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Delay-dependent robust H_∞ control for a class of uncertain switched systems with time delay*

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Abstract: For linear switched system with both parameter uncertainties and time delay, a delay-dependent sufficient condition for the existence of a new robust H_∞ feedback controller was formulated in nonlinear matrix inequalities solvable by an LMI-based iterative algorithm. Compared with the conventional state-feedback controller, the proposed controller can achieve better robust control performance since the delayed state is utilized as additional feedback information and the parameters of the proposed controllers are changed synchronously with the dynamical characteristic of the system. This design method was also extended to the case where only delayed state is available for the controller. The example of balancing an inverted pendulum on a cart demonstrates the effectiveness and applicability of the proposed design methods.

Key words: Linear switched systems, Robust H_∞ control, Matrix inequality, Time delay
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INTRODUCTION

With the increasing complexity and nonlinearity of control systems, linear switched models with parameter uncertainties were suggested by researchers for describing the nonlinear system (Hassibi, 2000). Typical examples of those switched systems include automobile transmission systems, stepper motor driver, computer disc driver (Hassibi *et al.*, 1999), flexible manufacturing systems and a wide variety of other engineering systems (Lennartson *et al.*, 1994). In recent years, increasing attention has been devoted to the performance analysis and controller synthesis of switched systems and some useful results had been obtained (Hassibi, 2000; Lee *et al.*, 2000; Skafidas *et al.*, 1999).

On the other hand, in most engineering practices time delay phenomena is hard to be avoided due to the transmission speed limitation of information or material, the time-consuming measurement or analysis of the online analyzer, etc. In the last decades, stability analysis and controller synthesis of time delay systems were an attractive field in control theory and received remarkable progress (Cao *et al.*, 1998; Li *et al.*, 2001; Moon *et al.*, 2001).

However, only a few studies on the control problem has been reported for switched systems with both parameter uncertainties and time delay arising in with time delay modeling nonlinear systems, such as remote control systems (Luo *et al.*, 2003) and networked control systems (Walsh *et al.*, 1999; Zhang *et al.*, 2001) for manufacturing, underwater manipulation, aerospace explorer, satellite repair, etc. Generally speaking, if local dy-

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namics of a nonlinear system with time delay around different work points can be described by linear models with parameter uncertainties and time delay, then the overall system will be described by a linear uncertain switched model with time delay. Thus, it is very important to extend the stability analysis and controller design issues to such kind of models.

In this paper, a delay-dependent sufficient condition for the existence of a new robust H_∞ feedback controller for uncertainty linear switched systems with time delay is presented in terms of nonlinear matrix inequalities which can be solved efficiently by an iterative algorithm. The delayed state of the system is taken as additional feedback information in the computation of the controller output. The parameters of the proposed controller are switched synchronously with the dynamical characteristics of the model to guarantee better robust control performance of the closed-loop system. Since the delayed system state is the only information source available in most networked control systems (Zhang *et al.*, 2001) and remote control systems (Luo *et al.*, 2003), a sufficient condition for the existence of a robust H_∞ state delay feedback controller is also discussed in this paper.

PROBLEM STATEMENT AND PRELIMINARIES

Consider an uncertain linear switched system with time delay given by

$$\dot{\boldsymbol{x}} = (\boldsymbol{A}_{0,q(t)} + \Delta\boldsymbol{A}_{0,q(t)})\boldsymbol{x} + (\boldsymbol{A}_{1,q(t)} + \Delta\boldsymbol{A}_{1,q(t)})\boldsymbol{x}_d + (\boldsymbol{B}_{0,q(t)} + \Delta\boldsymbol{B}_{0,q(t)})\boldsymbol{u} + \boldsymbol{B}_{1,q(t)}\boldsymbol{w} \quad (1a)$$

$$\boldsymbol{z} = \boldsymbol{C}_{0,q(t)}\boldsymbol{x} + \boldsymbol{C}_{1,q(t)}\boldsymbol{x}_d + \boldsymbol{D}_{0,q(t)}\boldsymbol{u} \quad (1b)$$

$$\boldsymbol{x}(t) = \varphi(t) \in \mathbb{C}[-d, 0] \quad (1c)$$

where $\boldsymbol{x}_d \triangleq x(t\text{-}d)$ and $\boldsymbol{x}$, $\boldsymbol{u}$, $\boldsymbol{z}$ is the state, control and controlled output of the system with appropriate dimensions, $\boldsymbol{w}$ is the external disturbance which belongs to $\boldsymbol{L}_2[0,\infty)$, $q(t)\colon \mathbb{R}^+ \to I_N \triangleq \{1, 2, \cdots, N\}$ is a piecewise constant switching function, $d>0$ is the constant size of time delay, and $\varphi(t)$ is the initial condition. For any $q(t)\in I_N$, $\boldsymbol{A}_{0,q(t)}$, $\boldsymbol{A}_{1,q(t)}$, $\boldsymbol{B}_{0,q(t)}$, $\boldsymbol{B}_{1,q(t)}$, $\boldsymbol{C}_{0,q(t)}$, $\boldsymbol{C}_{1,q(t)}$ and $\boldsymbol{D}_{0,q(t)}$ are real constant matrices with appropriate dimensions. $\Delta\boldsymbol{A}_{0,q(t)}$, $\Delta\boldsymbol{A}_{1,q(t)}$ and $\Delta\boldsymbol{B}_{0,q(t)}$ are admissible parameter uncertainties of the system that can be described as

$$\Delta\boldsymbol{A}_{0,i} = \boldsymbol{E}_{A_0,i}\Delta_i(t)\boldsymbol{F}_{A_0,i}; \quad \Delta\boldsymbol{A}_{1,i} = \boldsymbol{E}_{A_1,i}\Delta_i(t)\boldsymbol{F}_{A_1,i}$$

$$\Delta\boldsymbol{B}_{0,i} = \boldsymbol{E}_{B_0,i}\Delta_i(t)\boldsymbol{F}_{B_0,i}$$

where $\{\boldsymbol{E}_{A_0,i}, \boldsymbol{E}_{A_1,i}, \boldsymbol{E}_{B_0,i}\}_{i\in I_N}$ and $\{\boldsymbol{F}_{A_0,i}, \boldsymbol{F}_{A_1,i}, \boldsymbol{F}_{B_0,i}\}_{i\in I_N}$ are known real constant matrices characterizing the upper bounds of those uncertainties and $\{\Delta_i(t)\}_{i\in I_N}$ are time-varying uncertainties satisfying

$$\Delta_i^{\mathrm{T}}(t)\Delta_i(t) \le \boldsymbol{I}, \text{ for } \forall i \in I_N \quad (3)$$

According to Eq.(1), a nominal linear switched system with time delay can be written as

$$\dot{\boldsymbol{x}} = \boldsymbol{A}_{0,q(t)}\boldsymbol{x} + \boldsymbol{A}_{1,q(t)}\boldsymbol{x}_d + \boldsymbol{B}_{0,q(t)}\boldsymbol{u} + \boldsymbol{B}_{1,q(t)}\boldsymbol{w} \quad (3a)$$

$$\boldsymbol{z} = \boldsymbol{C}_{0,q(t)}\boldsymbol{x} + \boldsymbol{C}_{1,q(t)}\boldsymbol{x}_d + \boldsymbol{D}_{0,q(t)}\boldsymbol{u} \quad (3b)$$

Obviously, system Eq.(1) and Eq.(3) both consist of N dynamical linear subsystems with time delay and a switching function that determines the switching action among them. A system with such kind of architecture can be used to characterize many nonlinear systems with time delay (Hassibi, 2000).

In this paper, it is assumed that the switching function $q(t)$ is an available or measurable piecewise constant function and has finite discontinuous instants in any limited interval. Now, consider a switched controller with state delay feedback as follows

$$\boldsymbol{u} = \boldsymbol{K}_{0,q(t)}\boldsymbol{x} + \boldsymbol{K}_{1,q(t)}\boldsymbol{x}_d \quad (5)$$

where $\boldsymbol{K}_{0,q(t)}$ and $\boldsymbol{K}_{1,q(t)}$ are feedback gain matrices to be determined. Since the controller Eq.(4) utilizes not only real-time system state but also delayed system state as the feedback information, it may achieve better control performance than the conventional memoryless state-feedback controller (Li *et al.*, 2001). Furthermore, the controller Eq.(4) and the system Eq.(1) or Eq.(3) have the same switching function, i.e., the control law will be changed as soon as the dynamical characteristic of the plant is remarkably changed. Then, it may achieve certain control objectives which cannot be accomplished by a conventional memoryless state-feedback controller.

