Bröker and Zhou (2000) it was proved that, if $\sum = \{A_0, A_1\}$ with two real 2 × 2 matrices, then $det(A_0) \leqslant 0$ or $det(A_1) \leqslant 0$ implies

$$\rho_\infty(\textstyle\sum) = \begin{cases} \sup_{j\geqslant 0}(\rho(A_0A_1^j))^{\frac{1}{j+1}}, & det(A_0) \leqslant 0; \\ \sup_{j\geqslant 0}(\rho(A_0^jA_1))^{\frac{1}{j+1}}, & det(A_1) \leqslant 0. \end{cases} \quad (4)$$

Recently, Zhou (1998) obtained some interesting formulas for even p. To present his result, let us remember that the Kronecker product of $A = (a_{ij})_{1\leqslant i,j\leqslant n}$ and $B = (b_{ij})_{1\leqslant i,j\leqslant m}$ is denoted by $A \otimes B$ and is defined to be the block matrix

$$A \otimes B := \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & & a_{nn}B \end{pmatrix}.$$

The k-th Kronecker power $A^{[k]}$ is defined inductively for all positive integers k by $A^{[1]} := A$ and $A^{[k+1]} := A \otimes A^{[k]}$ for $k = 1, 2, \cdots$. Zhou (1998) proved that for $k = 1, 2, \cdots$, Eq.(5) holds

$$\rho_{2k}(\textstyle\sum) = \left\{\rho\left(\sum_{A\in\sum} \bar{A} \otimes A)^{[k]}\right)\right\}^{\frac{1}{2k}}, \quad (5)$$

where $\bar{A} = (\overline{a_{ij}})_{1\leqslant i,j\leqslant n}$. Thus, for even p we have a clear expression for $\rho_p(\sum)$.

In the present paper we shall investigate the p-norm joint spectral radius for integers. We will see later that for integer p the calculation of the p-norm joint spectral radius can be reduced to $\rho_1(\sum)$ with a proper $\sum$. Some basic results concerning the calculation of $\rho_p(\sum)$ will be presented in the next two sections. Moreover, our approach allows us to provide a simple proof of Berger-Wang's relation Eq.(3).

JOINT SPECTRAL RADIUS AND KRONECKER PRODUCT

We shall denote by $tr(A)$ the trace of the square matrix A. Let us write the following auxiliary results as

Lemma 1 If

$$\tilde{\rho}(\textstyle\sum) := \limsup_{n\to\infty} \sup_{A\in\mathscr{L}_n}(\rho(A))^{\frac{1}{n}}, \text{ then}$$

$$\tilde{\rho}(\textstyle\sum) = \limsup_{n\to\infty} \sup_{A\in\mathscr{L}_n} |tr(A)|^{\frac{1}{n}}. \quad (6)$$

For two square matrices $A = (a_{ij})_{i\leqslant i,j\leqslant n}$ and $B = (b_{ij})_{1\leqslant i,j\leqslant m}$, Eq.(7) holds

$$tr(A)tr(B) = tr(A \otimes B). \quad (7)$$

Let $A_1, \cdots, A_N$ be square matrices of size n and $B_1, \cdots, B_n$ of size m, then

$$(A_1 \cdots A_N) \otimes (B_1 \cdots B_N) = (A_1 \otimes B_1) \cdots (A_N \otimes B_N). \quad (8)$$

The first relation above can be found in Chen and Zhou (2000), and the other two formulas are in Horn and Johnson (1991) pages 250 and 251. We also note that Eq.(6) implies in particular for any square matrix A

$$\rho(A) = \limsup_{n\to\infty} |tr(A^n)|^{\frac{1}{n}}. \quad (9)$$

Let us look at Eq.(1) again. There $\|\cdot\|$ is an arbitrary matrix norm. We can of course consider $\|\cdot\|$ as a norm on a space of N^2 dimensions, where N is the size of the matrix. As any two norms on a given finite dimensional space are equivalent, we may replace the matrix norm $\|\cdot\|$ in Eq.(1) by the entrywise l_q-norm given by

