Journal of Zhejiang University SCIENCE
ISSN 1009-3095
http://www.zju.edu.cn/jzus
E-mail: jzus@zju.edu.cn

Local Lyapunov Exponents and characteristics of fixed/periodic points embedded within a chaotic attractor

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Received Aug. 12, 2004; revision accepted Jan. 21, 2005

Abstract: A chaotic dynamical system is characterized by a positive averaged exponential separation of two neighboring trajectories over a chaotic attractor. Knowledge of the Largest Lyapunov Exponent λ_1 of a dynamical system over a bounded attractor is necessary and sufficient for determining whether it is chaotic ($\lambda_1>0$) or not ($\lambda_1\leq 0$). We intended in this work to elaborate the connection between Local Lyapunov Exponents and the Largest Lyapunov Exponent where an alternative method to calculate λ_1 has emerged. Finally, we investigated some characteristics of the fixed points and periodic orbits embedded within a chaotic attractor which led to the conclusion of the existence of chaotic attractors that may not embed in any fixed point or periodic orbit within it.

Key words: Chaotic attractor, Largest Lyapunov Exponent, Local Lyapunov Exponents
doi:10.1631/jzus.2005.A0296 **Document code:** A **CLC number:** O221

INTRODUCTION

Lyapunov Exponents (Chen and Dong, 1998) are the most popular and well-known characteristics for identifying chaotic behaviors in a dynamical system. They are means for diagnosing the behavior of a dynamical system, but yet, not for predicting chaos. A full knowledge of the attractor is required in order to calculate these exponents.

A powerful method (Wolf *et al*., 1985) was introduced for calculating these exponents from the time series evolution of a dynamical system. Later, a method for calculating these exponents numerically was introduced (Sandri, 1996), which involved use of the exclusive Mathematica package for this purpose. The significance of those exponents can be easily noticed by the increasing recent researches (Grond *et al*., 2003; Grond and Diebner, 2005).

Local Lyapunov Exponents which are the subject of investigation in this paper are nothing new but a localized definition of Lyapunov Exponents at any initial point on an attractor in the phase-space of a dynamical system. They are dependent on the choice of the initial point, whereas Lyapunov Exponents possess the same set of values at all points on the same attractor. Their values at each point in the phase-space have no direct impact on the nature of the motion, but we believe that the distribution of their stationary values in the phase-space has direct impact on the nature of their motion.

LOCAL LYAPUNOV EXPONENTS

We recall the notion of Local Lyapunov Exponents $\lambda_i(x_0, n)$ of order n at a point x_0 in the phase space of a map P (Galias, 1999), which are the exponential rates of separation of trajectories due to a perturbation in the initial point x_0 over n steps (iterations) of this map. In this paper, we restrict this definition to be considered over only one step (iteration) of the map and we will denote it as $\lambda_P^{\boldsymbol{E}}(x_0)$; where $\boldsymbol{E}$

is the unit direction vector in the phase-space along which the perturbation of the initial point takes place.

Local Lyapunov Exponents for a map

Let us consider an N-dimensional map P:R^N→R^N; where N is an integer. This map will be defined by the following set of N difference equations

$$x_{i+1}=P(x_i);\quad i=0,1,2,\ldots \tag{1}$$

Let $\boldsymbol{E}\in$R^N be a unit vector in the phase-space and consider a small perturbation to the initial condition x_0 so that the new initial condition is $x_0^{\boldsymbol{E}}=x_0+\delta\boldsymbol{E}$; where $\delta\in$R and $0<\delta<<1$. Iterating the map Eq.(1), we get the points $x_1=P(x_0)$ and $x_1^{\boldsymbol{E}}=P(x_0^{\boldsymbol{E}})$ shown in Fig.1. Using Taylor Expansion of P near the point x_0, we get $x_1^{\boldsymbol{E}}-x_1=\delta\Im(x_0)\cdot\boldsymbol{E}$; where $\Im(x)$ is the Jacobian matrix of P at the point x. Then the Local Lyapunov Exponent at the point x_0 along the direction unit vector $\boldsymbol{E}$ in the phase-space can be defined as:

$$\lambda_P^{\boldsymbol{E}}(x_0)=\lim_{\delta\to 0}\log\left(\left\|x_1^{\boldsymbol{E}}-x_1\right\|/\delta\right).$$