Substituting Eq.(4) into Eq.(1) and Eq.(3), respectively, one obtain the corresponding closed-loop systems

$$\dot{\boldsymbol{x}} = \overline{\boldsymbol{A}}_{0,q(t)}\boldsymbol{x} + \overline{\boldsymbol{A}}_{1,q(t)}\boldsymbol{x}_d + \boldsymbol{B}_{1,q(t)}\boldsymbol{w} \tag{5a}$$

$$\boldsymbol{z} = \overline{\boldsymbol{C}}_{0,q(t)}\boldsymbol{x} + \overline{\boldsymbol{C}}_{1,q(t)}\boldsymbol{x}_d \tag{5b}$$

and

$$\dot{\boldsymbol{x}} = \widehat{\boldsymbol{A}}_{0,q(t)}\boldsymbol{x} + \widehat{\boldsymbol{A}}_{1,q(t)}\boldsymbol{x}_d + \boldsymbol{B}_{1,q(t)}\boldsymbol{w} \tag{6a}$$

$$\boldsymbol{z} = \overline{\boldsymbol{C}}_{0,q(t)}\boldsymbol{x} + \overline{\boldsymbol{C}}_{1,q(t)}\boldsymbol{x}_d \tag{6b}$$

where

$$\overline{\boldsymbol{A}}_{0,q(t)} = \tilde{\boldsymbol{A}}_{0,q(t)} + \tilde{\boldsymbol{B}}_{0,q(t)}\boldsymbol{K}_{0,q(t)} = \widehat{\boldsymbol{A}}_{0,q(t)} + \Delta\widehat{\boldsymbol{A}}_{0,q(t)},$$

$$\overline{\boldsymbol{A}}_{1,q(t)} = \tilde{\boldsymbol{A}}_{1,q(t)} + \tilde{\boldsymbol{B}}_{0,q(t)}\boldsymbol{K}_{1,q(t)} = \widehat{\boldsymbol{A}}_{1,q(t)} + \Delta\widehat{\boldsymbol{A}}_{1,q(t)}$$

$$\tilde{\boldsymbol{A}}_{0,q(t)} = \boldsymbol{A}_{0,q(t)} + \Delta\boldsymbol{A}_{0,q(t)},\ \tilde{\boldsymbol{A}}_{1,q(t)} = \boldsymbol{A}_{1,q(t)} + \Delta\boldsymbol{A}_{1,q(t)},$$

$$\tilde{\boldsymbol{B}}_{0,q(t)} = \boldsymbol{B}_{0,q(t)} + \Delta\boldsymbol{B}_{0,q(t)},\ \widehat{\boldsymbol{A}}_{0,q(t)} = \boldsymbol{A}_{0,q(t)} + \boldsymbol{B}_{0,q(t)}\boldsymbol{K}_{0,q(t)}$$

$$\widehat{\boldsymbol{A}}_{1,q(t)} = \boldsymbol{A}_{1,q(t)} + \boldsymbol{B}_{0,q(t)}\boldsymbol{K}_{1,q(t)},$$

$$\Delta\widehat{\boldsymbol{A}}_{0,q(t)} = \Delta\boldsymbol{A}_{0,q(t)} + \Delta\boldsymbol{B}_{0,q(t)}\boldsymbol{K}_{0,q(t)},$$

$$\Delta\widehat{\boldsymbol{A}}_{1,q(t)} = \Delta\boldsymbol{A}_{1,q(t)} + \Delta\boldsymbol{B}_{0,q(t)}\boldsymbol{K}_{1,q(t)}$$

$$\overline{\boldsymbol{C}}_{0,q(t)} = \boldsymbol{C}_{0,q(t)} + \boldsymbol{D}_{0,q(t)}\boldsymbol{K}_{0,q(t)},$$

$$\overline{\boldsymbol{C}}_{1,q(t)} = \boldsymbol{C}_{1,q(t)} + \boldsymbol{D}_{0,q(t)}\boldsymbol{K}_{1,q(t)}$$

Now, based on the closed-loop model Eq.(5), the definition of the robust stabilizable of the system Eq.(1) is proposed as follows.

Definition Given a scalar $\gamma(\gamma>0)$, the system Eq.(1) is said to be robust stabilizable with an H_∞-norm bound γ if there exists a control law Eq.(4) such that the closed-loop system Eq.(5) with $\boldsymbol{w}(t)=\boldsymbol{0}$ is globally asymptotically stable for all admissible uncertainties Eq.(2) and, under zero initial conditions, the following H_∞ performance is satisfied

$$\int_0^t (\boldsymbol{z}^{\mathrm{T}}(\tau)\boldsymbol{z}(\tau) - \gamma^2\boldsymbol{w}^{\mathrm{T}}(\tau)\boldsymbol{w}(\tau))\mathrm{d}\tau \le 0$$
$$\text{for } \forall t>0,\ \forall \boldsymbol{w}\in \boldsymbol{L}_2[0,\infty) \tag{7}$$

Moreover, the controller Eq.(4) is called a robust H_∞ switched controller with state delay feedback of the system Eq.(1).

In this paper, our aim is to find a delay- dependent condition from which a robust H_∞ switched controller with state delay feedback of system Eq.(1) can be derived.

In obtaining the main results in the next section, the following matrix inequalities play an important role.

Lemma 1 Assume that $\boldsymbol{x}_i, \boldsymbol{y}_i \in \mathbb{R}^n, i=1,2,\cdots,r$ and $\Delta\in\mathbb{R}^{n\times n}$ satisfying $\Delta\Delta^{\mathrm{T}} \le \boldsymbol{I}$. Then, for any scalars ε_i, i=1, 2, …, $r(r+1)/2$, the following inequality holds

$$\sum_{i=1}^{r}(\boldsymbol{x}_i^{\mathrm{T}}\Delta\boldsymbol{y}_i + \boldsymbol{y}_i^{\mathrm{T}}\Delta^{\mathrm{T}}\boldsymbol{x}_i) \le \boldsymbol{X}^{\mathrm{T}}\Delta\boldsymbol{X} + \boldsymbol{Y}^{\mathrm{T}}\Delta^{-1}\boldsymbol{Y} \tag{8}$$

where

$$\boldsymbol{\Theta}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{\frac{r(r+1)}{2}}) \triangleq \begin{bmatrix} \varepsilon_1\boldsymbol{I} & \varepsilon_2\boldsymbol{I} & \cdots & \varepsilon_r\boldsymbol{I} \\ \varepsilon_2\boldsymbol{I} & \varepsilon_{r+1}\boldsymbol{I} & \cdots & \varepsilon_{2r-1}\boldsymbol{I} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_r\boldsymbol{I} & \varepsilon_{2r-1}\boldsymbol{I} & \cdots & \varepsilon_{\frac{r(r+1)}{2}}\boldsymbol{I} \end{bmatrix} > 0$$

and

$$\boldsymbol{X}^{\mathrm{T}} \triangleq [\boldsymbol{x}_1^{\mathrm{T}}\quad \boldsymbol{x}_2^{\mathrm{T}}\quad \cdots\quad \boldsymbol{x}_r^{\mathrm{T}}],$$
$$\boldsymbol{Y}^{\mathrm{T}} \triangleq [\boldsymbol{y}_1^{\mathrm{T}}\quad \boldsymbol{y}_2^{\mathrm{T}}\quad \cdots\quad \boldsymbol{y}_r^{\mathrm{T}}].$$

The proof is given in Appendix.

Lemma 2 (Moon *et al.*, 2001) Assume $\boldsymbol{a}(\cdot)\in\mathbb{R}^{n_a}$ and $\boldsymbol{b}(\cdot)\in\mathbb{R}^{n_b}$ are real vector functions and $\boldsymbol{N}(\cdot)\in\mathbb{R}^{n_a\times n_b}$ is defined on the interval $\boldsymbol{\Omega}$. Then, for any matrices $\boldsymbol{X}\in\mathbb{R}^{n_a\times n_a}$, $\boldsymbol{Y}\in\mathbb{R}^{n_a\times n_b}$ and $\boldsymbol{Z}\in\mathbb{R}^{n_b\times n_b}$, the following inequality holds

$$-2\int_\Omega \boldsymbol{a}^{\mathrm{T}}(\upsilon)\boldsymbol{N}(\upsilon)\boldsymbol{b}(\upsilon)\mathrm{d}\upsilon$$
$$\le \int_\Omega \begin{bmatrix}\boldsymbol{a}(\upsilon)\\ \boldsymbol{b}(\upsilon)\end{bmatrix}^{\mathrm{T}} \begin{bmatrix}\boldsymbol{X} & \boldsymbol{Y}-\boldsymbol{N}(\upsilon)\\ \boldsymbol{Y}^{\mathrm{T}}-\boldsymbol{N}^{\mathrm{T}}(\upsilon) & \boldsymbol{Z}\end{bmatrix}\begin{bmatrix}\boldsymbol{a}(\upsilon)\\ \boldsymbol{b}(\upsilon)\end{bmatrix}\mathrm{d}\upsilon \tag{9}$$

where

$$\begin{bmatrix}\boldsymbol{X} & \boldsymbol{Y}\\ \boldsymbol{Y}^{\mathrm{T}} & \boldsymbol{Z}\end{bmatrix} \ge 0.$$

MAIN RESULTS

In this section, the control problem of the nominal system Eq.(3) is considered first and a delay-dependent sufficient condition for the existence of a H_∞ switched controller with state delay feedback is presented as follows.

