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The analytical solutions for orthotropic cantilever beams (II): Solutions for density functionally graded beams*

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Abstract: In this paper, the specific solutions of orthotropic plane problems with body forces are derived. Then, based on the general solution in the case of distinct eigenvalues and the specific solution for density functionally graded orthotropic media, a series of beam problem, including the problems of cantilever beam with body forces depending only on *z* or on *x* coordinate and expressed by *z* or *x* polynomial is solved by the principle of superposition and the trial-and-error method.

Key words: General solution, Functionally graded media, Analytical solutions, Cantilever beams

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INTRODUCTION

Functionally graded materials (FGM) possess elastic constants and density that vary gradually with location within the material. In one kind of FGM, only the elastic constants vary continuously along one or more directions, and in the other kind, only the density varies gradually with location. We call the later one density functionally graded materials.

Functionally graded materials have attracted much interest. Tutuncu and Ozturk (2001) obtained closed-form solutions for stress and displacements in functionally graded pressure vessels subjected to internal pressure alone by using the infinitesimal theory of elasticity. Sankar (2001) derived an elasticity solution for a functionally graded beam, in which the Young's modulus was assumed to vary exponentially through the thickness when subjected to transverse loads. Anderson (2003) presented an analytical three-dimensional elasticity solution for the stresses and displacements of a sandwich composite with a functionally graded core subjected to arbitrary transverse pressure distribution. Wu and Tsai (2004) obtained the three-dimensional solution for the static analysis of functionally graded annular spherical shells in conjunction with the methods of differential quadrature (DQ) and asymptotic expansion.

In this paper, the specific solutions of orthotropic plane problems with body forces are derived based on the basic equations. First, the specific solutions for cantilever beam with body forces depending only on *z* or on *x* coordinate are given by two integral functions. Then, a series of beam problems with density functionally graded orthotropic media is solved by the trial-and-error method. The problems include that of cantilever beam with body forces depending only on *z* or on *x* coordinate and expressed by *z* or *x* polynomial. Analytical solutions for various problems are obtained by the principle of superposition.

SPECIFIC SOLUTIONS TO DENSITY FUNCTIONALLY GRADED BEAM

Body forces depend only on *z* coordinate

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$$f_x = -Q(z),\ \ f_z = -P(z) \tag{1}$$

It is easy to see that Eqs.(3) and (4) in Jiang and Ding (2005) have the specific solution for displacement as follows

$$u^* = \frac{1}{c_{55}}G(z),\ w^* = \frac{1}{c_{33}}F(z) \tag{2}$$

where

$$G(\xi) = \int_0^{\xi}(\xi-\eta)Q(\eta)\mathrm{d}\eta,\ F(\xi) = \int_0^{\xi}(\xi-\eta)P(\eta)\mathrm{d}\eta \tag{3}$$

Substituting Eq.(2) into Eq.(2) in Jiang and Ding (2005) leads to the stress specific solution

$$\sigma_x^* = \frac{c_{13}}{c_{33}}F'(z),\ \sigma_z^* = F'(z),\ \tau_{xz}^* = G'(z) \tag{4}$$

From Eq.(4), we find that the beam has uniformly distributed stresses τ_{xz}^* and σ_z^* on surfaces ($z=\pm h/2$), and distributed loads τ_{xz}^* and σ_x^* along height at the two ends (x=0, L).

Body forces depend only on *x* coordinate

$$f_x = -Q(x),\ f_z = -P(x) \tag{5}$$

It is easy to verify that Eqs.(3) and (4) in Jiang and Ding (2005) have displacement specific solution as follows

$$u^* = \frac{1}{c_{11}}G(x),\ w^* = \frac{1}{c_{55}}F(x) \tag{6}$$

where $G(x)$ and $F(x)$ are expressed as Eq.(3).

Substituting Eq.(6) into Eq.(2) in Jiang and Ding (2005) leads to the stress specific solution

$$\sigma_x^* = G'(x),\ \sigma_z^* = \frac{c_{13}}{c_{11}}G'(x),\ \tau_{xz}^* = F'(x) \tag{7}$$

From Eq.(7), we find that the beam has distributed loads τ_{xz}^* and σ_z^* on surfaces ($z=\pm h/2$) along length, and uniformly distributed loads τ_{xz}^* and σ_x^* along height at the two ends (x=0, L).

ANALYTICAL SOLUTIONS FOR DENSITY FUNCTIONALLY GRADED CANTILEVER BEAMS

For orthotropic plane problem, the solution for Eqs.(1) and (2) in Jiang and Ding (2005) should be expressed by the superposition principle as follows

$$u = u_0 + u^*,\ w = w_0 + w^* \tag{8a}$$

$$\sigma_x = \sigma_{x0} + \sigma_x^*,\ \sigma_z = \sigma_{z0} + \sigma_z^*,\ \tau_{xz} = \tau_{xz0} + \tau_{xz}^* \tag{8b}$$

where u_0, w_0, σ_{x0}, σ_{z0} and τ_{xz0} are the general solutions expressed as Eq.(5) in Jiang and Ding (2005) for beams without body forces, and u^*, w^*, σ_x^*, σ_z^* and τ_{xz}^* are the specific solutions expressed as Eqs.(2) and (4) or Eqs.(6) and (7) for beams with body forces dependent only on z or on x coordinate.