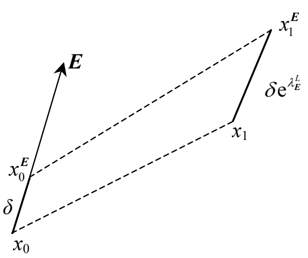

Fig.1 Evolution of a direction unit vector in the phase space of a map after one step

For simplicity, the norm $\|.\|$ is chosen to be the Euclidian norm defined on R^N. Hence,

$$\lambda_P^{\boldsymbol{E}}(x)=\log\left\|\Im(x)\cdot\boldsymbol{E}\right\| \tag{2}$$

From this relation, it is clear that $\lambda_P^{\boldsymbol{E}}(x)$ exists and is a real number for any point x and any unit direction vector $\boldsymbol{E}$.

Local Lyapunov Exponents for a flow

Let us consider a continuous N-dimensional dynamical system:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=F(x) \tag{3}$$

where $x\in$R^N and F:R^N→R^N. Starting with an initial condition $x=x_0$ at time $t=0$, we get a flow ϕ:R^N→R^N of system Eq.(3); where $\phi(0)=x_0$ and $\dfrac{\mathrm{d}\phi(t)}{\mathrm{d}t}=F(\phi(t))$.

Considering a very small time span $0<\mathrm{d}t<<1$, we may construct an orbit $\{x_i\}_{i=0}^{\infty}$; where $x_i=\phi(i\mathrm{d}t)$. This is an orbit of a map G:R^N→R^N associated with system Eq.(3) and defined by $x_{i+1}=G(x_i)=x_i+F(x_i)\mathrm{d}t$. The Jacobian of this map is $\Im_G(x)=\boldsymbol{I}+\Im(x)\mathrm{d}t$; where $\Im(x)$ is the Jacobian of F at the point x and $\boldsymbol{I}$ is the identity matrix.

Now, assume that $\mathrm{d}t=1/m$ for an integer m large enough and define the target map P:R^N→R^N as follows

$$x_{i+1}=P(x_i)=G^m(x_i)=(\underbrace{G\circ G\circ\ldots\circ G}_{m\text{ times}})(x_i).$$

The Jacobian of the target map P at the point x_0 can be given by

$$\Im_P(x_0)=\lim_{m\to\infty}(\boldsymbol{I}+\Im(x_{m-1})/m)(\boldsymbol{I}+\Im(x_{m-2})/m)\ldots(\boldsymbol{I}+\Im(x_0)/m)$$

where $x_i=x_{i-1}+F(x_{i-1})/m$; $i=1,2,\ldots,m$.

We define the Local Lyapunov Exponent of the given flow at the point x_0 along the direction unit vector $\boldsymbol{E}$ to be the same as that of the target map P, i.e.

$$\lambda_P^{\boldsymbol{E}}(x)=\lim_{m\to\infty}\log\left\|(\boldsymbol{I}+\Im(x_{m-1})/m)(\boldsymbol{I}+\Im(x_{m-2})/m)\ldots(\boldsymbol{I}+\Im(x_0)/m)\cdot\boldsymbol{E}\right\| \tag{4}$$

Fig.2 illustrates repeated evolutions of the perturbation vector $\delta\boldsymbol{E}$ in the phase-space over a unit of time.