Theorem 1 Given scalars $\bar{d}(\bar{d}>0)$ and $\gamma(\gamma>0)$, the system Eq.(3) is stabilizable with an H_∞-norm bound γ for any time delay constant d satisfying $0\le d\le \bar{d}$ if there exist matrices $\boldsymbol{Q}>0$, $\boldsymbol{W}>0$, $\boldsymbol{U}>0$, $\{\boldsymbol{M}_i>0\}_{i\in I_N}$ and $\{\boldsymbol{L}_{0,i},\boldsymbol{L}_{1,i},\boldsymbol{N}_i\}_{i\in I_N}$ such that the following matrix inequalities are satisfied

$$\begin{bmatrix} \boldsymbol{\Omega}_{11}(i) & \boldsymbol{\Omega}_{12}(i) & \boldsymbol{B}_{1,i} & \boldsymbol{\Omega}_{14}(i) & \boldsymbol{\Omega}_{15}(i) \\ * & -\boldsymbol{W} & \boldsymbol{0} & \boldsymbol{\Omega}_{24}(i) & \boldsymbol{\Omega}_{25}(i) \\ * & * & -\gamma\boldsymbol{I} & \boldsymbol{0} & \bar{d}\boldsymbol{B}_{1,i}^{\mathrm{T}} \\ * & * & * & -\gamma\boldsymbol{I} & \boldsymbol{0} \\ * & * & * & * & -\bar{d}\boldsymbol{U} \end{bmatrix}<0 \quad (10a)$$

$$\begin{bmatrix} \boldsymbol{M}_i & \boldsymbol{N}_i \\ \boldsymbol{N}_i^{\mathrm{T}} & \boldsymbol{Q}\boldsymbol{U}^{-1}\boldsymbol{Q} \end{bmatrix}\ge 0 \quad (12b)$$

where

$$\boldsymbol{\Omega}_{11}(i)=\boldsymbol{A}_{0,i}\boldsymbol{Q}+\boldsymbol{Q}\boldsymbol{A}_{0,i}^{\mathrm{T}}+\boldsymbol{B}_{0,i}\boldsymbol{L}_{0,i}+\boldsymbol{L}_{0,i}^{\mathrm{T}}\boldsymbol{B}_{0,i}^{\mathrm{T}} +\bar{d}\boldsymbol{M}_i+\boldsymbol{N}_i+\boldsymbol{N}_i^{\mathrm{T}}+\boldsymbol{W}$$

$\boldsymbol{\Omega}_{12}(i)=\boldsymbol{A}_{1,i}\boldsymbol{Q}+\boldsymbol{B}_{0,i}\boldsymbol{L}_{1,i}-\boldsymbol{N}_i$, $\boldsymbol{\Omega}_{14}(i)=\boldsymbol{Q}\boldsymbol{C}_{0,i}^{\mathrm{T}}+\boldsymbol{L}_{0,i}^{\mathrm{T}}\boldsymbol{D}_{0,\mathrm{i}}^{\mathrm{T}}$,

$\boldsymbol{\Omega}_{15}(i)=\bar{d}(\boldsymbol{Q}\boldsymbol{A}_{0,i}^{\mathrm{T}}+\boldsymbol{L}_{0,i}^{\mathrm{T}}\boldsymbol{B}_{0,i}^{\mathrm{T}})$

$\boldsymbol{\Omega}_{24}(i)=\boldsymbol{Q}\boldsymbol{C}_{1,i}^{\mathrm{T}}+\boldsymbol{L}_{1,i}^{\mathrm{T}}\boldsymbol{D}_{0,i}^{\mathrm{T}}$, $\boldsymbol{\Omega}_{25}(i)=\bar{d}(\boldsymbol{Q}\boldsymbol{A}_{1,i}^{\mathrm{T}}+\boldsymbol{L}_{1,i}^{\mathrm{T}}\boldsymbol{B}_{0,i}^{\mathrm{T}})$.

Moreover, the suitable feedback gain matrices of controller Eq.(4) can be constructed by $\boldsymbol{K}_{0,i}=\boldsymbol{L}_{0,i}\boldsymbol{Q}^{-1}$ and $\boldsymbol{K}_{1,i}=\boldsymbol{L}_{1,i}\boldsymbol{Q}^{-1}$.

Proof Since the switching function $q(t)$ is a piecewise constant function and has finite discontinuous instants in any limited interval, $\boldsymbol{x}(t)$ must be a strongly continuous function of t and piecewise differentiable. Based on real function theory, the following equation holds

$$\boldsymbol{x}_d=\boldsymbol{x}-\int_{t-d}^{t}\dot{\boldsymbol{x}}(\tau)\mathrm{d}\tau$$

Then, the closed-loop system Eq.(6a) can be rewritten as

$$\dot{\boldsymbol{x}}=(\widehat{\boldsymbol{A}}_{0,q(t)}+\widehat{\boldsymbol{A}}_{1,q(t)})\boldsymbol{x}+\boldsymbol{B}_{1,q(t)}\boldsymbol{w}-\widehat{\boldsymbol{A}}_{1,q(t)}\int_{t-d}^{t}\dot{\boldsymbol{x}}(\tau)\mathrm{d}\tau.$$

Choose the Lyapunov functional candidate as

$$V(\boldsymbol{x}(t))=V_1(t)+V_2(t)+V_3(t)$$

where

$$V_1(t)=\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$$

$$V_2(t)=\int_{-d}^{0}\int_{t+\tau}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(\upsilon)\boldsymbol{R}\dot{\boldsymbol{x}}(\upsilon)\mathrm{d}\upsilon\mathrm{d}\tau$$

$$V_3(t)=\int_{t-d}^{t}\boldsymbol{x}^{\mathrm{T}}(\upsilon)\boldsymbol{T}\boldsymbol{x}(\upsilon)\mathrm{d}\upsilon$$

and $\boldsymbol{P}$, $\boldsymbol{R}$, $\boldsymbol{T}$ are positive definite matrices. When the value of $q(t)$ is fixed, the derivative of $V_1(t)$ is

$$\dot{V}_1(t)\big|_{q(t)=i}=\boldsymbol{x}^{\mathrm{T}}((\widehat{\boldsymbol{A}}_{0,i}+\widehat{\boldsymbol{A}}_{1,i})^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}(\widehat{\boldsymbol{A}}_{0,i}+\widehat{\boldsymbol{A}}_{1,i}))\boldsymbol{x} +2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}_{1,i}\boldsymbol{w}-2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\widehat{\boldsymbol{A}}_{1,i}\int_{t-d}^{t}\dot{\boldsymbol{x}}(\tau)\mathrm{d}\tau$$

By defining $\boldsymbol{a}(\cdot)$, $\boldsymbol{b}(\cdot)$, and $\boldsymbol{N}(\cdot)$ in Lemma 2 as $\boldsymbol{a}(\upsilon)\triangleq\boldsymbol{x}(t)$, $\boldsymbol{b}(\upsilon)\triangleq\dot{\boldsymbol{x}}(t)$ and $\boldsymbol{N}(\upsilon)\triangleq\boldsymbol{P}\widehat{\boldsymbol{A}}_{1,i}$ for all $\upsilon\in[t-d,t]$ and applying matrix inequality Eq.(9), it is obtained that

$$\dot{V}_1(x)\big|_{q(t)=i} \le\boldsymbol{x}^{\mathrm{T}}(\widehat{\boldsymbol{A}}_{0,i}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}\widehat{\boldsymbol{A}}_{0,i}+\bar{d}\widehat{\boldsymbol{M}}_i+\widehat{\boldsymbol{N}}_i+\widehat{\boldsymbol{N}}_i^{\mathrm{T}})\boldsymbol{x}+2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}_{1,i}\boldsymbol{w} +2\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{P}\widehat{\boldsymbol{A}}_{1,i}-\widehat{\boldsymbol{N}}_i)\boldsymbol{x}_d+\int_{t-d}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(\tau)\boldsymbol{R}\dot{\boldsymbol{x}}(\tau)\mathrm{d}\tau$$

where $\widehat{\boldsymbol{M}}_i>0$ and $\widehat{\boldsymbol{N}}_i$ are matrices with appropriate dimensions and satisfy the following matrix inequalities for all $i\in I_N$

$$\begin{bmatrix} \widehat{\boldsymbol{M}}_i & \widehat{\boldsymbol{N}}_i \\ \widehat{\boldsymbol{N}}_i^{\mathrm{T}} & \boldsymbol{R} \end{bmatrix}\ge 0. \quad (11a)$$

Moreover,

$$\dot{V}_2(t)\big|_{q(t)=i} \le\bar{d}(\widehat{\boldsymbol{A}}_{0,i}\boldsymbol{x}+\widehat{\boldsymbol{A}}_{1,i}\boldsymbol{x}_d+\boldsymbol{B}_{1,i}\boldsymbol{w})^{\mathrm{T}}\boldsymbol{R}(\widehat{\boldsymbol{A}}_{0,i}\boldsymbol{x}+\widehat{\boldsymbol{A}}_{1,i}\boldsymbol{x}_d+\boldsymbol{B}_{1,i}\boldsymbol{w}) -\int_{t-d}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(\tau)\boldsymbol{R}\dot{\boldsymbol{x}}(\tau)\mathrm{d}\tau$$

$$\dot{V}_3(t)\big|_{q(t)=i}=\boldsymbol{x}^{\mathrm{T}}\boldsymbol{T}\boldsymbol{x}-\boldsymbol{x}_d^{\mathrm{T}}\boldsymbol{T}\boldsymbol{x}_d$$

