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Solution of a rigid disk on saturated soil considering consolidation and rheology

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Abstract: The problem of a rigid disk acting with normal force on saturated soil was studied using Biot consolidation theory and integral equation method and the Merchant model to describe the saturated soil rheology. Using integral transform techniques, general solutions of Biot consolidation functions and the dual integral equations of a rigid disk on saturated soil were established based on the boundary conditions. These equations can be simplified using Laplace-Hankel and Abel transform methods. The numerical solutions of the integral equations, and the corresponding inversion transform were used to obtain the settlement and contact stresses of the rigid disk. Numerical examples showed that the soil settlement is small if only consolidation is considered, so the soil rheology must be taken into account to calculate the soil settlement. Numerical solution of Hankel inverse transform is also given in this paper.

Key words: Saturated soil, Rigid disk, Biot consolidation theory, Hankel transform, Rheology
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INTRODUCTION

An important problem in foundation engineering is the determination of the rate of settlement of a raft foundation placed on a deep clay layer. Gibson and McNamee (1957), and McNamee and Gibson (1960a; 1960b) considered the problem of normal uniform loading applied to the surface of a half-space and showed that one-dimensional theory gives a marked underestimate of the rate of settlement. This theory corresponds to the case of a perfectly flexible footing. However, it is often found in practice that the raft is so stiff that it is more accurate to consider it as perfectly rigid, than to consider it as perfectly flexible. Chiarella and Booker (1975) used Biot consolidation theory and considered a pair of dual integral equations which are then reduced to a pair of double Fredholm-Volterra integral equations, solvable by application of Galerkin's technique, but the expressions of the disk settlement and the contact stress were complex. With computer development, another effective method to study a raft foundation placed on a deep clay layer is FEM (finite element method) (Yang and Yang, 1995). Jin (1999) and Jin and Liu (2000) tried to obtain the solution of this dynamic problem.

Soil is a highly non-linear material and soil deformation process includes consolidation and rheology aspects. While the soil is loaded, the porewater will drain out under the excess pore pressure gradient, and the volume of the soil will decrease. On the other hand, soil rheology in determining time-dependent stress-strain relationships is essential for prediction and analysis of structural changes in soils. In this paper we consider the case of a smooth circular rigid disk indenting fluid-saturated soil, with fully permeable surface. The governing dual integral equations are presented using Laplace-Hankel transform techniques. These governing integral equations are further reduced to systems of standard Fredholm integral equations of the second kind by Abel transform. The methods used in this paper overcome the difficulty of taking account of the time factor in the course of the consolidation, and the rheology in elastic theory in the calculation of a rigid disk on saturated soil. So the methods used in this paper can be used to calculate the settlement and contact stresses of the disk in poroelastic and viscoelastic soil with time change. In practice, the soil rheology has very remarkable effect on the settlement of the disk. So consolidation and rheology must be considered when we study the deformation behavior of a disk on saturated soil. In this paper the Merchant model is introduced to simulate the soil rheology characteristics.

GOVERNING EQUATIONS

According to the consolidation theory of Biot (1941), the governing equations can be written in terms of displacements and the excess pore pressure as follows

$$\nabla^2 u_r + (2\eta - 1)\frac{\partial e}{\partial r} - \frac{u_r}{r^2} + \frac{\partial p}{G\partial r} = 0 \tag{1a}$$

$$\nabla^2 u_z + (2\eta - 1)\frac{\partial e}{\partial z} + \frac{\partial p}{G\partial z} = 0 \tag{1b}$$

$$\nabla^2 e = -\frac{\partial e}{c\partial t} \tag{1c}$$

where u_r, u_z are the solid displacements, p is the excess pore pressure (taken as positive in tension), t is the time variable and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{2a}$$

$$e = -(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}) \tag{2b}$$

$$\Omega = \frac{1-\nu}{1-2\nu} \tag{2c}$$

$$c = \frac{2G\Omega k}{\rho} \tag{2d}$$

where ∇^2 is the Laplace operator; e is the dilatation; ν and G are Poisson's ratio and shear modulus of the bulk material respectively; c is the coefficient of consolidation; ρ is the unit weight of the water and k is coefficient of permeability of poroelastic soil.

