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Generalized solutions to the Benjamin-Ono equations in sense of Colombeau*

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Abstract: This paper discusses the existence and uniqueness of the generalized solution in the sense of Colombeau to the Benjamin-Ono (B-O) equation and the relationship between the new generalized solution and the classical solution.

Key words: B-O equation, Algebra of generalized solution, Hilbert transform
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INTRODUCTION

In the theory of partial differential equations, Schwartz distribution theory and Sobolev spaces play a very important role. But one shortcoming of distribution theory, however, is its inability to solve nonlinear problems because the product of distributions may not be a distribution. The need to define the product of distributions also arises with problems in Dirac's quantum theory of an electromagnetic field. It is impossible to introduce an associate multiplication in the space of distributions via the following example. For the Heaviside function H one observes that $H^m=H^n$ implies that $mH^{m-1}\delta=nH^{n-1}\delta$ implies $m=n$.

During the past 20 years a new theory of generalized functions was developed in which their product is defined and values of their other nonlinear functions can be calculated (Colombeau, 1991; 1984; 1983). These new generalized functions are defined as limits of smooth functions. The new space contains distributions so that δ^2 makes sense.

We are investigating generalized solutions in the sense of Colombeau (1984) to the Benjamin-Ono equation. The solutions will belong to algebra $\mathcal{G}_{2,2}$ of generalized functions. Many nonlinear problems can be dealt with successfully by the new generalized function. For example, for the Cauchy problem of the Kdv equation $\partial_t u+u\partial_x u+\partial_x^3 u=0$, the regularized long-wave equation $\partial_t u+\partial_x u+u\partial_x^2 u+\partial_x^2\partial_t u=0$, the Burgers equation $\partial_t u+u\partial_x u=v\partial_x^3 u$ and MKdv equation $\partial_t u+u^2\partial_x u+\partial_x^3 u=0$ with generalized functions as initial data, solutions were found in certain algebras of new generalized functions via the theory of Biaginioni and Oberguggenberger (1992a; 1992b), Bu (1995), and Colombeau (1983; 1990).

Our object here is to obtain a similar result for the Benjamin-Ono equation

$$\partial_t u+u\partial_x u+H\partial_x^2 u=0 \qquad x,t\in R \tag{1}$$

$$u(x,0)=g(x) \tag{2}$$

where H denotes Hilbert transform

$$(Hf)(x)=PV\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{f(y)}{x-y}\mathrm{d}y$$

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The Hilbert transform is linear and commutes with differentiation operators. Moreover, H is a bounded operator from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ for any $p>1$, and from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$ for all s.

This paper is organized as follows. The definition of space $\mathfrak{g}_{p,q}(\Omega)$ and the conservation laws of B-O equations are contained in Section 2. In Section 3, we prove the existence and uniqueness of generalized solution in sense of Colombeau. The relation between new generalized solutions and classical equations is given in Section 4.

THE SPACE $\mathfrak{g}_{p,q}(\Omega)$ AND CONSERVATION LAWS OF B-O EQUATIONS

Notation 1 In this paper all functions and distribution spaces are assumed to be real valued. Let $M\in\mathbb{N}$, $1\le p, q\le\infty$, $m\in\mathbb{Z}$ and Ω an open subset of $\mathbb{R}^n$. $W^{m,p}(\Omega)$ is the usual Sobolev space, $W^{\infty,p}(\Omega)=\cap_m W^{m,p}(\Omega)$, $W^{-\infty,p}(\Omega)=\cup_m W^{m,p}(\Omega)$, $H^m(\Omega)=W^{m,2}(\Omega)$. We set

$$\Psi(\Omega)=\{u:(0,\infty)\times\Omega\to\mathbb{R} \text{ such that } u(\varepsilon,\cdot)\in C^\infty(\Omega),\forall\varepsilon>0\} \tag{3}$$

$$\Psi_p(\Omega)=\{u\in\Psi(\Omega) \text{ such that } u(\varepsilon,\cdot)\in C^{\infty,p}(\Omega),\forall\varepsilon>0\} \tag{4}$$

$$\Psi_{M,p}(\Omega)=\{u\in\Psi_p(\Omega) \text{ such that } \forall\alpha\in\mathbb{N}^n \text{ there is } n\in\mathbb{N} \text{ such that } \|\partial^\alpha u(\varepsilon,\cdot)\|_p=O(\varepsilon^{-N}) \text{ as } \varepsilon\to 0\} \tag{5}$$

$$\mathcal{N}_{p,q}(\Omega)=\{u\in\Psi_{M,p}(\Omega)\cap\Psi_q(\Omega) \text{ such that } \forall\alpha\in\mathbb{N}^n \text{ and } M\in\mathbb{N}\ \|\partial^\alpha u(\varepsilon,\cdot)\|_q=O(\varepsilon^{M}) \text{ as } \varepsilon\to 0\} \tag{6}$$

where $\|\cdot\|_p$ denotes the L^p-norm.