Stationary Local Lyapunov Exponents

Relations Eqs.(2) and (4) show that, at any point x in the phase-space, the Local Lyapunov Exponent is

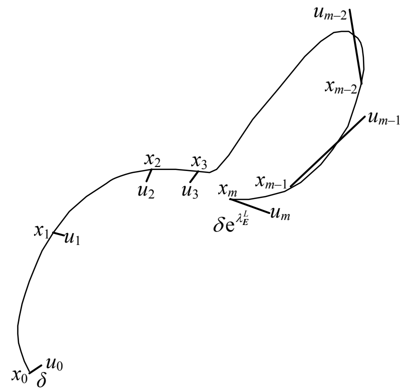

Fig.2 Evolution of a direction unit vector in the phase space of a flow over one unit of time

a function of the direction unit vector $\boldsymbol{E}$. Moreover, if the phase space is N-dimensional, where $N\geq 2$, then $\boldsymbol{E}$ is a function of $N-1$ variables (usually angular variables θ_i) which determine this vector in the phase-space. Therefore, solving the following set of $N-1$ equations

$$\frac{\partial \lambda_P^{\boldsymbol{E}}(x)}{\partial \theta_i}=0 \qquad (5)$$

we get the stationary directions from which we can calculate the Maximum Local Lyapunov Exponent $\lambda_P^{\max}(x)$ at the point x.

It is worth noticing that, if λ_1 is the Largest Lyapunov Exponent of a map P over an attractor A, then $n\lambda_1$ is the Largest Lyapunov Exponent of the map $P^n=\underbrace{P\circ P\circ \ldots \circ P}_{n \text{ times}}$ over the same attractor.

Examples

1. For one-dimensional map, the direction unit vector is one-dimensional (constant which is equal to one). Therefore, there is only one Stationary Local Lyapunov Exponent which will be regarded as the maximum. Hence, for a one dimensional map

$x_{i+1}=f(x_i)$; i=0,1,2,…,

we have

$$\lambda_f^{\max}(x_0)=\log\left|\frac{\mathrm{d}f}{\mathrm{d}x}(x_0)\right|; \quad \forall x_0\in \overline{B}.$$

2. For two-dimensional map, the phase-space is two-dimensional. Hence, the direction unit vector can be taken as $\boldsymbol{E}$=(cosθ,sinθ). Therefore, having a two-dimensional map

$$x_{i+1}=f(x_i,y_i), \qquad y_{i+1}=g(x_i,y_i); \qquad i=0,1,2,\ldots,$$

and after solving Eq.(5), we find

$$\lambda_P^{\max}(x,y)=\log\sqrt{(A+\sqrt{4C^2+B^2})/2};$$

where, $A=f_x^2+f_y^2+g_x^2+g_y^2$, $B=f_x^2-f_y^2+g_x^2-g_y^2$, and $C=f_xf_y+g_xg_y$.

CHARACTERISTICS OF MAXIMUM LOCAL EXPONENTS

Definition 1 Let A be a bounded attractor of a map P with a bounded subset B of its basin of attraction satisfying $B\supseteq A$ and $B\supseteq P(B)$. We may define

$$\chi_P(B)=\sup\{\lambda_P^{\max}(x);\ x\in\overline{B}\};$$

where $\overline{B}$ is the closure of B.

Since P is differentiable, $\lambda_{P^n}^{\max}(x)<\infty;\ \forall x\in\overline{B}$ and $\forall n\geq 1$. Hence, $\chi_{P^n}(B)<\infty;\ \forall n\geq 1$.

Note The condition $\det(\mathfrak{J}(x))\neq 0;\ \forall x\in\overline{B}$ is necessary and sufficient in order to have $\lambda_{P^n}^{\max}(x)>-\infty$; $\forall x\in\overline{B}$ and $\forall n\geq 1$. Therefore, from now on, we will assume that $\det(\mathfrak{J}(x))\neq 0;\ \forall x\in\overline{B}$ unless otherwise stated.