Adopting the McNamee displacement functions (McNamee and Gibson, 1960a), the governing equations can be simplified to two uncoupled partial differential equations of Φ and Ψ.

$$c\nabla^4\Phi = \nabla^2\frac{\partial\Phi}{\partial t} \tag{3a}$$

$$\nabla^2\Psi = 0 \tag{3b}$$

By the m-order Hankel transform with respect to the radial coordinate and the Laplace transform with respect to the time variable

$$\tilde{\phi}^m(\xi, z, s) = \int_0^\infty\int_0^\infty e^{-st} r J_m(r\xi)\phi(r, z, t)\mathrm{d}r\mathrm{d}t \tag{4a}$$

$$\phi(r, z, t) = \frac{1}{2\pi i}\int_0^\infty\int_{\alpha-i\infty}^{\alpha+i\infty} \xi e^{st} J_m(r\xi)\tilde{\phi}^m(\xi, z, s)\mathrm{d}s\mathrm{d}\xi \tag{4b}$$

where superscript (~) is designated parameter in Laplace transform domain and m is the order of Hankel transform. Application of zero-order Laplace-Hankel integral transform to Eq.(3a) and Eq.(3b), yielded the following solutions expressed as

$$\tilde{\Phi}^0(\xi, z, s) = Ae^{-z\xi} + Be^{-z\gamma} + Ce^{z\xi} + De^{z\gamma} \tag{5a}$$

$$\tilde{\Psi}^0(\xi, z, s) = Ee^{-z\xi} + Fe^{z\xi} \tag{5b}$$

where $\gamma = \sqrt{\xi^2 + s/c}$ and A, B, C, D, E and F are unknown arbitrary functions and determined from appropriate boundary and continuity conditions.

FREDHOLM INTEGRAL EQUATION

The rheology behavior of soil is considered. The Merchant Model shown in Fig.1 we used considered the instantaneous elastic response, creep and relaxation behavior of half-space. The rheology function of the Merchant Model can be expressed as

$$J(t) = \frac{1}{E_0} + \frac{1}{E_1}(1 - e^{-\frac{E_1}{\eta_0}t}) \tag{6}$$

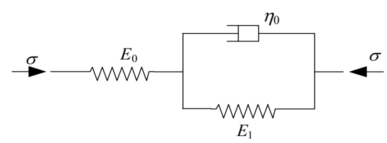

Fig.1 The rheology model of soil (Merchant model)

where η_0 is the soil damping factor.

By using Laplace transform, Eq.(6) can be expressed as

$$s\tilde{J}(s)=\frac{1}{E_0}+\frac{1}{E_1}\left(1-\frac{s}{s+(E_1/\eta_0)}\right) \quad (7)$$

Fig.2 shows a focused force P applied to a unit radius (a=1) rigid disk embedded on saturated soil. The disk settlement is $\delta(t)$. Because displacements and stresses have their numerical limits in the remote domain, C, D, and F in Eq.(5) must be zero, there are only three indetermined unknown arbitrary functions: A, B and E. Because the Schapery (1962) formula is employed for Laplace reverse transform, the elastic modulus of soil E_s must be replaced by $1/(s\tilde{J}(s))$.

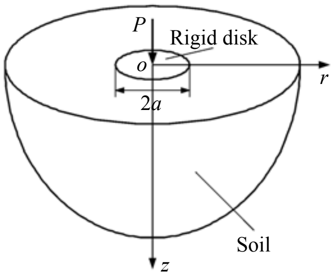

Fig.2 The Model of fluid-saturated soil

Assuming that the rigid disk is fully permeable and that the contact between the disk and the soil surface where the boundary conditions are as follows:

$$u_z(r,0,t)=\delta(t),\ 0\le r\le 1 \quad (8a)$$

$$\sigma_{zz}(r,0,t)=0,\ \ r>1 \quad (8b)$$

$$p(r,0,t)=0,\ \ \ 0\le r<\infty \quad (8c)$$

$$\sigma_{zr}(r,0,t)=0,\ \ 0\le r<\infty \quad (8d)$$

Using Eqs.(8c)–(8d) and the expressions for p and σ_{zr}, we obtain:

$$E=\frac{\eta}{\xi}(\xi^2-\gamma^2)B \quad (9a)$$

$$A=-\frac{\gamma}{\xi}B \quad (9b)$$

From Eqs.(8a)–(8b) and Eq.(4b), we obtain two dual integral equations:

$$\int_0^\infty \xi\frac{\Delta_1}{\Delta}\tilde{\sigma}_{zz}^0(\xi,0,s)J_0(\xi r)\mathrm{d}\xi=\tilde{\delta}(s)\quad (0\le r<1) \quad (10a)$$

$$\int_0^\infty \xi\tilde{\sigma}_{zz}^0(\xi,0,s)J_0(\xi r)\mathrm{d}\xi=0\quad (1<r<\infty) \quad (10b)$$

where

$$\Delta_1=-\frac{\eta(-\gamma^2+\xi^2)}{\xi},$$

$$\Delta=2[(\xi^2-\xi\gamma)-\eta(-\gamma^2+\xi^2)],$$

and $\tilde{\delta}(s)$ is the Laplace transform of $\delta(t)$.

Use of Abel transforms (Gladwell, 1980) on Eqs.(10a)–(10b) yield:

$$\int_0^\infty \xi\frac{\Delta_1}{\Delta}\tilde{\sigma}_{zz}^0(\xi,0,s)\cos(\xi x)\mathrm{d}\xi=\tilde{\delta}(s)\ \ (0\le r<1) \quad (11a)$$

$$\int_0^\infty \xi\tilde{\sigma}_{zz}^0(\xi,0,s)\cos(\xi x)\mathrm{d}\xi=0\ \ (1<r<\infty) \quad (11b)$$

For the inversion of the Laplace transform, the Schapery (1962) formula was employed

$$\Gamma(t)=[s\tilde{\Gamma}(s)]_{s=(1/2t)} \quad (12)$$

where $\Gamma(t)$ is unknown function in the time domain, $\tilde{\Gamma}(s)$ is unknown function in the Laplace transform domain. Chau (1996) held that Schapery's method is suitable for consolidation problem. Eqs.(11a)–(11b) can be expressed by using the Laplace inverse transforms

$$\int_0^\infty \xi\frac{\Delta_1}{\Delta}\sigma_{zz}^0(\xi,0,t)\cos(\xi x)\mathrm{d}\xi=\delta(t)\ \ (0\le r<1) \quad (13a)$$

$$\int_0^\infty \xi\sigma_{zz}^0(\xi,0,t)\cos(\xi x)\mathrm{d}\xi=0\ \ (1<r<\infty) \quad (13b)$$

We obtain:

$$\lim_{\xi\to\infty}\frac{\xi\Delta_1}{\Delta}=\frac{\eta}{-2+2\eta}=l \quad (14)$$

Eq.(13a) can be written as

$$\int_0^\infty \left(\xi\frac{\Delta_1}{\Delta l}-1\right)\sigma_{zz}^0(\xi,0,t)\cos(\xi x)\mathrm{d}\xi$$
$$+\int_0^\infty \sigma_{zz}^0(\xi,0,t)\cos(\xi x)\mathrm{d}\xi=\frac{\delta(t)}{l} \quad (15)$$

Based on the procedure suggested by Sneddon (1972), we define the following integral representation:

$$\sigma_{zz}^0(\xi,0,t)=\frac{2}{\pi}\frac{\delta(t)}{l}\int_0^1\phi(y)\cos(\xi y)\mathrm{d}y \quad (16)$$

i.e.:

$$\int_0^\infty \sigma_{zz}^0(\xi,0,t)\cos(\xi y)\mathrm{d}\xi=\frac{\delta(t)}{l}\phi(y)H(1-y) \quad (17)$$

where $H(1-y)$ is the Heaviside step function. Eq.(13b) satisfies automatically.