Thus, it results

$$\dot{V}(\boldsymbol{x}(t))\big|_{q(t)=i}+\gamma^{-1}\boldsymbol{z}^{\mathrm{T}}\boldsymbol{z}-\gamma\boldsymbol{w}^{\mathrm{T}}\boldsymbol{w} =\dot{V}_1(t)\big|_{q(t)=i}+\dot{V}_2(t)\big|_{q(t)=i}+\dot{V}_3(t)\big|_{q(t)=i}+\gamma^{-1}\boldsymbol{z}^{\mathrm{T}}\boldsymbol{z}-\gamma\boldsymbol{w}^{\mathrm{T}}\boldsymbol{w}$$

$$\le \begin{bmatrix} x \\ x_d \\ w \end{bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \hat{\Omega}(i) & -\hat{N}_i + P\hat{A}_{1,i} & PB_{1,i} \\ * & -T & 0 \\ * & * & -\gamma I \end{bmatrix} + \bar{d} \begin{bmatrix} \hat{A}_{0,i}^{\mathrm{T}} \\ \hat{A}_{1,i}^{\mathrm{T}} \\ B_{1,i}^{\mathrm{T}} \end{bmatrix} R \begin{bmatrix} \hat{A}_{0,i} & \hat{A}_{1,i} & B_{1,i} \end{bmatrix} + \gamma^{-1} \begin{bmatrix} \bar{C}_{0,i}^{T} \\ \bar{C}_{1,i}^{T} \\ 0 \end{bmatrix} \begin{bmatrix} \bar{C}_{0,i} & \bar{C}_{1,i} & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ x_d \\ w \end{bmatrix}$$

where , $\hat{\Omega}(i) = \hat{A}_{0,i}^{\mathrm{T}} P + P\hat{A}_{0,i} + \bar{d}\hat{M}_i + \hat{N}_i + \hat{N}_i^{\mathrm{T}} + T$.

By using Schur-Complement (Boyd *et al.*, 1994), it is concluded that a sufficient condition for $\dot{V}(x(t))|_{q(t)=i} + \gamma^{-1} z^{\mathrm{T}} z - \gamma w^{\mathrm{T}} w < 0$ for all $i \in I_N$ is the following matrix inequalities and Eq.(11a) are satisfied for all $i \in I_N$

$$\begin{bmatrix} \hat{\Omega}(i) & -\hat{N}_i + P\hat{A}_{1,i} & PB_{1,i} & \bar{C}_{0,i}^{\mathrm{T}} & \bar{d}\hat{A}_{0,i}^{\mathrm{T}} \\ * & -T & 0 & \bar{C}_{1,i}^{\mathrm{T}} & \bar{d}\hat{A}_{1,i}^{\mathrm{T}} \\ * & * & -\gamma I & 0 & \bar{d}B_{1,i}^{\mathrm{T}} \\ * & * & * & -\gamma I & 0 \\ * & * & * & * & \bar{d}R^{-1} \end{bmatrix} < 0 \quad (11b)$$

Now, by pre- and post-multiplying diag$\{P^{-1}, P^{-1}\}$ and diag$\{P^{-1}, P^{-1}, I, I, I\}$ to Eq.(11a) and Eq.(11b), respectively, and setting $Q=P^{-1}$, $W=P^{-1}TP^{-1}$, $L_{0,i}=K_{0,i}P^{-1}$, $L_{1,i}=K_{1,i}P^{-1}$, $U=R^{-1}$, $M_i=P^{-1}\hat{M}_iP^{-1}$ and $N_i = P^{-1}\hat{N}_iP^{-1}$, it can be concluded that Eq.(11) $\Leftrightarrow$ Eq.(10), i.e., the satisfying of Eq.(10) implies that the following inequality holds almost everywhere on t

$$\dot{V}(x(t)) + \gamma^{-1} z^{\mathrm{T}} z - \gamma w^{\mathrm{T}} w < 0_{a.e.} \quad (12)$$

Considering the continuity of $x(t)$ one can conclude that $V(x(t))$ is a globally Lyapunov function of the closed-loop system Eq.(6) with $w(t)=0$. Under the zero initial conditions and by integrating the left-hand of Eq.(12) from 0 to t, the following inequality is obtained

$$\int_0^t (\gamma^{-1} z^{\mathrm{T}}(\tau) z(\tau) - \gamma w^{\mathrm{T}}(\tau) w(\tau)) \mathrm{d}\tau < -V(x(t)) \le 0$$

which implies that the closed-loop system Eq.(6) satisfies the given H_∞ performance Eq.(7).

Now, we extend Theorem 1 to designing a robust H_∞ switched controller with state delay feedback for the uncertain linear switched system with time delay described by Eq.(1).

Theorem 2 Given scalars $\bar{d}(\bar{d} > 0)$ and $\gamma(\gamma>0)$, the system Eq.(1) is robustly stabilizable with an H_∞-norm bound γ for any time delay constant d satisfying $0 \le d \le \bar{d}$ if there exist scalars $\varepsilon_{j,i}>0$, for $j=1,2,\ldots,16$, $i \in I_N$ and matrices $Q>0$, $W>0$, $U>0$, $\{M_i > 0\}_{i \in I_N}$ and $\{L_{0,i}, L_{1,i}, N_i\}_{i \in I_N}$ such that the following matrix inequalities are satisfied.

$$\begin{bmatrix} \Omega_{11}(i) & \Omega_{12}(i) & B_{1,i} & \Omega_{14}(i) & \Omega_{15}(i) & \hat{\Omega}_{16}(i) \\ * & -W & 0 & \Omega_{24}(i) & \Omega_{25}(i) & \hat{\Omega}_{26}(i) \\ * & * & -\gamma I & 0 & \bar{d}B_{1,i}^{\mathrm{T}} & 0 \\ * & * & * & -\gamma I & 0 & 0 \\ * & * & * & * & \Omega_{55}(i) & 0 \\ * & * & * & * & * & \hat{\Omega}_{66}(i) \end{bmatrix} < 0 \quad (13a)$$

$$\begin{bmatrix} M_i & N_i \\ N_i^{\mathrm{T}} & QU^{-1}Q \end{bmatrix} \ge 0 \text{ for } \forall i \in I_N \quad (13b)$$

where

$$\Omega_{11}(i) = A_{0,i}Q + QA_{0,i}^{\mathrm{T}} + B_{0,i}L_{0,i} + L_{0,i}^{\mathrm{T}}B_{0,i}^{\mathrm{T}} + \bar{d}M_i + N_i + N_i^{\mathrm{T}} + W + \varepsilon_{1,i}E_{A_0,i}E_{A_0,i}^{\mathrm{T}} + \varepsilon_{4,i}E_{A_1,i}E_{A_1,i}^{\mathrm{T}} + (\varepsilon_{7,i} + 2\varepsilon_{8,i} + \varepsilon_{11,i})E_{B_0,i}E_{B_0,i}^{\mathrm{T}},$$

$$\Omega_{12}(i) = A_{1,i}Q + B_{0,i}L_{1,i} - N_i, \quad \Omega_{14}(i) = QC_{0,i}^{\mathrm{T}} + L_{0,i}^{\mathrm{T}}D_{0,i}^{\mathrm{T}},$$

$$\Omega_{24}(i) = QC_{1,i}^{\mathrm{T}} + L_{1,i}^{\mathrm{T}}D_{0,i}^{\mathrm{T}}, \quad \Omega_{25}(i) = \bar{d}(QA_{1,i}^{\mathrm{T}} + L_{1,i}^{\mathrm{T}}B_{0,i}^{T}),$$

$$\Omega_{15}(i) = \bar{d}(QA_{0,i}^{\mathrm{T}} + L_{0,i}^{\mathrm{T}}B_{0,i}^{\mathrm{T}}) + \varepsilon_{2,i}E_{A_0,i}E_{A_0,i}^{\mathrm{T}} + \varepsilon_{5,i}E_{A_1,i}E_{A_1,i}^{\mathrm{T}} + (\varepsilon_{9,i} + \varepsilon_{10,i} + \varepsilon_{12,i} + \varepsilon_{13,i})E_{B_0,i}E_{B_0,i}^{\mathrm{T}},$$

$$\Omega_{55}(i) = -\bar{d}U + \varepsilon_{3,i}E_{A_0,i}E_{A_0,i}^{\mathrm{T}} + \varepsilon_{6,i}E_{A_1,i}E_{A_1,i}^{\mathrm{T}} + (\varepsilon_{14,i} + 2\varepsilon_{15,i} + \varepsilon_{16,i})E_{B_0,i}E_{B_0,i}^{\mathrm{T}},$$

$$\hat{\Omega}_{16}(i) = [\Omega_{16}(i) \quad 0 \quad \Omega_{18}(i)],$$

$$\Omega_{16}(i) = [QF_{A_0,i}^{\mathrm{T}} \quad \bar{d}QF_{A_0,i}^{\mathrm{T}}],$$

$$\Omega_{18}(i) = [L_{0,i}^{\mathrm{T}}F_{B_0,i}^{\mathrm{T}} \quad 0 \quad \bar{d}L_{0,i}^{\mathrm{T}}F_{B_0,i}^{\mathrm{T}} \quad 0],$$

$$\hat{\Omega}_{26}(i)=[\mathbf{0}\quad \Omega_{27}(i)\quad \Omega_{28}(i)],$$
$$\Omega_{27}(i)=[\boldsymbol{QF}_{A_1,i}^{\mathrm{T}}\quad \bar{d}\boldsymbol{QF}_{A_1,i}^{\mathrm{T}}],$$
$$\Omega_{28}(i)=[\mathbf{0}\quad \boldsymbol{L}_{1,i}^{\mathrm{T}}\boldsymbol{F}_{B_0,i}^{\mathrm{T}}\quad \mathbf{0}\quad \bar{d}\boldsymbol{L}_{1,i}^{\mathrm{T}}\boldsymbol{F}_{B_0,i}^{\mathrm{T}}],$$
$$\hat{\Omega}_{66}(i)=\mathrm{diag}(\Omega_{66}(i),\Omega_{77}(i),\Omega_{88}(i)),$$
$$\Omega_{66}(i)=-\Theta(\varepsilon_{1,i},\varepsilon_{2,i},\varepsilon_{3,i}),$$
$$\Omega_{77}(i)=-\Theta(\varepsilon_{4,i},\varepsilon_{5,i},\varepsilon_{6,i}),$$
$$\Omega_{88}(i)=-\Theta(\varepsilon_{7,i},\varepsilon_{8,i},\cdots\varepsilon_{16,i}),$$