Let x_0 be any point from the basin of attraction of a given attractor A of a map P, the Largest Lyapunov Exponent of this map over the attractor is defined as $\lambda_1=\lim\limits_{n\to\infty}\frac{1}{n}\log\left|\mu_1(\mathfrak{J}_n(x_0))\right|$; where $\mathfrak{J}_n$ is the Jacobian of the map P^n and $\mu_1(\cdot)$ denotes the largest (in magnitude) eigenvalue of the concerning matrix (Chen and Dong, 1998), and it represents the rate of the exponential stretch in the direction in which the maximum stretch occurs. Therefore, we may write

$$\lambda_1 = \lim_{n\to\infty}\frac{1}{n}\lambda_{P^n}^{\max}(x_0).$$

Corollary 1 One may easily observe that $n\lambda_1 \le \chi_{P^n}(B);\ \forall n \ge 1$. Hence, if there exists an integer n_1 such that $\chi_{P^{n_1}}(B) < 0$, then the attractor A is non-chaotic. Moreover, if A is chaotic, then $\chi_{P^n}(B) > 0;\ \forall n \ge 1$.

Lemma 1 The Maximum Local Lyapunov Exponents of the maps P^n, n=1,2,3,…; at a point x_0 from the basin of attraction of an attractor A, will tend to be distributed (located) along both sides of a straight line of slope λ_1 in the (n–λ) plane. i.e. there exists an integer n_0, such that

$$\frac{1}{m}(\lambda_{P^{m+n_0}}^{\max}(x_0) - \lambda_{P^{n_0}}^{\max}(x_0)) \xrightarrow[m\to\infty]{} \lambda_1.$$

Proof For any ε>0, there exists an integer n_1≥1; such that

$$\left|\frac{1}{n}\lambda_{P^n}^{\max}(x_0) - \lambda_1\right| < \varepsilon;\quad \forall n \ge n_1 \tag{6}$$

Therefore,

$$\left|\frac{1}{m}(\lambda_{P^{n+m}}^{\max}(x_0) - \lambda_{P^n}^{\max}(x_0)) - \lambda_1\right| < \left(\frac{2n}{m}+1\right)\varepsilon;$$
$$\forall n \ge n_1 \text{ and } \forall m \ge 1 \tag{7}$$

Taking n=n_1 and m≥n_1, the right hand side of Eq.(7) can be made arbitrarily small which means that the slope of the line joining the point $(n_1, \lambda_{P^{n_1}}^{\max}(x_0))$ to the point $\left(n_1+m, \lambda_{P^{n_1+m}}^{\max}(x_0)\right)$ tends to λ_1 as m tends to infinity. Hence, the points $\lambda_{P^n}^{\max}(x_0)$; n≥2n_1 will be distributed along both sides of the line joining $(n_1, \lambda_{P^{n_1}}^{\max}(x_0))$ and $(2n_1, \lambda_{P^{2n_1}}^{\max}(x_0))$.

Lemma 2 An attractor A of a map P is chaotic (non-chaotic) if and only if, for any $x_0 \in B$ there exists an integer k such that $\lambda_{P^n}^{\max}(x_0) > 0$ (<0); $\forall n \ge k$.

Proof Using Eq.(7) and assuming $\varepsilon < |\lambda_1|$, n=n_1 and m≥n_1, we find:

Case 1 If the attractor is chaotic, i.e. λ_1>0, Eq.(7) implies that
Either:

$$\begin{aligned}\lambda_{P^{n_1}}^{\max}(x_0) + m\lambda_1 &\le \lambda_{P^{n_1+m}}^{\max}(x_0)\\ &< \lambda_{P^{n_1}}^{\max}(x_0) + m(\lambda_1+\varepsilon) + 2n_1\varepsilon\end{aligned} \tag{8}$$

and the left hand side of this inequality proves the lemma in this case after taking k=n_1+m_1; where m_1 is sufficiently large so that $\lambda_{P^{n_1}}^{\max}(x_0) + m_1\lambda_1 > 0$.