Substituting Eqs.(16) and (17) into Eq.(15), we have

$$\phi(x)+\int_0^1 K(x,y)\phi(y)\mathrm{d}y=1 \quad (0\le x<1) \quad (18)$$

where

$$K(x,y)=\frac{2}{\pi}\int_0^\infty\left(\frac{\xi\Delta_1}{\Delta l}-1\right)\cos(\xi x)\cos(\xi y)\mathrm{d}\xi \quad (19)$$

The force-displacement relationship for the rigid disk can then be obtained from the resultant contact stress:

$$-P=\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r\sigma_{zz}(r,0,t)\mathrm{d}r$$
$$=2\pi\int_0^1 r\sigma_{zz}(r,0,t)\mathrm{d}r=2\pi\sigma_{zz}^0(0,0,t) \quad (20)$$

Using Eq.(16), we obtain:

$$-P=\frac{4\delta(t)}{l}\int_0^1\phi(y)\mathrm{d}y \quad (21)$$

That is:

$$\delta(t)=-\Psi(t)P \quad (22)$$

where

$$\Psi(t)=\frac{1}{\frac{4}{l}\int_0^1\phi(y)\mathrm{d}y} \quad (23)$$

NUMERICAL SOLUTION OF HANKEL INVERSE TRANSFORM

Eq.(4b) is an infinite vibrated integral. In order to get the numerical result of Eq.(4b), we must find the zero points of the Bessel function $J_m(\xi r)$, then integrate the integral equation between two zero points. Usually, precise enough results can be obtained after 20 segments have been calculated. In this paper, Euler transform methods (Davis and Rabinowitz, 1984) were used to obtain a more precise result.

Assume $J_m(\xi r)$ is m-order Bessel function of the first kind, with zero points ξ_0, ξ_1, ξ_2, …, ξ_n, …, the following expression can be obtained:

$$\phi(r)=\int_0^\infty \xi\phi(\xi)J_m(\xi r)\mathrm{d}\xi$$
$$=\int_0^{\xi_0/r}\varphi(\xi)\mathrm{d}\xi+\int_{\xi_0/r}^{\xi_1/r}\varphi(\xi)\mathrm{d}\xi+\ldots+\int_{\xi_{n-1}/r}^{\xi_n/r}\varphi(\xi)\mathrm{d}\xi+\ldots$$
$$=u_0-u_1+\ldots+(-1)^i u_i+\ldots \quad (24)$$

where

$$\varphi(\xi)=\xi\phi(\xi)J_m(\xi r)$$
$$u_i=\left|\int_{\xi_{i-1}/r}^{\xi_i/r}\varphi(\xi)\mathrm{d}\xi\right|.$$

Using Euler transformation from the kth term in Eq.(24), this equation becomes

$$\phi(r)=\sum_{i=0}^{k-1}u_i+(-1)^k\left[\frac{1}{2}u_k-\frac{1}{2}\Delta u_k+\frac{1}{8}\Delta^2 u_k-\ldots\right] \quad (25)$$

where

$$\Delta u_k=u_{k+1}-u_k$$
$$\Delta^2 u_k=\Delta(\Delta u_k)=u_{k+2}-2u_{k+1}+u_k$$

NUMERICAL RESULTS

For long enough time it is clear that excess pore pressure p will have dissipated and thus, as $t\to\infty$, the solution will reduce to that of a purely elastic material indented by a rigid disk. This problem was solved by Boussinesq using classical potential theory and he showed that the contact pressure σ_{zF} and deflection of the die δ_F are given by the expressions

$$\sigma_{zF}=\frac{P}{2\pi a^2}\frac{a}{(a^2-r^2)^{1/2}} \tag{26a}$$

$$\delta_F=\frac{(1-\nu)P}{4Ga} \tag{26b}$$

where v denotes Poisson's ratio of the soil skeleton.