Moreover, the suitable feedback gain matrices of controller Eq.(4) can be constructed by $\boldsymbol{K}_{0,i}=\boldsymbol{L}_{0,i}\boldsymbol{Q}^{-1}$ and $\boldsymbol{K}_{1,i}=\boldsymbol{L}_{1,i}\boldsymbol{Q}^{-1}$.

Proof According to Theorem 1 and by replacing $\boldsymbol{A}_{0,i}$, $\boldsymbol{A}_{1,i}$ and $\boldsymbol{B}_{0,i}$ in matrix inequalities Eq.(10a) with $\tilde{\boldsymbol{A}}_{0,i}$, $\tilde{\boldsymbol{A}}_{1,i}$ and $\tilde{\boldsymbol{B}}_{0,i}$, respectively, it is obtained that the sufficient conditions for the system Eq.(1) being robustly stabilizable with an H_∞-norm bound γ are as follows

$$\begin{bmatrix}\boldsymbol{x}_1\\ \boldsymbol{x}_2\\ \boldsymbol{x}_3\\ \boldsymbol{x}_4\\ \boldsymbol{x}_5\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\tilde{\Omega}_{11}(i) & \tilde{\Omega}_{12}(i) & \boldsymbol{B}_{1,i} & \Omega_{14}(i) & \tilde{\Omega}_{15}(i)\\ * & -\boldsymbol{W} & \mathbf{0} & \Omega_{24}(i) & \tilde{\Omega}_{25}(i)\\ * & * & -\gamma\boldsymbol{I} & \mathbf{0} & \bar{d}\boldsymbol{B}_{1,i}^{\mathrm{T}}\\ * & * & * & -\gamma\boldsymbol{I} & \mathbf{0}\\ * & * & * & * & -\bar{d}\boldsymbol{U}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_1\\ \boldsymbol{x}_2\\ \boldsymbol{x}_3\\ \boldsymbol{x}_4\\ \boldsymbol{x}_5\end{bmatrix}<0 \qquad (14a)$$

$$\begin{bmatrix}\boldsymbol{M}_i & \boldsymbol{N}_i\\ \boldsymbol{N}_i^{\mathrm{T}} & \boldsymbol{QU}^{-1}\boldsymbol{Q}\end{bmatrix}\ge 0,\ \text{for } \forall i\in I_N \qquad (14b)$$

where

$$\tilde{\Omega}_{11}(i)=\tilde{\boldsymbol{A}}_{0,i}\boldsymbol{Q}+\boldsymbol{Q}\tilde{\boldsymbol{A}}_{0,i}^{\mathrm{T}}+\tilde{\boldsymbol{B}}_{0,i}\boldsymbol{L}_{0,i}+\boldsymbol{L}_{0,i}^{\mathrm{T}}\tilde{\boldsymbol{B}}_{0,i}^{\mathrm{T}}+\bar{d}\boldsymbol{M}_i+\boldsymbol{N}_i+\boldsymbol{N}_i^{\mathrm{T}}+\boldsymbol{W},$$
$$\tilde{\Omega}_{12}(i)=\tilde{\boldsymbol{A}}_{1,i}\boldsymbol{Q}+\tilde{\boldsymbol{B}}_{0,i}\boldsymbol{L}_{1,i}-\boldsymbol{N}_i,$$
$$\tilde{\Omega}_{15}(i)=\bar{d}(\boldsymbol{Q}\tilde{\boldsymbol{A}}_{0,i}^{\mathrm{T}}+\boldsymbol{L}_{0,i}^{\mathrm{T}}\tilde{\boldsymbol{B}}_{0,i}^{\mathrm{T}}),$$
$$\tilde{\Omega}_{25}(i)=\bar{d}(\boldsymbol{Q}\tilde{\boldsymbol{A}}_{1,i}^{\mathrm{T}}+\boldsymbol{L}_{1,i}^{\mathrm{T}}\tilde{\boldsymbol{B}}_{0,i}^{\mathrm{T}}),$$

and $\boldsymbol{x}_i$, $i=1,2,\ldots,5$ are arbitrary real vectors with appropriate dimensions. The matrix inequality Eq.(14a) can be rewritten as

$$\begin{bmatrix}\boldsymbol{x}_1\\ \boldsymbol{x}_2\\ \boldsymbol{x}_3\\ \boldsymbol{x}_4\\ \boldsymbol{x}_5\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\Omega'_{11}(i) & \Omega_{12}(i) & \boldsymbol{B}_{1,i} & \Omega_{14}(i) & \Omega'_{15}(i)\\ * & -\boldsymbol{W} & \mathbf{0} & \Omega_{24}(i) & \Omega_{25}(i)\\ * & * & -\gamma\boldsymbol{I} & \mathbf{0} & \bar{d}\boldsymbol{B}_{1,i}^{\mathrm{T}}\\ * & * & * & -\gamma\boldsymbol{I} & \mathbf{0}\\ * & * & * & * & -\bar{d}\boldsymbol{U}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_1\\ \boldsymbol{x}_2\\ \boldsymbol{x}_3\\ \boldsymbol{x}_4\\ \boldsymbol{x}_5\end{bmatrix}+\Phi_1(i)+\Phi_2(i)+\Phi_3(i)<0$$

where

$$\Phi_1(i)=2\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{E}_{A_0,i}\Delta_i\boldsymbol{F}_{A_0,i}\boldsymbol{Qx}_1+2\boldsymbol{x}_5^{\mathrm{T}}\bar{d}\boldsymbol{E}_{A_0,i}\Delta_i\boldsymbol{F}_{A_0,i}\boldsymbol{Qx}_1,$$
$$\Phi_2(i)=2\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{E}_{A_1,i}\Delta_i\boldsymbol{F}_{A_1,i}\boldsymbol{Qx}_2+2\boldsymbol{x}_5^{\mathrm{T}}\bar{d}\boldsymbol{E}_{A_1,i}\Delta_i\boldsymbol{F}_{A_1,i}\boldsymbol{Qx}_2,$$
$$\Phi_3(i)=2\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{E}_{B_0,i}\Delta_i\boldsymbol{F}_{B_0,i}\boldsymbol{L}_{0,i}\boldsymbol{x}_1+2\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{E}_{B_0,i}\Delta_i\boldsymbol{F}_{B_0,i}\boldsymbol{L}_{1,i}\boldsymbol{x}_2+2\boldsymbol{x}_5^{\mathrm{T}}\bar{d}\boldsymbol{E}_{B_0,i}\Delta_i\boldsymbol{F}_{B_0,i}\boldsymbol{L}_{0,i}\boldsymbol{x}_1+2\boldsymbol{x}_5^{\mathrm{T}}\bar{d}\boldsymbol{E}_{B_0,i}\Delta_i\boldsymbol{F}_{B_0,i}\boldsymbol{L}_{1,i}\boldsymbol{x}_2,$$
$$\Omega'_{11}(i)=\boldsymbol{A}_{0,i}\boldsymbol{Q}+\boldsymbol{QA}_{0,i}^{\mathrm{T}}+\boldsymbol{B}_{0,i}\boldsymbol{L}_{0,i}+\boldsymbol{L}_{0,i}^{\mathrm{T}}\boldsymbol{B}_{0,i}^{\mathrm{T}}+\bar{d}\boldsymbol{M}_i+\boldsymbol{N}_i+\boldsymbol{N}_i^{\mathrm{T}}+\boldsymbol{W},$$
$$\Omega'_{15}(i)=\bar{d}(\boldsymbol{QA}_{0,i}^{\mathrm{T}}+\boldsymbol{L}_{0,i}^{\mathrm{T}}\boldsymbol{B}_{0,i}^{\mathrm{T}}),$$

Now, according to Schur-Complement (Boyd *et al*., 1994) and using Lemma 1, one can understand that Eq.(14a)⇐Eq.(13a), which completes the proof.

Considering the real-time state information cannot be available in most applications, such as networked control systems (Zhang *et al*., 2001) and remote control systems (Luo *et al*., 2003), we have to design a state delay feedback controller to guarantee the robust stability of the closed-loop system. By setting $\boldsymbol{L}_{0,i}=\mathbf{0}$, Theorem 2 can be extended to this case.