In the next sections, we will consider two kinds of density functionally graded cantilever beam shown in Fig.1 of Jiang and Ding (2005). The boundary conditions are

$$z = \pm h/2: \sigma_z = 0,\ \tau_{xz} = 0 \tag{9a}$$

$$x = 0: \int_{-h/2}^{h/2}\sigma_x \mathrm{d}z = 0,\quad \int_{-h/2}^{h/2}\sigma_x z\mathrm{d}z = 0,\quad \int_{-h/2}^{h/2}\tau_{xz}\mathrm{d}z = 0 \tag{9b}$$

$$(x = L, z = 0): u = 0,\ w = 0,\ \partial w/\partial x = 0 \tag{9c}$$

The solution for the first kind of cantilever beam with body forces depending only on *z* coordinate

$$f_x = 0,\ f_z = \rho g,\ \rho = \sum_{n=0}^{3} d_n (z/h)^n \tag{10}$$

where ρ is the density, g is the acceleration of gravity, and d_n (n=0,1,2,3) are material constants.

Substituting Eq.(10) into Eq.(3) leads to

$$F(z) = -\sum_{n=0}^{3}\frac{d_n g}{(n+1)(n+2)h^n}z^{n+2},\ G(z) = 0 \tag{11}$$

The corresponding specific solution can be obtained by substituting Eq.(11) into Eqs.(2) and (4)

$$u^{*}=0,\ w^{*}=-\frac{1}{c_{33}}\sum_{n=0}^{3}\frac{d_n g}{(n+1)(n+2)h^n}z^{n+2} \tag{12}$$

$$\sigma_x^{*}=-\frac{c_{13}}{c_{33}}\sum_{n=0}^{3}\frac{d_n g}{(n+1)h^n}z^{n+1},\ \sigma_z^{*}=-\sum_{n=0}^{3}\frac{d_n g}{(n+1)h^n}z^{n+1},$$
$$\tau_{xz}^{*}=0 \tag{13}$$

It is apparent that the boundary displacement conditions Eq.(9c) at the fixed end ($x=L$) are satisfied by Eq.(12). At the same time, we find that the specific solution Eq.(13) may cause normal surface tractions ($z=\pm h/2$)

$$\begin{aligned}\sigma_z^{*}=P_0&=-\sum_{n=0}^{3}\frac{d_n g}{(n+1)h^n}\left(\pm\frac{h}{2}\right)^{n+1}\\&=-\frac{gh}{8}\left(d_1+\frac{1}{8}d_3\right)\pm\frac{gh}{2}\left(-d_0-\frac{1}{12}d_2\right)\end{aligned} \tag{14}$$

To satisfy the surface tractions conditions Eq.(9a), we only need to superpose the specific solution Eqs.(12) and (13) on the solution Eq.(10) in Jiang and Ding (2005) for cantilever beam without body forces and under uniform loads on upper and bottom surfaces ($z=\pm h/2$), where $\beta_1=\frac{gh}{8}\left(d_1+\frac{1}{8}d_3\right)$, $C_1=\frac{gh}{2}\left(d_0+\frac{1}{12}d_2\right)$. In order to satisfy the tractions conditions Eq.(9b), the above solution should be superposed on the solution Eq.(19) in Jiang and Ding (2005), where $N=-\int_{-h/2}^{+h/2}\sigma_x^{*}\mathrm{d}z=\frac{c_{13}h^2 g}{c_{33}}\left(\frac{d_1}{24}+\frac{d_3}{320}\right)$, $M=-\int_{-h/2}^{+h/2}\sigma_x^{*}z\mathrm{d}z=\frac{c_{13}h^3 g}{c_{33}}\left(\frac{d_0}{24}+\frac{d_2}{240}\right)$.

The solution for the second kind of cantilever beam with body forces depending only on *x* coordinate

$$f_x=0,\ f_z=\rho g,\ \rho=\sum_{n=0}^{3}c_n(x/L)^n \tag{15}$$

From Eq.(3), we have

$$F(x)=-\sum_{n=0}^{3}\frac{c_n g}{(n+1)(n+2)L^n}x^{n+2},\ G(x)=0 \tag{16}$$

Substituting Eq.(16) into Eqs.(6) and (7) yields the corresponding specific solution

$$u^{*}=0,\ w^{*}=-\frac{1}{c_{55}}\sum_{n=0}^{3}\frac{c_n g}{(n+1)(n+2)L^n}x^{n+2} \tag{17}$$

$$\tau_{xz}^{*}=-\sum_{n=0}^{3}\frac{c_n g}{(n+1)L^n}x^{n+1},\ \sigma_x^{*}=\sigma_z^{*}=0 \tag{18}$$