$$\|A\|_q := \left(\sum_{i,j} |a_{ij}|^q\right)^{\frac{1}{q}}.$$

Clearly, we have

$$\|A\|_q = \left(\sum_{i,j} |tr(AE_{ij})|^q\right)^{\frac{1}{q}},$$

where each E_{ij} has entry 1 in position (i, j) and all other entries are zero. In this way we obtain

$$\rho_p(\textstyle\sum) = \begin{cases} \limsup_{n\to\infty} \sup_{A\in\mathscr{L}_n}\left(\sum_{i,j} |tr(AE_{ij})|\right)^{\frac{1}{n}}, & p = \infty; \\ \limsup_{n\to\infty}\left(\sum_{A\in\mathscr{L}_n}\sum_{i,j} |tr(AE_{ij})|^p\right)^{\frac{1}{pn}}, & 1 \leqslant p < \infty. \end{cases} \quad (10)$$

Later we will see that for our approach the Eq.(10) is quite useful. In fact, almost all our results presented in this paper rely on this formula. We note also that for $p=\infty$, the quantity $\tilde{\rho}(\sum)$ defined in Lemma 1 is exactly $\rho_\infty(\sum)$ (see Eq.(3)).

In order to study the p-norm joint spectral radius for other integers, we shall use the following correspondence:

$$A = R + iH \leftrightarrow B = \begin{pmatrix} R & H \\ -H & R \end{pmatrix}.$$

Clearly, by this correspondence, the set $\sum = \{A_1, \cdots, A_d\}$ of complex $n\times n$ matrices becomes the set $\sum' = \{B_1, B_2, \cdots, B_d\}$ of real $2n\times 2n$ matrices. Moreover, the p-norm joint spectral radius for complex matrices is preserved under this correspondence. Indeed, we can define two norms $\|\cdot\|'$ and $\|\cdot\|''$ via

$\|A\|' := \|R\|_q + \|H\|_q$ and $\|B\|'' := \|B\|_q$.

Obviously, there exists $c>0$ such that

$c^{-1}\|A\|' \leqslant \|B\|'' \leqslant c\|A\|'$.

As $\|\cdot\|'$ is equivalent to a given matrix norm $\|\cdot\|$, Eq.(1) tells us $\rho_p(\sum) = \rho_p(\sum')$. We remark also that the spectral radius of a matrix product does not change under this correspondence, i.e.,

$$\rho(A_{\epsilon_1}\cdots A_{\epsilon_n}) = \rho(B_{\epsilon_1}\cdots B_{\epsilon_n}). \quad (11)$$

With these notations we have

Theorem 2 Let $\sum = \{A_1, \cdots, A_d\}$ be a set of complex square matrices of the same size. Then for $k=1, 2, \cdots$, Eq.(12) holds

$$\rho_k(\sum) = \rho_1(B_1^{[k]}, \cdots, B_d^{[k]})^{\frac{1}{k}}, \quad (12)$$

where B_j corresponds to A_j. Furthermore, if k is even, then

$$\rho_1(B_1^{[k]}, \cdots, B_d^{[k]}) = \rho\Big(\sum_{j=1}^{d} B_j^{[k]}\Big). \quad (13)$$

We shall denote $E_{i,j,n}$ to be the $n\times n$ matrix, which has entry 1 in position (i, j) and all other entries are zero. Let us first write the following simple fact as

Lemma 3 Let $n = m\times N$. Then there exist some $1\leqslant i_1, j_1\leqslant m$ and $1\leqslant i_2, j_2\leqslant N$ such that

$$E_{i,j,n} = E_{i_1,j_1,m}\otimes E_{i_2,j_2,N}.$$

Proof: Indeed, as $1\leqslant i, j\leqslant n$, we have for some $1\leqslant i_1, j_1\leqslant m$ and $1\leqslant i_2, j_2\leqslant N$

$$i=(i_1-1)N+i_2 \quad \text{and} \quad j=(j_1-1)N+j_2.$$

Then it follows from the definition of the Kronecker product that $E_{i,j,n} = E_{i_1,j_1,m}\otimes E_{i_2,j_2,N}$.