Theorem 3 Given scalars $\bar{d}\,(\bar{d}>0)$ and $\gamma(\gamma>0)$. If there exist scalars $\varepsilon_{j,i}>0$, $j=1,2,\ldots,9$ and matrices $\boldsymbol{Q}>0$, $\boldsymbol{W}>0$, $\boldsymbol{U}>0$, $\{\boldsymbol{M}_i>0\}_{i\in I_N}$ and $\{\boldsymbol{L}_{0,i},\boldsymbol{L}_{1,i},\boldsymbol{N}_i\}_{i\in I_N}$ such that the following matrix inequalities are satisfied for all $i\in I_N$

$$\begin{bmatrix}\Omega_{11}(i) & \Omega_{12}(i) & \boldsymbol{B}_{1,i} & \Omega_{14}(i) & \Omega_{15}(i) & \hat{\Omega}_{16}(i)\\ * & -\boldsymbol{W} & \mathbf{0} & \Omega_{24}(i) & \Omega_{25}(i) & \hat{\Omega}_{26}(i)\\ * & * & -\gamma\boldsymbol{I} & \mathbf{0} & \bar{d}\boldsymbol{B}_{1,i}^{\mathrm{T}} & \mathbf{0}\\ * & * & * & -\gamma\boldsymbol{I} & \mathbf{0} & \mathbf{0}\\ * & * & * & * & \Omega_{55}(i) & \mathbf{0}\\ * & * & * & * & * & \hat{\Omega}_{66}(i)\end{bmatrix}<0 \qquad (15a)$$

$$\begin{bmatrix} \boldsymbol{M}_i & \boldsymbol{N}_i \\ \boldsymbol{N}_i^{\mathrm{T}} & \boldsymbol{Q}\boldsymbol{U}^{-1}\boldsymbol{Q} \end{bmatrix} \geq 0 \tag{15b}$$

where

$$\boldsymbol{\Omega}_{11}(i) = \boldsymbol{A}_{0,i}\boldsymbol{Q} + \boldsymbol{Q}\boldsymbol{A}_{0,i}^{\mathrm{T}} + \bar{d}\boldsymbol{M}_i + \boldsymbol{N}_i + \boldsymbol{N}_i^{\mathrm{T}} + \boldsymbol{W}$$
$$+\varepsilon_{1,i}\boldsymbol{E}_{A_0,i}\boldsymbol{E}_{A_0,i}^{\mathrm{T}} + \varepsilon_{4,i}\boldsymbol{E}_{A_1,i}\boldsymbol{E}_{A_1,i}^{\mathrm{T}} + \varepsilon_{7,i}\boldsymbol{E}_{B_0,i}\boldsymbol{E}_{B_0,i}^{\mathrm{T}}$$

$$\boldsymbol{\Omega}_{12}(i) = -\boldsymbol{N}_i + \boldsymbol{A}_{1,i}\boldsymbol{Q} + \boldsymbol{B}_{0,i}\boldsymbol{L}_{1,i}, \quad \boldsymbol{\Omega}_{14}(i) = \boldsymbol{Q}\boldsymbol{C}_{0,i}^{\mathrm{T}},$$

$$\boldsymbol{\Omega}_{24}(i) = \boldsymbol{Q}\boldsymbol{C}_{1,i}^{\mathrm{T}} + \boldsymbol{L}_{1,i}^{\mathrm{T}}\boldsymbol{D}_{0,i}^{\mathrm{T}}, \quad \boldsymbol{\Omega}_{25}(i) = \bar{d}(\boldsymbol{Q}\boldsymbol{A}_{1,i}^{\mathrm{T}} + \boldsymbol{L}_{1,i}^{\mathrm{T}}\boldsymbol{B}_{0,i}^{\mathrm{T}}),$$

$$\boldsymbol{\Omega}_{15}(i) = \bar{d}\boldsymbol{Q}\boldsymbol{A}_{0,i}^{\mathrm{T}} + \varepsilon_{2,i}\boldsymbol{E}_{A_0,i}\boldsymbol{E}_{A_0,i}^{\mathrm{T}} + \varepsilon_{5,i}\boldsymbol{E}_{A_1,i}\boldsymbol{E}_{A_1,i}^{\mathrm{T}}$$
$$+\varepsilon_{8,i}\boldsymbol{E}_{B_0,i}\boldsymbol{E}_{B_0,i}^{\mathrm{T}},$$

$$\boldsymbol{\Omega}_{55}(i) = -\bar{d}\boldsymbol{U} + \varepsilon_{3,i}\boldsymbol{E}_{A_0,i}\boldsymbol{E}_{A_0,i}^{\mathrm{T}} + \varepsilon_{6,i}\boldsymbol{E}_{A_1,i}\boldsymbol{E}_{A_1,i}^{\mathrm{T}} + \varepsilon_{9,i}\boldsymbol{E}_{B_0,i}\boldsymbol{E}_{B_0,i}^{\mathrm{T}},$$

$$\hat{\boldsymbol{\Omega}}_{16}(i) = [\boldsymbol{\Omega}_{16}(i) \quad \boldsymbol{0} \quad \boldsymbol{0}], \quad \boldsymbol{\Omega}_{16}(i) = [\boldsymbol{Q}\boldsymbol{F}_{A_0,i}^{\mathrm{T}} \quad \bar{d}\boldsymbol{Q}\boldsymbol{F}_{A_0,i}^{\mathrm{T}}],$$

$$\hat{\boldsymbol{\Omega}}_{26}(i) = [\boldsymbol{0} \quad \boldsymbol{\Omega}_{27}(i) \quad \boldsymbol{\Omega}_{28}(i)],$$

$$\boldsymbol{\Omega}_{27}(i) = [\boldsymbol{Q}\boldsymbol{F}_{A_1,i}^{\mathrm{T}} \quad \bar{d}\boldsymbol{Q}\boldsymbol{F}_{A_1,i}^{\mathrm{T}}],$$

$$\boldsymbol{\Omega}_{28}(i) = [\boldsymbol{L}_{1,i}^{\mathrm{T}}\boldsymbol{F}_{B_0,i}^{\mathrm{T}} \quad \bar{d}\boldsymbol{L}_{1,i}^{\mathrm{T}}\boldsymbol{F}_{B_0,i}^{\mathrm{T}}],$$

$$\hat{\boldsymbol{\Omega}}_{66}(i) = \mathrm{diag}(\boldsymbol{\Omega}_{66}(i), \boldsymbol{\Omega}_{77}(i), \boldsymbol{\Omega}_{88}(i)),$$

$$\boldsymbol{\Omega}_{66}(i) = -\boldsymbol{\Theta}(\varepsilon_{1,i}, \varepsilon_{2,i}, \varepsilon_{3,i}), \quad \boldsymbol{\Omega}_{77}(i) = -\boldsymbol{\Theta}(\varepsilon_{4,i}, \varepsilon_{5,i}, \varepsilon_{6,i}),$$

$$\boldsymbol{\Omega}_{88}(i) = -\boldsymbol{\Theta}(\varepsilon_{7,i}, \varepsilon_{8,i}, \varepsilon_{9,i}),$$

then the system Eq.(1) is robustly stabilizable with an H_∞-norm bound γ for any time delay constant d satisfying $0 \leq d \leq \bar{d}$ and a suitable state delay feedback controller is $\boldsymbol{u}(t)=\boldsymbol{L}_{1,i}\boldsymbol{Q}^{-1}\boldsymbol{x}_d$.

Remark 1 All the stabilization conditions proposed in this section are delay-dependent since the matrix inequalities Eq.(10a), Eq.(13a) and Eq.(15a) are dependent upon the upper bound of time delay $\bar{d}$. In general, the delay-dependent stabilization is less conservative than delay-independent stabilization except for some special case where the system is stabilizable in nature independent of the size of time delay (Lennartson *et al*., 1994; Luo *et al*., 2003). Moreover, when $q(t)\equiv 1$, i.e., system Eq.(1) is an uncertain linear system with time delay, the proposed conclusions above are still applicable.

Remark 2 Note that all of stabilization conditions proposed in this section are not linear matrix inequalities (LMIs) because of the nonlinear term $\boldsymbol{Q}\boldsymbol{U}^{-1}\boldsymbol{Q}$ in Eq.(10b), Eq.(13b) and Eq.(15b). Then, the available LMI tools cannot be used directly to obtain a feasible solution of Eq.(10), Eq.(13) or Eq.(15). An easy, but some conservative way to deal with this problem is simply to set $\boldsymbol{Q}=\boldsymbol{U}$ in Eq.(10b), Eq.(13b) and Eq.(15b), which converts the nonlinear matrix inequalities Eq.(10b), Eq.(13b) and Eq.(15b) to a set of LMIs. If one can afford more computational costs, however, better results can be obtained by using an LMI-based iterative algorithm developed by Moon *et al*.(2001). Since the related transform of the nonlinear terms $\boldsymbol{Q}\boldsymbol{U}^{-1}\boldsymbol{Q}$ and the detail steps of the iterative algorithm used in the numerical example in next section are very similar to the case in Moon *et al*.(2001), we will not discuss them in this paper.