Or:

$$\begin{aligned}\lambda_{P^{n_1}}^{\max}(x_0) + m\lambda_1 &\ge \lambda_{P^{n_1+m}}^{\max}(x_0)\\ &> \lambda_{P^{n_1}}^{\max}(x_0) + m(\lambda_1-\varepsilon) - 2n_1\varepsilon\end{aligned} \tag{9}$$

and the right hand side of this inequality proves the lemma in this case after taking k=n_1+m_1; where m_1 is sufficiently large so that

$$\lambda_{P^{n_1}}^{\max}(x_0) + m_1(\lambda_1-\varepsilon) - 2n_1\varepsilon > 0.$$

Case 2 If the attractor is non-chaotic, i.e. λ_1<0, then

(i) For the first sub-case, the right hand side of Eq.(8) proves the lemma in this case after taking k=n_1+m_1; where m_1 is sufficiently large so that $\lambda_{P^{n_1}}^{\max}(x_0) + m_1(\lambda_1-\varepsilon) - 2n_1\varepsilon < 0$.

(ii) For the second sub-case, the left hand side of Eq.(9) proves the lemma in this case after taking k=n_1+m_1; where m_1 is sufficiently large so that $\lambda_{P^{n_1}}^{\max}(x_0) + m_1\lambda_1 < 0$.

The proof of the inverse case is trivial as we know that $\frac{1}{n}\lambda_{P^n}^{\max}(x_0) \xrightarrow[n\to\infty]{} \lambda_1$, and therefore, λ_1 will have the same sign of $\lambda_{P^n}^{\max}(x_0)$ as $n\to\infty$.

Note If $\det(\Im(x_0)) = 0$; for some $x_0 \in \overline{B}$, both Lemma 1 and Lemma 2 remain valid so long as x_0 is not the initial point of the orbit concerned. Moreover, $\det(\Im_{P^n}(x_0)) = 0;\ \forall n \ge 1$. i.e. $\lambda_{P^n}^{\max}(x_0) = -\infty;\ \forall n \ge 1$.

Examples

1. Logistic map

Many properties of this map were investigated earlier in (Kaplan and Glass, 1995). This map is given by $x_{i+1}=\alpha x_i(1-x_i)$; where $0\leq x_i\leq 1$ and $0\leq\alpha\leq 4$.

Fig.3 shows the chaotic attractor observed for the parameter value α=3.6, and the corresponding Maximum Local Lyapunov Exponents at the point x_0=0.2 for the composite maps P^n; n=1,2,…,1000. This figure's curve clearly resembles a straight line. To find an approximate value of λ_1 of the map for the given parameter's value, we calculated the slope of the line joining the points $(600, \lambda_{P^{600}}^{\max}(x_0))$ and $(1000, \lambda_{P^{1000}}^{\max}(x_0))$ in the graph and found that it was equal to the value of the Largest Lyapunov Exponent λ_1=0.1768 which was calculated in an ordinary way; where x_0=0.1. Accordingly, Fig.4 shows the same for a non-chaotic attractor of the logistic map observed for α=3.5; where the Largest Lyapunov Exponent was found to be λ_1=–0.8725 in this case.

2. Henon map

This is a map proposed by a French astronomer (Henon, 1976) and given by the following two equations of discrete form:

$$x_{i+1}=1-\alpha x_i^2+y_i,\quad y_{i+1}=\beta x_i.$$

When $|\beta|<1$, the map contracts, stretches and bends areas in the phase-plane, so it is called a horse-shoe map.