Remark 2 (i) If Ω has strong local Lipschtz property and $u\in\Psi_p(\Omega)$, then $u(\varepsilon,\cdot)\in C^\infty(\bar{\Omega})$ for every ε.

(ii) If $\Omega=\mathbb{R}^n$, $p<\infty$, and $u\in\Psi_p(\Omega)$, then for every $\varepsilon>0$, $\lim_{|x|\to\infty}u(\varepsilon,x)=0$.

Proposition 3 Let Ω have cone property. Then

(i) if $p_1\le p_2$, $\Psi_{M,p_1}(\Omega)\subseteq\Psi_{M,p_2}(\Omega)$;

(ii) $\Psi_{M,p_1}(\Omega)$ is an algebra with partial derivatives;

(iii) $\mathcal{N}_{p,q}(\Omega)$ is an ideal in $\Psi_{M,p_1}(\Omega)$ which is invariant under partial derivatives.

Proof See Biaginioni and Obergaggerberger (1992a) proposition 2.3.

Definition 4 We define, for $1\le p, q\le\infty$,

$$\mathfrak{g}_{p,q}(\Omega)=\Psi_{M,p}(\Omega)/\mathcal{N}_{p,q}(\Omega)$$

Proposition 5 (i) There is an imbedding of $W^{-\infty,p}(\mathbb{R}^n)$ into $\mathfrak{g}_{p,q}(\mathbb{R}^n)$.

(ii) If $q\le p$, this embedding turns $W^{\infty,p}(\mathbb{R}^n)$ into a subalgebra of $\mathfrak{g}_{p,q}(\mathbb{R}^n)$.

Proof See Biaginioni and Obergaggerberger (1992a). Notice that the imbedding map $W^{-\infty,p}(\mathbb{R}^n)\to\Psi_p(\mathbb{R}^n)$:

Set $\rho(x)\in S(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n}x^i\rho(x)\mathrm{d}x=0$ for all $i\in\mathbb{N}^n$, and $\int_{\mathbb{R}^n}\rho(x)\mathrm{d}x=1$. The imbedding map can be defined by $l(w)(\varepsilon, x)=(w*\rho_\varepsilon)(x)$.

Definition 6 We say that $u\in\mathfrak{g}_{p,q}(\Omega)$ is associated with the distribution $u\in\mathfrak{D}'(\Omega)$ if there is a representative $\hat{u}$ of u such that $\hat{u}(\varepsilon,\cdot)\to w$ in $\mathfrak{D}'(\Omega)$ as $\varepsilon\to 0$. Notation: $u\approx w$.

Definition 7 We say that $u\in\mathfrak{g}_{p,q}(\Omega)$ is of $r-\sqrt[j]{\log}-type$, $r\ge p$, $j\ge 1$, if it has a representative $\hat{u}\in\Psi_{M,p}(\Omega)$ such that

$$\|\hat{u}(\varepsilon,\cdot)\|_r=O(\sqrt[j]{|\log\varepsilon|}) \quad \text{as } \varepsilon\to 0 \tag{7}$$

Definition 8 Let $u\in\mathfrak{g}_{p,q}(\mathbb{R}\times(0,T))$. We define the restriction of u to $\mathbb{R}\times\{0\}$ as follows: Let $\hat{u}$ be the representative of u. By Remark 2(i) $\hat{u}(\varepsilon,\cdot,\cdot)\in C^\infty(\mathbb{R}\times(0,T))$, $\forall\varepsilon>0$. Since the restriction map $W^{m+1,p}(\mathbb{R}\times(0,T))\to W^{m,p}(\mathbb{R})$ is continuous, we have that $\hat{u}(\varepsilon,\cdot,0)$ belongs to $\Psi_{M,p}(\mathbb{R})$. Also, $\hat{u}(\varepsilon,\cdot,0)\in\mathcal{N}_{p,q}(\mathbb{R})$ if $\hat{u}\in\mathcal{N}_{p,q}(\mathbb{R}\times(0,T))$. Thus we may define the restriction of u to $\mathbb{R}\times\{0\}$ as the class of $\hat{u}(\varepsilon,\cdot,0)$ in $\mathfrak{g}_{p,q}(\mathbb{R})$.