NUMERICAL EXAMPLES

In this section, to illustrate the utilization of the results presented in the previous section, we consider the following dynamical model of the inverted-pendulum system where the effect of the friction on the hinge is approximately described as a time delay term depending on the angular velocity of the pendulum, the mass of the pendulum and the cart and the nature of the friction represented by a friction coefficient.

$$\begin{cases} \dot{x}_1 = x_2 + 0.25(m+M)x_2(t-d) + w(t) \\ \dot{x}_2 = \dfrac{g\sin(x_1) - 0.5aml\,x_2^2\sin(2x_1)}{(\frac{4}{3}l - aml\cos^2(x_1))} \\ \qquad + \dfrac{-a\cos(x_1)u + 1.2aml\,x_2(t-d)}{(\frac{4}{3}l - aml\cos^2(x_1))} + w(t) \end{cases} \tag{16}$$

where x_1 denotes the angle of the pendulum from the vertical, x_2 is the angular velocity, u is the force applied to the cart, and w is the external disturbance. Parameter g stands for the gravity constant, m and M are the mass of the pendulum and the cart respectively, $a=1/(m+M)$, $2l$ is the length of the pendulum. The values of these parameters can be found in Feng (2002).

Since the system Eq.(16) is a nonlinear plant with time delay, it should be transformed into an uncertain linear switched model with time delay before our results can be used. This will be achieved by using the locally linearization methods and taking the differences between the linearized local models and the original nonlinear model as the system uncertainties. In this paper, the range of the pendulum swing is supposed to be $|x_1|\le\pi/3+\pi/60$ and is divided into 11 sub-ranges S_i, i=1,2,…, 11 as follows

$$S_i \triangleq \{\theta \big| (\pi(i-1)/30-\pi/60)<|\theta|\le(\pi(i-1)/30+\pi/60)\}.$$

By linearizing the plant Eq.(16) around the central points of sub-ranges, i.e., $(x_1,x_2)=(\theta_i,0)$, where $\theta_i=\pi(i\text{-}1)/30$, i =1,2,…,11, the linear switched model Eq.(1) is obtained, where

$$\boldsymbol{A}_{0,i}=\begin{bmatrix}0 & 1\\ \Delta_{A_0}(\theta_i) & 0\end{bmatrix},\ \boldsymbol{A}_{1,i}=\begin{bmatrix}0 & 2.5(m+M)\\ 0 & \Delta_{A_1}(\theta_i)\end{bmatrix},$$

$$\boldsymbol{B}_{0,i}=\begin{bmatrix}0\\ \Delta_{B_0}(\theta_i)\end{bmatrix},\ \boldsymbol{B}_{1,i}=\begin{bmatrix}1\\ 1\end{bmatrix},$$

$$\boldsymbol{C}_{0,i}=[1\quad 0],\quad i=1,2,\cdots,11$$

$$\Delta_{A_0}(\theta)\triangleq\begin{cases}\dfrac{g}{4l/3-aml}, & \theta=0\\[2mm] \dfrac{g\sin(\theta)}{\theta\left(4l/3-aml\cos^2(\theta)\right)}, & \theta\neq 0\end{cases}$$

$$\Delta_{A_1}(\theta)\triangleq\frac{1.2aml}{4l/3-aml\cos^2(\theta)},$$

$$\Delta_{B_0}(\theta)\triangleq-\frac{a\cos(\theta)}{4l/3-aml\cos^2(\theta)}.$$

Obviously, the upper bounds for those uncertainties can be chosen as follows

$$\boldsymbol{E}_{A_0,i}=\begin{bmatrix}0 & 0\\ \max\left\{\left|\Delta_{A_0}(\theta_i)-\Delta_{A_0}(\theta_i\pm\pi/60)\right|\right\} & 0\end{bmatrix},$$

$$\boldsymbol{E}_{A_1,i}=\begin{bmatrix}0 & 0\\ 0 & \max\left\{\left|\Delta_{A_1}(\theta_i)-\Delta_{A_1}(\theta_i\pm\pi/60)\right|\right\}\end{bmatrix},$$

$$\boldsymbol{E}_{B_0,i}=\begin{bmatrix}0 & 0\\ 0 & \max\left\{\left|\Delta_{B_0}(\theta_i)-\Delta_{B_0}(\theta_i\pm\pi/60)\right|\right\}\end{bmatrix},$$

$$\boldsymbol{F}_{A_0,i}=\boldsymbol{F}_{A_1,i}=\boldsymbol{I},\ \boldsymbol{F}_{B_0,i}=\begin{bmatrix}0\\ 1\end{bmatrix},\ \text{for } i=1,2,\cdots,11$$

Let the admissible H_∞ performance bound γ=0.2 and the upper bound of time delay is equal to the constant time delay of real system, i.e. $\bar{d}=d$ =0.1 s. According to the Theorem 2 and using LMI-based iterative algorithm developed by Moon *et al*.(2001), after 22 rounds of iteration, we can obtain the robust H_∞ switched controller Eq.(4) with state delay feedback as follows

$$\boldsymbol{K}_{0,1}=[1267.1\quad 436.5],\quad \boldsymbol{K}_{1,1}=[-0.178\quad 128.285],$$
$$\boldsymbol{K}_{0,2}=[1285.1\quad 443.4],\quad \boldsymbol{K}_{1,2}=[-0.125\quad 129.277],$$
$$\boldsymbol{K}_{0,3}=[1316.2\quad 454.7],\quad \boldsymbol{K}_{1,3}=[-0.113\quad 132.177],$$
$$\boldsymbol{K}_{0,4}=[1364.6\quad 472.1],\quad \boldsymbol{K}_{1,4}=[-0.108\quad 137.163],$$
$$\boldsymbol{K}_{0,5}=[1435.1\quad 497.4],\quad \boldsymbol{K}_{1,5}=[-0.101\quad 144.500],$$
$$\boldsymbol{K}_{0,6}=[1532.2\quad 532.2],\quad \boldsymbol{K}_{1,6}=[-0.087\quad 154.609],$$
$$\boldsymbol{K}_{0,7}=[1662.2\quad 578.9],\quad \boldsymbol{K}_{1,7}=[-0.055\quad 168.150],$$
$$\boldsymbol{K}_{0,8}=[1835.0\quad 640.8],\quad \boldsymbol{K}_{1,8}=[0.018\quad 186.160],$$
$$\boldsymbol{K}_{0,9}=[2066.8\quad 723.9],\quad \boldsymbol{K}_{1,9}=[0.192\quad 210.327],$$
$$\boldsymbol{K}_{0,10}=[2385.1\quad 837.9],\quad \boldsymbol{K}_{1,10}=[0.581\quad 243.521],$$
$$\boldsymbol{K}_{0,11}=[2839.9\quad 1000.8],\quad \boldsymbol{K}_{1,11}=[1.090\quad 290.854].$$

Obviously, this robust H_∞ switched controller with state delay feedback has a piecewise constant feedback gain depending on the sub-range in which x_1 is. In addition, the entry values of those feedback gain matrices indicate that the larger deviation between x_1 and original results in a stronger control force. The solid line in Fig.1 shows the angle response of the closed-loop system without external disturbance for the initial state $x_1=\pi/3$, $x_2=0$. If we cancel the state delay feedback loop, the closed-loop system is still stable, but the performance deteriorates as being shown by the dash line in Fig.1. The dash-dot line in Fig.1 shows the angle response of the closed-loop systems with the external disturbance $\boldsymbol{w}(t)=\sin(2\pi t)$, which is also showed by the dotted line in Fig.1. This example clearly demonstrated that the controller obtained from Theorem 2 can not only stabilize the uncertain linear switched systems with time delay but also achieve the given H_∞ performance.

Now, consider the case where the time delay is introduced by the measurement components, the transmission of the system state information, i.e., the real-time state of the plant is not available for the controller. Let the admissible H_∞ performance bound γ=0.1. By applying Theorem 3, a suitable state delay feedback controller is obtained when $\bar{d}=0.014\ \text{s}$. Three simulation results of the closed-loop system under the different initial conditions, external disturbances and time delay constants are shown in Fig.2, which indicates that the state delay feedback controller stabilizes the uncertain linear switched systems with time delay when $d \le \bar{d}$, but $\bar{d}$ is only the suboptimal upper bound of all admissible time delay of the closed-loop system since only a sufficient condition is presented in Theorem 3.