Proof of Theorem 2: As $\rho_k(\sum) = \rho_k(\sum')$ we need only show the assertions for $\sum = \{B_1, \cdots, B_d\}$ of real matrices. Having Eq.(12), then Eq.(13) follows from Eq.(5). Thus, it remains to verify Eq.(12). Writing $T = E_{ij}$ and denoting by N the size of B_l, we have by Eq.(7) and Eq.(8)

$$\left|tr(B_{\epsilon_1}\cdots B_{\epsilon_n}T)\right|^k = \left|tr(B_{\epsilon_1}^{[k]}\cdots B_{\epsilon_n}^{[k]}T^{[k]})\right|,$$

which leads to

$$\sum_{1\leqslant i,j\leqslant N}\sum_{\epsilon_1,\cdots,\epsilon_n}\left|tr(B_{\epsilon_1}\cdots B_{\epsilon_n}E_{ij})\right|^k = \sum_{1\leqslant i,j\leqslant N}\sum_{\epsilon_1,\cdots,\epsilon_n}\left|tr(B_{\epsilon_1}^{[k]}\cdots B_{\epsilon_n}^{[k]}E_{i,j}^{[k]})\right| \leqslant \sum_{1\leqslant i,j\leqslant N^k}\sum_{\epsilon_1,\cdots,\epsilon_n}\left|tr(B_{\epsilon_1}^{[k]}\cdots B_{\epsilon_n}^{[k]}E_{i,j})\right|,$$

where E_{ij} on the right hand side of the last inequality is an $N^k\times N^k$ matrix, which has entry 1 in position (i, j) and all other entries are zero. By Eq.(10) the above inequality implies

$$\rho_k(\sum) \leqslant (\rho_1(B_1^{[k]}, \cdots, B_d^{[k]}))^{\frac{1}{k}}.$$

For the opposite direction we use Lemma 3. We consider the $N^k\times N^k$ matrix E_{i,j,N^k}. By using Lemma 3 we obtain inductively that for some $1\leqslant i_1, i_2, \cdots, i_k\leqslant N$ and $1\leqslant j_1, j_2, \cdots, j_k\leqslant N$ the following equation holds

$$E_{i,j,N^k} = E_{i_1,j_1,N}\otimes E_{i_2,j_2,N}\otimes\cdots\otimes E_{i_k,j_k,N}.$$

Using this factorization of E_{i,j,N^k} we deduce from Eq.(7) and Eq.(8)

$$\sum_{1\leqslant i,j\leqslant N^k}\sum_{\epsilon_1,\cdots,\epsilon_n}\left|tr(B_{\epsilon_1}^{[k]}\cdots B_{\epsilon_n}^{[k]}E_{i,j,N^k})\right| \leqslant \sum_{\substack{1\leqslant i_1,\cdots,i_k\leqslant N\\ 1\leqslant j_1,\cdots,j_k\leqslant N}}\sum_{\epsilon_1,\cdots,\epsilon_n}\left|tr(B_{\epsilon_1}^{[k]}\cdots B_{\epsilon_n}^{[k]}(E_{i_1,j_1,N}\otimes E_{i_2,j_2,N}\otimes\cdots\otimes E_{i_k,j_k,N}))\right| = \sum_{\substack{1\leqslant i_1,\cdots,i_k\leqslant N\\ 1\leqslant j_1,\cdots,j_k\leqslant N}}\sum_{\epsilon_1,\cdots,\epsilon_n}\times\prod_{\tau=1}^{k}\left|tr(B_{\epsilon_1}\cdots B_{\epsilon_n}E_{i_\tau,j_\tau,N})\right|.$$