In (Bock and Kruskal, 1979; Biaginioni and Obergaggenberger, 1992a), we know that the B-O equation has a sequence of conserved laws and Nakamura (1979) computed the infinite numbers of conservation laws by using Bäcklund transform. These conservation laws are in the form of

$$I_n=\int(\frac{u^n}{n}+P)\mathrm{d}x$$

where P is a polynomial consisting of u, $\partial_x u$, …, $\partial_x^n u$ and their Hilbert transform. Furthermore, we also have the following estimate:

Proposition 9 The B-O equation has a sequence of conserved laws I_n, and I_{2k} which are in the form of

$$I_{2k}=\int[(\partial_x^{k-1}u)^2+(\partial_x^{k-1}u)Q_k+\tilde{Q}_k]\,\mathrm{d}x,\ k=0,1,2,... \quad (8)$$

where Q_k and $\tilde{Q}_k$ are polynomials of order less than k–2 in u, its derivative with respect to x, and the Hilbert transforms of these.

EXISTENCE AND UNIQUENESS OF GENERALIZED SOLUTION

Theorem 10 Let $g\in\mathfrak{g}_{p,q}(\mathbb{R})$. Then there is a solution u of Eqs.(1) and (2) in $g\in\mathfrak{g}_{p,q}(\mathbb{R}\times(0,\infty))$ such that

$$u|_{t=0}=g \quad (9)$$

and, for every $T>0$, $u|_{\mathbb{R}\times(0,T)}\in\mathfrak{g}_{2,2}(\mathbb{R}\times(0,T))$.

Proof Let $\hat{g}\in\Psi_{M,2}(\mathbb{R})$ be a representative of g. Since $\hat{g}(\varepsilon,\cdot)\in H^\infty(\mathbb{R})$ for each $\varepsilon>0$, according to the existence theory of Iorio (1986) (Thm. 4.4), there is a unique solution $\hat{u}_\varepsilon$ of Eq.(1) with initial data $\hat{g}(\varepsilon,\cdot)$ belonging to $\bigcap_{0\le j\le[s/3]}C_B^j\{[0,\infty);H^{s-3j}(\mathbb{R})\}$ for arbitrary s. Furthermore, if $\hat{g}\in H^\infty(\mathbb{R})$, then $\hat{u}(\varepsilon,\cdot,t)\in H^\infty(\mathbb{R})$.

For this, it suffices to prove that for all $k\in\mathbb{N}$ there are $c>0$ and $\eta>0$ such that

$$\sup_{t\ge0}\left\|\partial_x^k\hat{u}(\varepsilon,\cdot,t)\right\|_2\le\frac{c}{\varepsilon^N},\quad 0<\varepsilon<\eta \quad (10)$$

If we have Eq.(10), since $\hat{u}(\varepsilon,\cdot)$ satisfies Eq.(1), we get an analogous estimate for $\partial_t\hat{u}$ and then, by successive differentiations in the equation we get, for all $\alpha,\gamma\in\mathbb{N}$, there are $c>0$ and $\eta>0$ such that

$$\sup_{t\ge0}\left\|\partial_x^\alpha\partial_t^\gamma\hat{u}(\varepsilon,\cdot,t)\right\|_2\le\frac{c}{\varepsilon^N},\quad 0<\varepsilon<\eta \quad (11)$$

Accordingly, this implies an analogous estimate for $\left\|\partial_x^\alpha\partial_t^\gamma\hat{u}(\varepsilon,\cdot,t)\right\|_{L^\infty_{(\mathbb{R}\times(0,\infty))}}$ and $\left\|\partial_x^\alpha\partial_t^\gamma\hat{u}(\varepsilon,\cdot,t)\right\|_{L^2_{(\mathbb{R}\times(0,T))}}$ for $\forall T>0$. Therefore, $g\in\mathfrak{g}_{2,2}(\mathbb{R})$ and let $\hat{g}$ be a representative of g, then for $\forall\varepsilon>0$, $\hat{g}\in H^\infty(\mathbb{R})$. Then the class of $\hat{u}$ will be an element of $\mathfrak{g}_{\infty,2}(\mathbb{R}\times(0,\infty))$, a solution to Eqs.(1) and (9), whose restriction to any strip belongs to $\mathfrak{g}_{2,2}(\mathbb{R}\times(0,T))$.