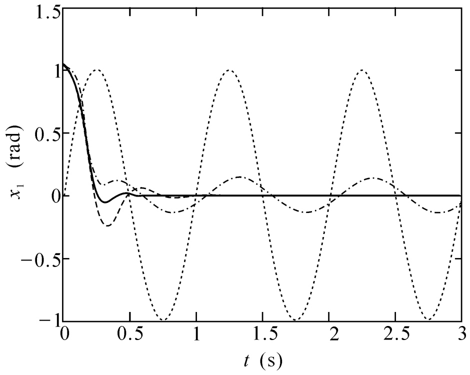

Fig.1 Angle responses of the inverted-pendulum system with different controllers

——: Robust H_∞ switched controller with state delay feedback for the system without external disturbance;
----: Robust H_∞ switched controller without state delay feedback for the system without external disturbance;
-·-·-: Robust H_∞ switched controller with state delay feedback for the system with external disturbance $\boldsymbol{w}(t)=\sin(2\pi t)$;
….: External disturbance $\boldsymbol{w}(t)=\sin(2\pi t)$

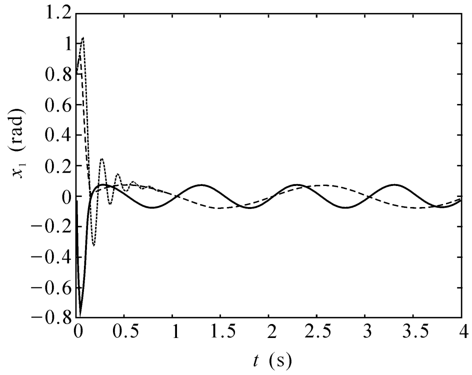

Fig.2 Angle responses of the inverted-pendulum system with state delay feedback controller for the different conditions

——: $x_1=0$, $x_2=-10$, $\boldsymbol{w}(t)=\sin(2\pi t)$, d=0.01;
----: $x_1=\pi/4$, $x_2=2$, $\boldsymbol{w}(t)=\sin(\pi t)$, d=0.014;
….: $x_1=\pi/4$, $x_2=2$, $\boldsymbol{w}(t)=\sin(\pi t)$, d=0.03

CONCLUSIONS

This paper addresses a new delay-dependent robust H_∞ controller design method for the uncertain linear switched systems with time delay. The new robust controller, which can be obtained by an iterative algorithm, utilizes the delayed state as additional feedback control information and has switched gain matrices to guarantee better robust control performance. The result is also extended to the case where only delayed state information is available for the controller. The example of balancing an inverted pendulum on a cart demonstrates the effectiveness and applicability of the proposed design methods.

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APPENDIX: THE PROOF OF LEMMA 1

We will base our proof on the following Lemma.

Lemma A.1 (Cao *et al.*, 1998) For any $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^n$ and any positive definite matrix $\boldsymbol{Q}\in\mathbb{R}^{n\times n}$

$$-2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}\le\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x}+\boldsymbol{y}^{\mathrm{T}}\boldsymbol{Q}^{-1}\boldsymbol{y}$$

Proof of Lemma 1 For the sake of simplicity and without loss of generality, we do the proof only for $r=2$. For the more general case, the proof is similar. Note that

$$\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{\Delta}\boldsymbol{y}_1+\boldsymbol{y}_1^{\mathrm{T}}\boldsymbol{\Delta}^{\mathrm{T}}\boldsymbol{x}_1+\boldsymbol{x}_2^{\mathrm{T}}\boldsymbol{\Delta}\boldsymbol{y}_2+\boldsymbol{y}_2^{\mathrm{T}}\boldsymbol{\Delta}^{\mathrm{T}}\boldsymbol{x}_2
=\begin{bmatrix}\boldsymbol{x}_1\\\boldsymbol{x}_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{\Delta}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{\Delta}\end{bmatrix}\begin{bmatrix}\boldsymbol{y}_1\\\boldsymbol{y}_2\end{bmatrix}+\begin{bmatrix}\boldsymbol{y}_1\\\boldsymbol{y}_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{\Delta}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{\Delta}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{x}_1\\\boldsymbol{x}_2\end{bmatrix}$$

Let $\boldsymbol{Q}\triangleq\begin{bmatrix}\varepsilon_1\boldsymbol{I}&\varepsilon_2\boldsymbol{I}\\\varepsilon_2\boldsymbol{I}&\varepsilon_3\boldsymbol{I}\end{bmatrix}>0$.

Based on the matrix theory, $\boldsymbol{Q}$ can be decomposed into

$$\begin{bmatrix}\varepsilon_1\boldsymbol{I}&\varepsilon_2\boldsymbol{I}\\\varepsilon_2\boldsymbol{I}&\varepsilon_3\boldsymbol{I}\end{bmatrix}=\begin{bmatrix}\boldsymbol{I}&\alpha_2\boldsymbol{I}\\\boldsymbol{0}&\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\alpha_1\boldsymbol{I}&\boldsymbol{0}\\\boldsymbol{0}&\alpha_3\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\boldsymbol{I}&\alpha_2\boldsymbol{I}\\\boldsymbol{0}&\boldsymbol{I}\end{bmatrix}^{\mathrm{T}}$$

Applying Lemma A.1, we have

$$\begin{aligned}
&\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{\Delta}\boldsymbol{y}_1+\boldsymbol{y}_1^{\mathrm{T}}\boldsymbol{\Delta}^{\mathrm{T}}\boldsymbol{x}_1+\boldsymbol{x}_2^{\mathrm{T}}\boldsymbol{\Delta}\boldsymbol{y}_2+\boldsymbol{y}_2^{\mathrm{T}}\boldsymbol{\Delta}^{\mathrm{T}}\boldsymbol{x}_2\\
&\le\begin{bmatrix}\boldsymbol{x}_1\\\boldsymbol{x}_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{\Delta}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{\Delta}\end{bmatrix}\begin{bmatrix}\varepsilon_1\boldsymbol{I}&\varepsilon_2\boldsymbol{I}\\\varepsilon_2\boldsymbol{I}&\varepsilon_3\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\boldsymbol{\Delta}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{\Delta}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{x}_1\\\boldsymbol{x}_2\end{bmatrix}
+\begin{bmatrix}\boldsymbol{y}_1\\\boldsymbol{y}_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\varepsilon_1\boldsymbol{I}&\varepsilon_2\boldsymbol{I}\\\varepsilon_2\boldsymbol{I}&\varepsilon_3\boldsymbol{I}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{y}_1\\\boldsymbol{y}_2\end{bmatrix}\\
&=\begin{bmatrix}\boldsymbol{x}_1\\\boldsymbol{x}_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{\Delta}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{\Delta}\end{bmatrix}\begin{bmatrix}\boldsymbol{I}&\alpha_2\boldsymbol{I}\\\boldsymbol{0}&\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\alpha_1\boldsymbol{I}&\boldsymbol{0}\\\boldsymbol{0}&\alpha_3\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\boldsymbol{I}&\alpha_2\boldsymbol{I}\\\boldsymbol{0}&\boldsymbol{I}\end{bmatrix}^{\mathrm{T}}\cdot
\begin{bmatrix}\boldsymbol{\Delta}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{\Delta}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{x}_1\\\boldsymbol{x}_2\end{bmatrix}+\begin{bmatrix}\boldsymbol{y}_1\\\boldsymbol{y}_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\varepsilon_1\boldsymbol{I}&\varepsilon_2\boldsymbol{I}\\\varepsilon_2\boldsymbol{I}&\varepsilon_3\boldsymbol{I}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{y}_1\\\boldsymbol{y}_2\end{bmatrix}\\
&=\begin{bmatrix}\boldsymbol{x}_1\\\boldsymbol{x}_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{I}&\alpha_2\boldsymbol{I}\\\boldsymbol{0}&\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\alpha_1\boldsymbol{\Delta}\boldsymbol{\Delta}^{\mathrm{T}}&\boldsymbol{0}\\\boldsymbol{0}&\alpha_3\boldsymbol{\Delta}\boldsymbol{\Delta}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\boldsymbol{I}&\alpha_2\boldsymbol{I}\\\boldsymbol{0}&\boldsymbol{I}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{x}_1\\\boldsymbol{x}_2\end{bmatrix}
+\begin{bmatrix}\boldsymbol{y}_1\\\boldsymbol{y}_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\varepsilon_1\boldsymbol{I}&\varepsilon_2\boldsymbol{I}\\\varepsilon_2\boldsymbol{I}&\varepsilon_3\boldsymbol{I}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{y}_1\\\boldsymbol{y}_2\end{bmatrix}\\
&\le\begin{bmatrix}\boldsymbol{x}_1\\\boldsymbol{x}_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{I}&\alpha_2\boldsymbol{I}\\\boldsymbol{0}&\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\alpha_1\boldsymbol{I}&\boldsymbol{0}\\\boldsymbol{0}&\alpha_3\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\boldsymbol{I}&\alpha_2\boldsymbol{I}\\\boldsymbol{0}&\boldsymbol{I}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{x}_1\\\boldsymbol{x}_2\end{bmatrix}
+\begin{bmatrix}\boldsymbol{y}_1\\\boldsymbol{y}_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\varepsilon_1\boldsymbol{I}&\varepsilon_2\boldsymbol{I}\\\varepsilon_2\boldsymbol{I}&\varepsilon_3\boldsymbol{I}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{y}_1\\\boldsymbol{y}_2\end{bmatrix}\\
&=\begin{bmatrix}\boldsymbol{x}_1\\\boldsymbol{x}_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\varepsilon_1\boldsymbol{I}&\varepsilon_2\boldsymbol{I}\\\varepsilon_2\boldsymbol{I}&\varepsilon_3\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_1\\\boldsymbol{x}_2\end{bmatrix}+\begin{bmatrix}\boldsymbol{y}_1\\\boldsymbol{y}_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\varepsilon_1\boldsymbol{I}&\varepsilon_2\boldsymbol{I}\\\varepsilon_2\boldsymbol{I}&\varepsilon_3\boldsymbol{I}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{y}_1\\\boldsymbol{y}_2\end{bmatrix}
\end{aligned}$$

This completes the proof.