The arithmetic-mean and geometric-mean inequality tells us

$$\prod_{\tau=1}^{k}\left|tr(\boldsymbol{B}_{\epsilon_1}\cdots\boldsymbol{B}_{\epsilon_n}\boldsymbol{E}_{i_\tau,j_\tau,N})\right| \leqslant \sum_{\tau=1}^{k}\left|tr(\boldsymbol{B}_{\epsilon_1}\cdots\boldsymbol{B}_{\epsilon_n}\boldsymbol{E}_{i_\tau,j_\tau,N})\right|^{k}$$

Thus, for some C_k independent of n we obtain from the last two inequalities

$$\sum_{1\leqslant i,j\leqslant N^k}\sum_{\epsilon_1,\cdots,\epsilon_n}\left|tr(\boldsymbol{B}_{\epsilon_1}^{[k]}\cdots\boldsymbol{B}_{\epsilon_n}^{[k]}\boldsymbol{E}_{i,j,N^k})\right| \leqslant C_k\sum_{1\leqslant i,j\leqslant N}\sum_{\epsilon_1,\cdots,\epsilon_n}\left|tr(\boldsymbol{B}_{\epsilon_1}\cdots\boldsymbol{B}_{\epsilon_n}\boldsymbol{E}_{i,j,N})\right|^{k}.$$

Combining this estimate with Eq.(10) we get

$$[\rho_1(\boldsymbol{B}_1^{[k]},\ \boldsymbol{B}_2^{[k]},\cdots,\ \boldsymbol{B}_d^{[k]})]^{\frac{1}{k}} \leqslant \rho_k(\boldsymbol{B}_1,\ \boldsymbol{B}_2,\cdots,\ \boldsymbol{B}_d).$$

This is the desired estimate.

Corollary 4 Let $\sum = \{\boldsymbol{A}_1,\cdots,\ \boldsymbol{A}_d\}$ be a set of non-negative square matrices of the same size. Then for $k = 1, 2, \cdots$, the following equation holds

$$\rho_k(\sum) = \left[\rho\left(\sum_{j=1}^{d}\boldsymbol{A}_j^{[k]}\right)\right]^{\frac{1}{k}}$$

The Eq.(3) is proved by Berger and Wang (1992). A second proof of this assertion is given by Elsner (1995); his proof is based mainly on analytic and geometric tools. We shall give a constructive proof of Eq.(3).

Proof: Clearly we need only verify

$$\rho_\infty(\sum) \leqslant \lim_{n\to\infty}\sup\ \sup_{A\in\mathscr{D}_n}(\rho(\boldsymbol{A}))^{\frac{1}{n}}. \quad (14)$$

First, the definition of the p-norm joint spectral radius tells us that for any $k = 1, 2, \cdots$, we have

$$\rho_\infty(\sum) \leqslant \rho_k(\sum).$$

If $\sum = \{\boldsymbol{A}_1,\ \cdots,\ \boldsymbol{A}_d\}$, then the corresponding set of real matrices is $\sum' = \{\boldsymbol{B}_1,\cdots,\ \boldsymbol{B}_d\}$. Thus, Theorem 2 tells us that

$$\rho_\infty(\sum) \leqslant \left[\rho\left(\sum_{i=1}^{d}\boldsymbol{B}_i^{[2k]}\right)\right]^{\frac{1}{2k}}.\ \forall\, k = 1, 2, \cdots.$$

Second, for given k we have, due to Eq.(9), a number n_k such that

$$\rho\left[\left(\sum_{i=1}^{d}\boldsymbol{B}_i^{[2k]}\right)^{n_k}\right] \leqslant 2\left|tr\left[\left(\sum_{i=1}^{d}\boldsymbol{B}_i^{[2k]}\right)^{n_k}\right]\right| \quad (15)$$

and $\lim_{k\to\infty} n_k = \infty$. On the other hand, one has by Eq.(7) and Eq.(8)