Fixing β=0.3, a typical chaotic attractor of this map for α=1.4 is depicted in Fig.5 along with the corresponding Maximum Local Lyapunov Exponents at a point x_0 of the maps P^n; n=1,2,…,1000. An additional straight line of slope λ_1=0.35569 equal to the Largest Lyapunov Exponent in this case is shown in the graph in order to show how the curve is distributed on both sides of that line. The initial point was taken as x_0=(–0.348,0.311). Accordingly, we plotted the same in Fig.6 for the non-chaotic attractor observed for α=1; in which this case the Largest Lyapunov Exponent was estimated λ_1=–0.161132 and the initial point was taken as x_0=(0.9517,–0.1969).

3. Lorenz system

Drazin (1992) detailedly investigated this system. It is a three-dimensional continuous flow given by the equations

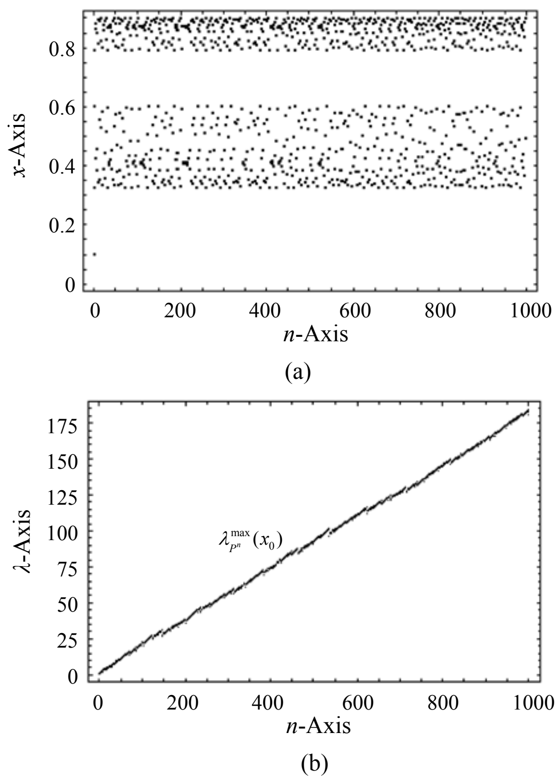

Fig.3 (a) Chaotic attractor of the logistic map for α=3.6; (b) The corresponding Maximum Local Lyapunov Exponents of P^n; n=1,2,…,1000

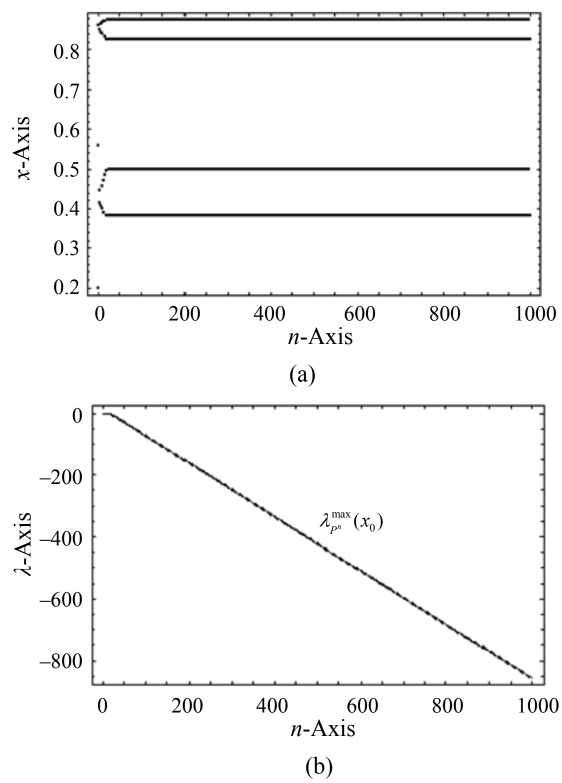

Fig.4 (a) Periodic attractor of the logistic map for α=3.5; (b) The corresponding Maximum Local Lyapunov Exponents of P^n; n=1,2,…,1000