It is apparent that the specific stress solution Eq.(18) satisfy the traction boundary conditions Eq.(9b) at the free end ($x=0$) automatically, and may cause the fourth power of x tangential tractions on the two surfaces ($z=\pm h/2$)

$$\tau_{xz}^{*}=\sum_{n=0}^{3}H_n(x)=-\sum_{n=0}^{3}\frac{c_n g}{(n+1)L^n}x^{n+1} \tag{19}$$

To satisfy the surface tractions conditions Eq.(9a), we should superpose the solution Eq.(18) on the solutions Eqs.(27a), (27b), (27c), (27d), (27e) and (28a), (28b), (28c), (28d), (28e) in Jiang and Ding (2005), where $T_1=c_0 g$, $T_2=\frac{c_1 g}{2L}$, $T_3=\frac{c_2 g}{3L^2}$ and $T_4=\frac{c_3 g}{4L^3}$.

To satisfy the displacement conditions Eq.(9c) at the fixed end ($x=L$), we should superpose the specific solution Eq.(17) on the rigid body displacements solutions as follows

$$u=\omega_0 z,\ w=w_0-\omega_0 x \tag{20}$$

where

$$\omega_0=-\frac{1}{c_{55}}\sum_{n=0}^{3}\frac{c_n Lg}{n+1},\ w_0=-\frac{1}{c_{55}}\sum_{n=0}^{3}\frac{c_n L^2 g}{n+2} \tag{21}$$

EXAMPLES

In the following analysis, the weight of the beam $\rho_0 hLg$ is assumed to be a constant, i.e.

$$\rho_0 Lh=\int_0^L\int_{-h/2}^{+h/2}\rho\mathrm{d}z\mathrm{d}x \tag{22}$$

where ρ_0 is the average density. Substituting Eqs.(10)

and (15) into Eq.(22), we have $\rho_0 = d_0 + \frac{d_2}{12}$ for the first kind of beam and $\rho_0 = c_0 + \frac{c_1}{2} + \frac{c_2}{3} + \frac{c_3}{4}$ for the second kind of beam.

Based on the above equations, all the displacements and stresses at any inner or boundary point of the cantilever beam can be obtained. In the calculation, we set L=150 mm, h=6 mm and ρ_0=7800 kg/m^3. The material constants are shown in Table 1, and the deflections of the beam w (x=z=0) are listed in Table 2 showing that the deflection w caused by the six kinds of functionally graded density cantilever beam depend only on x coordinate and z coordinate, respectively, i.e., when density depends on x coordinate: (case 1) $c_0=\rho_0$, $c_1=c_2=c_3=0$; (case 2) $c_1=2\rho_0$, $c_0=c_2=c_3=0$; (case 3) $c_2=3\rho_0$, $c_0=c_1=c_3=0$; (case 4) $c_3=4\rho_0$, $c_0=c_1=c_2=0$; (case 5) $c_0=c_1=c_2=c_3=\frac{12}{25}\rho_0$ and (case 6) $c_0 = \frac{1}{4}\rho_0$, $c_1 = \frac{1}{2}\rho_0$, $c_2 = \frac{3}{4}\rho_0$, $c_3 = \rho_0$. When density depends on z coordinate: (case 1) $d_0=\rho_0$, $d_1=d_2=d_3=0$; (case 2) $d_0=d_1=\rho_0$, $d_2=d_3=0$; (case 3) $d_2=12\rho_0$, $d_0=d_1=d_3=0$; (case 4) $d_0=d_3=\rho_0$, $d_1=d_2=0$; (case 5) $d_0 = d_1 = d_2 = d_3 = \frac{12}{13}\rho_0$ and (case 6) $d_0 = d_1 = d_3 = \frac{1}{2}\rho_0$, $d_2 = 6\rho_0$.

It is obvious that the deflections of the orthotropic cantilever beam caused by body forces are different, whereas the results of body force depending on z coordinate are the same as that of a homogeneous beam as the body force of each unit length is a constant value, and the deflection caused by body force depending only on x coordinate for the fourth case is the minimum, which is nearly twenty three percent of the homogeneous one.

CONCLUSION

The analytical solutions for orthotropic density functionally graded cantilever beams derived in this paper by the superposition principle and the trial-and-error method are very explicit and convenient, and are also useful for study of other problems with more complicated loads and boundary conditions. Moreover, these analytical solutions can serve as benchmarks for numerical methods such as the finite element method, the boundary element method, etc.

Table 1 Material properties

c_{11} (N/m^2)	c_{13} (N/m^2)	c_{33} (N/m^2)	c_{55} (N/m^2)
1.66×10^{11}	7.8×10^{10}	1.62×10^{11}	4.3×10^{10}

Table 2 Deflection *w* of cantilever beam with body force

Density ρ case	(1)	(2)	(3)	(4)	(5)	(6)
On x (m)	0.1253E-4	0.6663E-5	0.4148E-5	0.2830E-5	0.8618E-5	0.6544E-5
On z (m)	0.1253E-4	0.1253E-4	0.1253E-4	0.1253E-4	0.1253E-4	0.1253E-4

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