But the inequality (10) is an immediate consequence of the following lemma, and the proof is completed.

Lemma 11 For every $k\in\mathbb{N}$, there is a polynomial Q_k such that

$$\|\hat{u}(\varepsilon,\cdot,t)\|_{H^k(\mathbb{R})}\le Q_k\left(\|\hat{g}(\varepsilon,\cdot)\|_{H^k(\mathbb{R})}\right) \quad (12)$$

Proof In order to simplify the notation we drop the ε and the "hat" on the representatives of u and g. By Proposition 9, the B-O equation has a sequence of conserved quantities in the form of Eq.(8).

We will prove the assertion by induction over k. $k=0$ is true since $I_0=\int_{-\infty}^{+\infty}u^2(x,t)\mathrm{d}x$, i.e. $\mathrm{d}(\int_{-\infty}^{+\infty}u^2(x,t)\mathrm{d}x)/\mathrm{d}t=0$, and $\|u(\cdot,t)\|_2=\|g\|_2$. Assume that Eq.(12) holds for $j\le k-1$. Thus the expression below also holds.

$$\|\hat{u}(\varepsilon,\cdot,t)\|_{H^j(\mathbb{R})}\le Q_j\left(\|\hat{g}(\varepsilon,\cdot)\|_{H^j(\mathbb{R})}\right) \quad (13)$$

By Eq.(8) we have

$$\int_{-\infty}^{+\infty}(\partial_x^k u)^2\mathrm{d}x=\int_{-\infty}^{+\infty}(\partial_x^k u\cdot Q_k+\tilde{Q}_k)\mathrm{d}x+c \quad (14)$$

where, since $(\mathrm{d}/\mathrm{d}t)I_k(u)=0$, $I_{2(k+1)}(u)=I_{2(k+1)}(g)=C$ for all $t\geq0$. By Young inequality,

$$\int_{-\infty}^{+\infty}\partial_x^k u\cdot Q_k\,\mathrm{d}x\leq\frac{\varepsilon^2}{2}\int_{-\infty}^{+\infty}(\partial_x^k u)^2\,\mathrm{d}x+\frac{2}{\varepsilon^2}\int_{-\infty}^{+\infty}Q_k^2\mathrm{d}x \quad (15)$$

Combining Eqs.(14) and (15) yields

$$\int_{-\infty}^{+\infty}(\partial_x^k u)^2\,\mathrm{d}x\leq\int_{-\infty}^{+\infty}\tilde{Q}_k\mathrm{d}x+c \quad (16)$$

By Gagliardo-Nirenberg estimate, we have

$$\left\|\partial_x^k u\right\|_\infty\leq c\left\|u\right\|_{H^k(\mathbb{R})} \quad i=0, 1, \ldots, k-1 \quad (17)$$

By Eqs.(16) and (17) and the inductive hypothesis, then $\left\|\partial_x^k u\right\|_2^2$ can be controlled by $\left\|\partial_x^i u\right\|_2^2$ (i=0, 1, …, k−1) and $\left\|\partial_x^k g\right\|_2^2$. Therefore inequality (12) holds for $j=k$. Thus we have proved Eq.(12).

Remark 12 Theorem 10 establishes the existence of a representative $\hat{u}\in\Psi_{M,2}(\mathbb{R}\times(0,T))$. Since the ideal does not enter in the proof, it is inferred that there exists a solution in $\mathfrak{g}_{2,q}(\mathbb{R}\times(0,T))$ for every $T>0$ and $q\geq1$, if the initial data belong to $\mathfrak{g}_{2,q}(\mathbb{R})$.

Theorem 13 Let $g\in\mathfrak{g}_{2,q}(\mathbb{R})$. Then for every $T>0$ there is at most one solution $u\in\mathfrak{g}_{2,2}(\mathbb{R}\times(0,T))$ of Eqs.(1) and (9) such that $\partial_x u$ is of ∞-log-type.