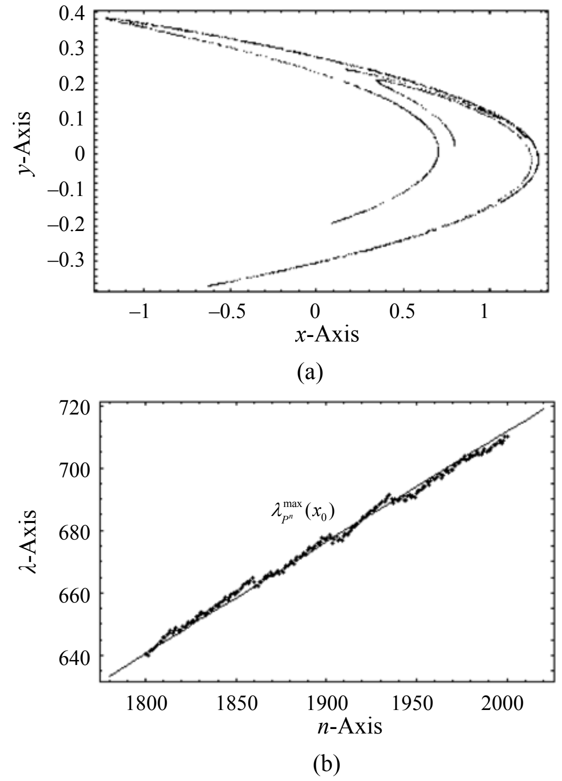

Fig.5 (a) Chaotic attractor of Henon map for α=1.4; (b) The corresponding Maximum Local Lyapunov Exponents of P^n; n=1,2,…,350

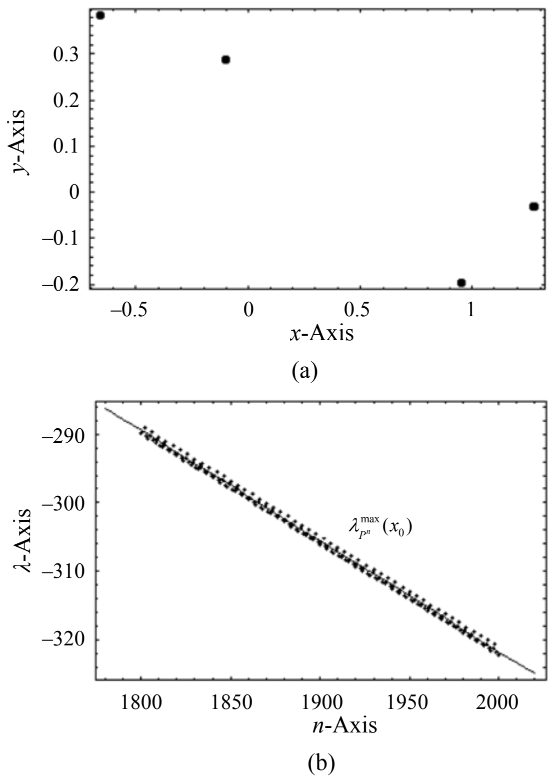

Fig.6 (a) Periodic attractor of Henon map for α=1; (b) The corresponding Maximum Local Lyapunov Exponents of P^n; n=1,2,…,350

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = p(y-x) \\ \frac{\mathrm{d}y}{\mathrm{d}t} = -xz + rx - y \\ \frac{\mathrm{d}z}{\mathrm{d}t} = xy - qz \end{cases}$$

where p, q, and r are system parameters.

Since any flow has one zero Lyapunov Exponent along the flow direction (i.e. the tangent $\boldsymbol{\tau}$), we may and without loss of generality, assume the direction unit vector to be in the plane normal to the flow at the given point, i.e. $\boldsymbol{E}=\cos(\theta)\boldsymbol{n}+\sin(\theta)\boldsymbol{b}$; where $\boldsymbol{n}$ and $\boldsymbol{b}$ are the normal and binormal unit vectors at the given point. After finding the direction unit vector $\boldsymbol{E}$, we proceed in the same way as described in subsection 2.2, then we calculate the Maximum Local Lyapunov Exponents at the given point of the maps P^n in the phase-space.