For small period of time, Polous and Davis (1980) showed that the disk placements and stresses in a consolidating material are identical with those of an incompressible elastic material (v=1/2) with a shear modulus G. Thus the initial constant pressure σ_{zI} and deflection of the disk δ_I can be deduced from Eqs.(26a)–(26b) and are given by

$$\sigma_{zI}=\frac{P}{2\pi a^2}\frac{a}{(a^2-r^2)^{1/2}} \tag{27a}$$

$$\delta_I=\frac{P}{8Ga} \tag{27b}$$

The method developed in the previous sections was used to calculate the time-settlement behavior of a rigid disk footing for a series of Poisson's ratio v=0(0.1)0.5. The results are plotted in Fig.3 as a degree of settlement U, where

$$U=\frac{\omega_0(t)-\delta_F}{\delta_I-\delta_F} \tag{28}$$

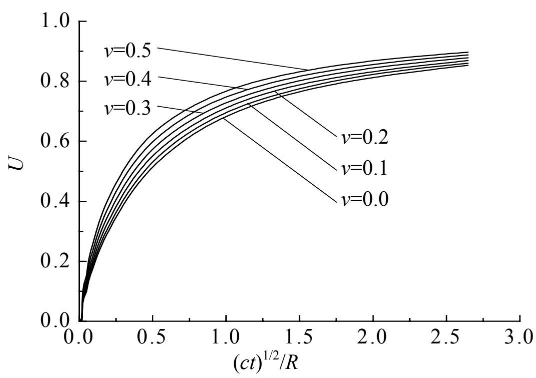

Fig.3 The time-settlement of the rigid circular footing

Fig.4 shows the stress distribution of the disk in z-direction along its radius, where the Poisson's ratio is equal to 0.3, and $\sigma_{zz}(0,0,t)$ is the z-stress of the rigid disk center at initial time.

Fig.5 shows the settlement without the saturated soil rheology compared with those reported by Chiarella and Booker (1975) and the soil Poisson's ratio is equal to 0.3. Fig.5 also shows the settlements comparison between the present solution and ABAQUS solution. It is evident from Fig.5 that the numerical algorithm used in the present study yields highly accurate numerical solutions.

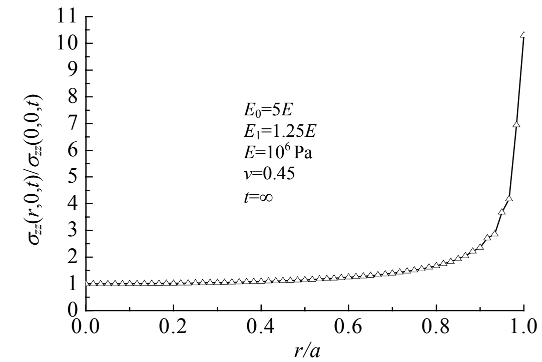

Fig.4 The contact z-stress of the rigid circular footing

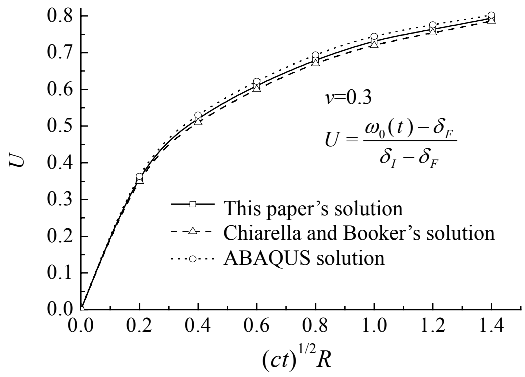

Fig.5 Comparison of present results with those of Chiarella and Booker's and FEA

To prove the solutions considering consolidation and rheology is correct, the FEM (ABAQUS) is used. There is not Merchant model in ABAQUS, so the user subroutine is used to define a material's mechanical behavior. The material's Jacobian matrix ($\partial\Delta\sigma/\partial\Delta\varepsilon$) must be given. The Jacobian matrix of Merchant model has the terms