Proof Let $u_1,u_2\in\mathfrak{g}_{2,2}(\mathbb{R})$ be the solution to Eqs.(1) and (9) with respective representatives $\hat{u}_1$, $\hat{u}_2\in\Psi_{M,2}(\mathbb{R}\times(0,T))$ such that $\partial_x\hat{u}_i$ satisfies Eq.(7) with $r=\infty$, $j=1$, $i=1$, 2. There are $N\in\mathcal{N}_{2,2}(\mathbb{R}\times(0,T))$ such that, setting $w=\hat{u}_1-\hat{u}_2$, $h=(\hat{u}_1+\hat{u}_2)/2$:

$$[w_t+(hw)_x+w_{xxx}](\varepsilon,x,t)=N(\varepsilon,x,t)$$
$$w(\varepsilon,x,0)=n(\varepsilon,x) \quad (18)$$

By changing representatives, we may assume that $n(\varepsilon, x)\equiv0$. For simplicity we drop the ε in our notation. Multiplying Eq.(18) by w and integrating with respect to x, we obtain

$$\int_{-\infty}^{+\infty}ww_t\mathrm{d}x+\int_{-\infty}^{+\infty}(hw)_x w\mathrm{d}x+\int_{-\infty}^{+\infty}w_{xxx}w\mathrm{d}x=\int_{-\infty}^{+\infty}wN\mathrm{d}x \quad (19)$$

By Remark 2(ii) and the properties of Hilbert transform we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty}w^2+\frac{1}{2}\int_{-\infty}^{+\infty}h_x w^2\mathrm{d}x=\int_{-\infty}^{+\infty}wN\mathrm{d}x$$

Integrating from zero to $t\leq T$ we have, since $w(x, 0)\equiv0$,

$$\begin{aligned}\int_{-\infty}^{+\infty}w^2(x,t)\mathrm{d}x&=2\int_0^t\int_{-\infty}^{+\infty}w(x,\tau)N(x,\tau)\mathrm{d}x\mathrm{d}\tau\\&-\int_0^t\int_{-\infty}^{+\infty}h_x(x,\tau)w^2(x,\tau)\mathrm{d}x\mathrm{d}\tau\\&\leq2\left\|w\right\|_{L^2(\mathbb{R}\times(0,T))}\left\|N\right\|_{L^2(\mathbb{R}\times(0,T))}\\&+\left\|h_x\right\|_\infty\int_0^t\int_{-\infty}^{+\infty}w^2(x,\tau)\mathrm{d}x\mathrm{d}\tau\end{aligned} \quad (20)$$

By Cronwall's inequality, we get

$$\left\|w(\cdot,t)\right\|_2^2\leq2\left\|w\right\|_{L^2(\mathbb{R}\times(0,T))}\left\|N\right\|_{L^2(\mathbb{R}\times(0,T))}\exp(T\left\|h_x\right\|_\infty) \quad (21)$$

Since $w\in\Psi_{M,2}(\mathbb{R}\times(0,T))$, $N\in\mathcal{N}_{2,2}(\mathbb{R}\times(0,T))$, and h_x is of ∞-log-type, it follows that, for every $M>0$,

$$\sup_{t\in[0,T]}\left\|w(\cdot,t)\right\|_2=O(\varepsilon^M)\ as\ \varepsilon\to0.$$

For the x-derivatives, we get, by differentiation of Eq.(18):

$$\partial_x^k w_t+\partial_x^{k+1}(hw)+\partial_x^{k+3}w=\partial_x^k N$$

Multiplying the above equation by $\partial_x^k w$ and integrating with respect to x we get

$$\begin{aligned}&\int_{-\infty}^{+\infty}\partial_x^k w_t\partial_x^k w\mathrm{d}x+\int_{-\infty}^{+\infty}\sum_{j=0}^{k+1}\binom{k+1}{j}\partial_x^{k+1-j}h\partial_x^j w\partial_x^k w\mathrm{d}x\\&+\int_{-\infty}^{+\infty}H\partial_x^{k+2}w\partial_x^k w\mathrm{d}x=\int_{-\infty}^{+\infty}\partial_x^k N\partial_x^k\partial w\mathrm{d}x\end{aligned} \quad (22)$$

The third term of the left hand side is zero and the second one equals

$$\begin{aligned}&(k+\frac{1}{2})\int_{-\infty}^{+\infty}\partial_x h(\partial_x^k w)^2\mathrm{d}x\\&+\int_{-\infty}^{+\infty}\sum_{j=0}^{k-1}\binom{k+1}{j}\partial_x^{k+1-j}h\partial_x^j w\partial_x^k w\mathrm{d}x\end{aligned} \quad (23)$$

Then Eq.(22) becomes

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty}(\partial_x^k w)^2\mathrm{d}x\\&=-(k+\frac{1}{2})\int_{-\infty}^{+\infty}\partial_x h(\partial_x^k w)^2\mathrm{d}x+\int_{-\infty}^{+\infty}\partial_x^k N\partial_x^k w\mathrm{d}x\end{aligned}$$

$$-\sum_{j=0}^{k-1}\binom{k+1}{j}\int_{-\infty}^{+\infty}\partial_x^{k+1-j}h\partial_x^j w\partial_x^k w\mathrm{d}x \quad (24)$$