$$\left|tr\left[\left(\sum_{i=1}^{d}\boldsymbol{B}_i^{[2k]}\right)^{n_k}\right]\right| = \left|\sum_{\epsilon_1,\cdots,\epsilon_{n_k}} tr(\boldsymbol{B}_{\epsilon_1}^{[2k]}\cdots\boldsymbol{B}_{\epsilon_{n_k}}^{[2k]})\right| = \sum_{\epsilon_1,\cdots,\epsilon_{n_k}}[tr(\boldsymbol{B}_{\epsilon_1}\cdots\boldsymbol{B}_{\epsilon_{n_k}})]^{2k} \leqslant d^{n_k}\sup_{\epsilon_1,\cdots,\epsilon_{n_k}}\left|tr(\boldsymbol{B}_{\epsilon_1}\cdots\boldsymbol{B}_{\epsilon_{n_k}})\right|^{2k}.$$

Assuming the size of $\boldsymbol{A}_j$ is N, the size of $\boldsymbol{B}_j$ is $2N$. Since the trace is the sum of eigenvalues we have

$$\sup_{\epsilon_1,\cdots,\epsilon_{n_k}}\left|tr(\boldsymbol{B}_{\epsilon_1}\cdots\boldsymbol{B}_{\epsilon_{n_k}})\right| \leqslant 2N\sup_{\epsilon_1,\cdots,\epsilon_{n_k}}\rho(\boldsymbol{B}_{\epsilon_1}\cdots\boldsymbol{B}_{\epsilon_{n_k}}).$$

Now, as see Eq.(11)

$$\rho(\boldsymbol{B}_{\epsilon_1}\cdots\boldsymbol{B}_{\epsilon_{n_k}}) = \rho(\boldsymbol{A}_{\epsilon_1}\cdots\boldsymbol{A}_{\epsilon_{n_k}}),$$

we get from above

$$\left|tr\left[\left(\sum_{i=1}^{d}\boldsymbol{B}_j^{[2k]}\right)^{n_k}\right]\right| \leqslant d^{n_k}(2N)^{2k}\left[\sup_{\epsilon_1,\cdots,\epsilon_{n_k}}\rho(\boldsymbol{A}_{\epsilon_1}\cdots\boldsymbol{A}_{\epsilon_{n_k}})\right]^{2k}.$$

Combining this with Eq.(15) we obtain

$$\rho_\infty(\sum) \leqslant 2^{\frac{1}{2kn_k}}d^{\frac{1}{2k}}(2N)^{\frac{1}{n_k}}\left[\sup_{\epsilon_1,\cdots,\epsilon_{n_k}}\rho(\boldsymbol{A}_{\epsilon_1}\cdots\boldsymbol{A}_{\epsilon_{n_k}})\right]^{\frac{1}{n_k}}.$$

The choice of n_k implies

$$\lim_{k\to\infty}2^{\frac{1}{2kn_k}}d^{\frac{1}{2k}}(2N)^{\frac{1}{n_k}} = 1.$$

Thus we conclude

$$\rho_\infty(\sum) \leqslant \lim_{k\to\infty}\sup\left[\sup_{\epsilon_1,\cdots,\epsilon_{n_k}}\rho(\boldsymbol{A}_{\epsilon_1}\cdots\boldsymbol{A}_{\epsilon_{n_k}})\right]^{\frac{1}{n_k}}.$$

This finishes the proof of Eq.(3).

The following result may be useful for the estimate of joint spectral radii. Let $\sum_j = \{\boldsymbol{A}_{1,j},\cdots,\ \boldsymbol{A}_{d,j}\}$ be a finite set of $n_j\times n_j$ complex matrices. Denote $\sum_1\otimes\sum_2\otimes\cdots\otimes\sum_m := \{\boldsymbol{A}_{1,1}\otimes\boldsymbol{A}_{1,2}\otimes\cdots\boldsymbol{A}_{1,m},\cdots,\ \boldsymbol{A}_{d,1}\otimes\boldsymbol{A}_{d,2}\otimes\cdots\boldsymbol{A}_{d,m}\}$. We have

Theorem 5 Let $1\leqslant p\leqslant\infty$ and $1\leqslant r_j\leqslant\infty$ such that $\sum_{j=1}^{m}1/r_j = 1$, then

$$\rho_p(\sum_1\otimes\sum_2\otimes\cdots\otimes\sum_m) \leqslant \prod_{j=1}^{m}\rho_{prj}(\sum_j).$$