$$\frac{\partial\Delta\sigma_{ii}}{\partial\Delta\varepsilon_{ii}}=\frac{1}{\left(\frac{\Delta t}{2}+\tilde{\nu}\right)}\left[\Delta t\left(\frac{\lambda}{2}+\mu\right)+\tilde{\lambda}+2\tilde{\mu}\right]$$

$$\frac{\partial \Delta\sigma_{ii}}{\partial \Delta\varepsilon_{jj}} = \frac{1}{\left(\dfrac{\Delta t}{2}+\tilde{v}\right)}\left[\Delta t\frac{\lambda}{2}+\tilde{\lambda}\right]$$

$$\frac{\partial \Delta\sigma_{ij}}{\partial \Delta\gamma_{ij}} = \frac{1}{\left(\dfrac{\Delta t}{2}+\tilde{v}\right)}\left[\Delta t\frac{\mu}{2}+\tilde{\mu}\right]$$

$$i = x, y, z \qquad (29)$$

where λ, $\bar{\lambda}$, μ, $\bar{\mu}$, $\bar{v}$ are functions of v, E_0, E_1, η of Eq.(2) and Eq.(6). The comparisons between the present solution and ABAQUS solution is shown in Fig.6. The normalized properties are: $E_0=1\times10^7$ Pa, $E_1=2\times10^7$ Pa, $\eta_0=10^{12}$ and 5×10^{13}. It also showed that the solution of consolidation solution considering rheology is correct.

Fig.7 shows the settlements of disk with time change because of the different parameters of therheology model. The final elastic modulus of soil in different rheology models is $E=8\times10^6$ Pa and the soil damping factor is $\eta_0=10^{13}$. It also shows that the soil deformation does not change during the initial time quantum (0–10^5 s); the consolidation and rheology deformation of soil occurs during the middle time quantum (10^5 s–10^8 s) and there is no deformation change after 10^8 s.

The effect of soil Poisson's ratio v on the disk settlements are shown in Fig.8. The consolidation deformation of soil increases with decreasing of Poisson's ratio.

Fig.9 shows the contact z-stress of different points on the disk with time change without regard to soil rheology. $\sigma_{zz}(r,0,0)$ is the z-stress of the rigid disk at initial time. It shows that the contact stresses on the disk change with increasing time and only the initial and final contact z-stress are equal. The z-stress of the point near the centre of the disk is greater at the middle time than at the initial and final time, but it has reverse result on the point near the border of the disk. Heinrich and Desoyer assumed that the contact stress distribution is constant and equal to the correct initial and final value. However, their method will not give any idea of the variation of contact stress distribution with time and is not applicable to more complicated problems. Fig.10 compares the contact z-stress of different points on the disk with time change with and