Integrating from zero to t, we have

$$\int_{-\infty}^{+\infty}(\partial_x^k w)^2\,\mathrm{d}x$$
$$\leq 2(k+\frac{1}{2})\|\partial_x h\|_\infty\int_0^t\int_{-\infty}^{+\infty}(\partial_x^k w(x,s))^2\,\mathrm{d}x\mathrm{d}s$$
$$+2\int_0^t\int_{-\infty}^{+\infty}\left|(\partial_x^k N(x,s)\partial_x^k w(x,s))\right|\mathrm{d}x\mathrm{d}s$$
$$+2\sum_{j=0}^{k-1}\binom{k+1}{j}\int_0^t\int_{-\infty}^{+\infty}\left|\partial_x^{k+1-j}h\partial_x^j w\partial_x^k w\right|\mathrm{d}x\mathrm{d}s \quad (25)$$

The last two terms on the right-hand side can be estimated as

$$2\left[\left\|\partial_x^k N\right\|_2\left\|\partial_x^k w\right\|_2 + c\sum_{j=0}^{k-1}\left\|\partial_x^{k+1-j}h\right\|_\infty\left\|\partial_x^k w\right\|_2\sup_{0\leq t\leq T}\left\|\partial_x^j w(\cdot,t)\right\|_2\right] \quad (26)$$

Then, if we assume that $\sup_{0\leq t\leq T}\left\|\partial_x^k w(\cdot,t)\right\|_2 = O(\varepsilon^M)$ for any given M>0 and 0≤j≤k, we get, since all derivatives of h and w satisfy Eq.(5)

$$\sup_{0\leq t\leq T}\left\|\partial_x^k w(\cdot,t)\right\|_2 \leq c\varepsilon^M\exp\left[(2k+1)\|\partial_x h\|_\infty\cdot T\right] \quad (27)$$

Since $\partial_x h$ is of ∞-log-type we infer that $\sup_t\left\|\partial_x^j w(\cdot,t)\right\|_2 = O(\varepsilon^M)$. For the mixed derivatives the result follows from the equation as in the proof of Theorem 10.

Remark 14 The solutions to the B-O equation are unique in the algebra $\mathfrak{g}_s(\mathbb{R}\times[0,T])$. But we cannot obtain the uniqueness in the algebra $\mathfrak{g}_s\left(\mathbb{R}\times[0,\infty)\right)$ because the bounds in the proof are only required to hold locally.

RELATIONSHIP BETWEEN GENERALIZED SOLUTIONS AND CLASSICALSOLUTIONS

Theorem 15 If $g\in H^2(\mathbb{R})$, then the solution to Eqs.(1) and (9) in $\mathfrak{g}_{2,2}(\mathbb{R}\times(0,T))$ in Theorem 10 is associated with the classical solution $v\in C([0,t];H^2(\mathbb{R}))$ given by Iorio (1986).

Proof Consider the imbedding map τ in Proposition 5, the L^2-norms of $\tau(g)(\varepsilon,\cdot)$, $\tau(g')(\varepsilon,\cdot)$ and $\tau(g'')(\varepsilon,\cdot)$, are bounded independently of τ. By Theorem 13 there is a unique solution $u\in\mathfrak{g}_{2,2}(\mathbb{R}\times(0,T))$ to Eq.(1) with initial data in the class of $\tau(g)$ in $\mathfrak{g}_{2,2}(\mathbb{R})$.

By Theorem 4.3 in Iorio (1986), there is a unique classical solution $\hat{u}_\varepsilon$ to Eq.(1) with initial data the class of $\tau(g)(\varepsilon,\cdot)$, which is in $C([0,T];H^\infty(\mathbb{R}))$. Our generalized solution u has, by construction, $\hat{u}_\varepsilon$ as a representative. Since $\tau(g)(\varepsilon,\cdot)\to g$ in $H^2(\mathbb{R})$ as $\varepsilon\to 0$, it follows from the continuous dependence result in Kenig *et al.*(1994) [Them. 1.1 (IV)] that $\hat{u}_\varepsilon$ converges to v in $C([0,T];H^\infty(\mathbb{R}))$, hence also in $\mathfrak{D}'(\mathbb{R}\times(0,T))$.

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