In particular,

$$\rho_\infty(\sum_1 \otimes \sum_2 \otimes \cdots \otimes \sum_m) \leqslant \prod_{j=1}^{m} \rho_\infty(\sum_j).$$

Proof: For convenience we write $\sum := \sum_1 \otimes \sum_2 \otimes \cdots \otimes \sum_m$. Now let $A \in \mathscr{L}_n(\sum)$, thus, Eq.(8) yields for some $(\epsilon_1, \cdots, \epsilon_n) \in \{1, \cdots, d\}^n$,

$$A = (A_{\epsilon_1,1} \otimes A_{\epsilon_1,2} \otimes \cdots \otimes A_{\epsilon_1,m}) \cdots (A_{\epsilon_n,1} \otimes A_{\epsilon_n,2} \otimes \cdots \otimes A_{\epsilon_n,m}) = (A_{\epsilon_1,1} A_{\epsilon_2,1} \cdots A_{\epsilon_n,1}) \otimes \cdots \otimes (A_{\epsilon_1,m} A_{\epsilon_2,m} \cdots A_{\epsilon_n,m}) =: B_1 \otimes B_2 \otimes \cdots \otimes B_m,$$

where $B_j \in \mathscr{L}_n(\sum_j)$. On the other hand, by Lemma 3 we can write the $\prod_{\tau=1}^{m} n_\tau \times \prod_{\tau=1}^{m} n_\tau$ matrix E_{ij} as

$$E_{ij} = E_{i_1, j_1, n_1} \otimes \cdots \otimes E_{i_m, j_m, n_m},$$

where $1 \leqslant i_\tau, j_\tau \leqslant n_\tau$ for $\tau = 1, 2, \cdots, m$. Now again using Eq.(8) we obtain

$$AE_{ij} = (B_1 \otimes B_2 \otimes \cdots \otimes B_m)(E_{i_1, j_1, n_1} \otimes \cdots \otimes E_{i_m, j_m, n_m}) = (B_1 E_{i_1, j_1, n_1}) \otimes \cdots \otimes (B_m E_{i_m, j_m, n_m}).$$

Hence, Eq.(7) implies

$$tr(AE_{ij}) = \prod_{\tau=1}^{m} tr(B_\tau E_{i_\tau, j_\tau, n_\tau}).$$

It follows from this formula and Hölder's inequality

$$\sum_j \left| a_{j,1} a_{j,2} \cdots a_{j,m} \right|^p \leqslant \prod_{\tau=1}^{m} \left(\sum_j | a_{j,\tau} |^{pr_\tau} \right)^{\frac{1}{r_\tau}}$$

that

$$\sum_{A \in \mathscr{L}_n(\sum)} \left| tr(AE_{ij}) \right|^p \leqslant \prod_{\tau=1}^{m} \left(\sum_{B_\tau \in \mathscr{L}_n(\sum_\tau)} \left| tr(B_\tau E_{i_\tau, j_\tau, n_\tau}) \right|^{pr_\tau} \right)^{\frac{1}{r_\tau}}.$$

Therefore, for some C independent of n, the last inequality yields

$$\sum_{i,j} \sum_{A \in \mathscr{L}_n(\sum)} \left| tr(AE_{ij}) \right|^p \leqslant C \prod_{\tau=1}^{m} \left(\sum_{1 \leqslant i_\tau, j_\tau \leqslant n_\tau} \sum_{B_\tau \in \mathscr{L}_n(\sum_\tau)} \left| tr(B_\tau E_{i_\tau, j_\tau, n_\tau}) \right|^{pr_\tau} \right)^{\frac{1}{r_\tau}},$$

where the sum $\sum_{i,j}$ runs over $1 \leqslant i, j \leqslant \prod_{\tau=1}^{m} n_\tau$. The desired estimate follows from this inequality and Eq.(10).

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