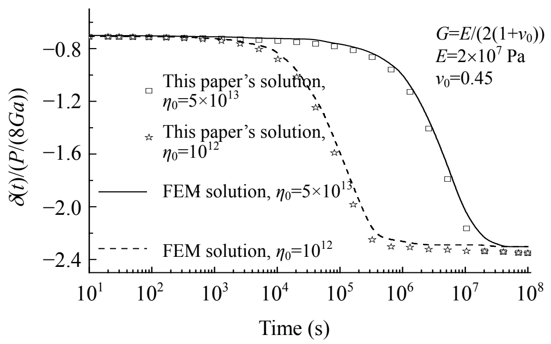

Fig.6 Comparison of present results with those of FEA considering rheology

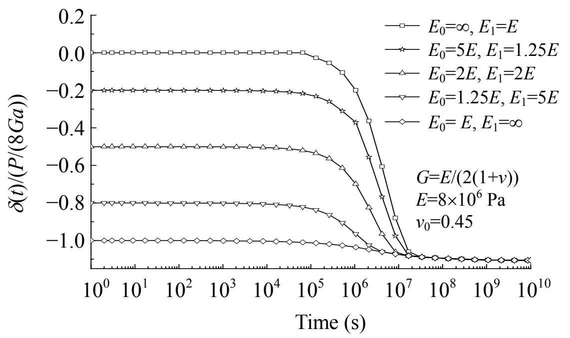

Fig.7 Effect of parameters of rheology model on disk displacements

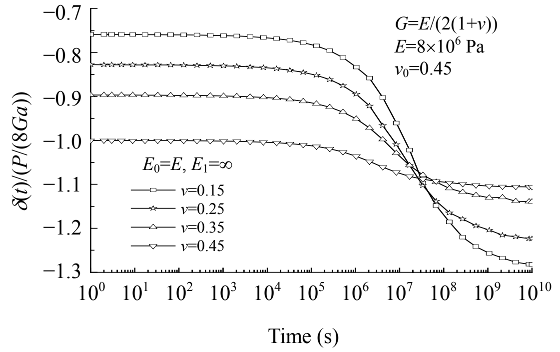

Fig.8 Effect of saturated soil Poisson's ratios on disk's displacements

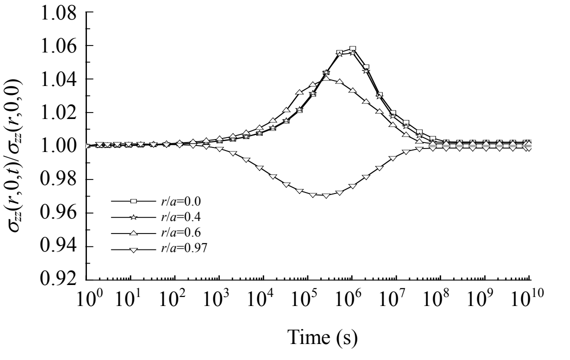

Fig.9 The contact z-stress of different points on disk with time change without regard to soil rheology

without regard to soil rheology. We can find that their z-stress almost have the same change rule.

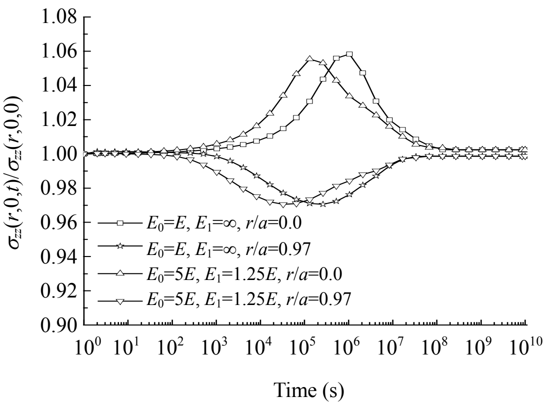

Fig.10 The contact z-stress of different points on disk with time change with regard to soil rheology

Fig.11 compares the settlement with and without regard to the soil rheology. As to taking no account of the soil rheology, the long-term settlement is only a small part of total settlement and does not accord with the practical considerations (Gurinsky, 2002). It was found that it is more reasonable to calculate the disk settlement taking account of the soil rheology than taking no account of the soil rheology.

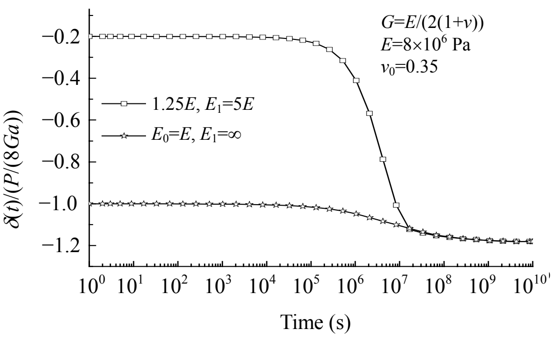

Fig.11 The displacement comparison of with and without rheology

CONCLUSION

In this work, the settlement and contact stresses of a rigid disk on saturated soil were obtained considering consolidation and rheology using Biot consolidation theory and integral equation method. Numerical examples showed that the disk settlement only considering consolidation is small, so the soil rheology must be taken into account to calculate the problem of a disk on saturated soil. It was also shown that the disk settlement and contact z-stress change with time and that the numerical method to solve the Fredholm integral equation in this paper is accurate. With the aid of the theory of plates and shells, the method can be naturally extended to the problem of the elastic disk on saturated soil considering consolidation and rheology.

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