The idea of choosing $\boldsymbol{E}$ in this way is to reduce the number of trigonometric equations obtained from relation Eq.(5) which are required to be solved in order to obtain the Maximum Local Lyapunov Exponent at the given point.

Fig.7 shows the chaotic attractor of Lorenz system obtained for the parameters' values p=10, q=8/3 and r=27, along with the Maximum Local Lyapunov Exponents of the maps P^n; where P is the map associated with this flow. This figure's curve clearly resembles a straight line. To find an approximate value of λ_1 of the map for the given parameter's value, we calculated the slope of the line joining the points $(150, \lambda_{P^{150}}^{\max}(x_0))$ and $(300, \lambda_{P^{300}}^{\max}(x_0))$ in the graph and found it to be equal to the value of the Largest Lyapunov Exponent λ_1=0.895116 calculated by using the Mathematica package (Sandri, 1996); where the initial point was taken as x_0=(0.0001,0.0001,0.0001). Whereas, we depict in Fig.8 the same for a non-chaotic attractor obtained for the parameters' values p=10, q=8/3 and r=21; where in this case the largest Lyapunov exponent was calculated and found to be λ_1=−0.114787. The time step of integration in both cases was taken as dt=0.0001. Accordingly, the flow was discretised to yield the following formula:

$$\lambda_P^{\max}(x,y) = \log\sqrt{A+\sqrt{C^2+B^2}}\ ;$$

where $(\Im(x_0)\cdot\boldsymbol{E})^2 = B\cos(2\theta)+C\sin(2\theta)+A$, which was derived in order to calculate the maximum local Lyapunov exponents.

CHARACTERISTICS OF A FIXED POINT EMBEDDED WITHIN A CHAOTIC ATTRACTOR

Given the Euclidian norm $\|\cdot\|$ defined on R^N, one may define a distance function d:$\mathrm{R}^N\times\mathrm{R}^N\rightarrow\mathrm{R}^N$; where for any $x_1,x_2\in\mathrm{R}^N$, $d(x_1,x_2)=\|x_1-x_2\|$. Accordingly, given a point $x_1\in\mathrm{R}^N$ and a set $A\subseteq\mathrm{R}^N$, the distance between x_1 and A is defined by $d(x_1,A)=\inf\{d(x_1,x_2),x_2\in A\}$.

A point $x\in\mathrm{R}^N$ is said to be embedded within a set $A\subseteq\mathrm{R}^N$ if and only if $x\notin A$ and $d(x,A)=0$. In other words, any neighborhood of the point x in R^N must intersect with A provided that $x\notin A$.

Let $x^*\in\mathrm{R}^N$ be a fixed point of the map Eq.(1) and $A\subseteq\mathrm{R}^N$ be a chaotic attractor (if any) of this map, then $x^*\notin A$.

Note A fixed point x^* of a map P is stable if and only if $\lambda_P^{\boldsymbol{E}}(x^*)\leq 0$; for any direction unit vector $\boldsymbol{E}\in\mathrm{R}^N$. Moreover, this condition is equivalent to $\lambda_{P^n}^{\boldsymbol{E}}(x^*)\leq 0$; for any direction unit vector $\boldsymbol{E}\in\mathrm{R}^N$ and $\forall n=1,2,\ldots$

Corollary 2 A stable fixed point x^* of a dynamical system cannot be embedded within a chaotic attractor A of the same system.

Proof To the contrary, suppose that x^* is embedded within A. Choosing appropriately a sufficiently small neighborhood of the fixed point, all points in this neighborhood must stay in it under the sequential application of the map. On the other hand, this neighborhood must contain points from the chaotic attractor which will wander under the sequential application of the same map in the whole area containing

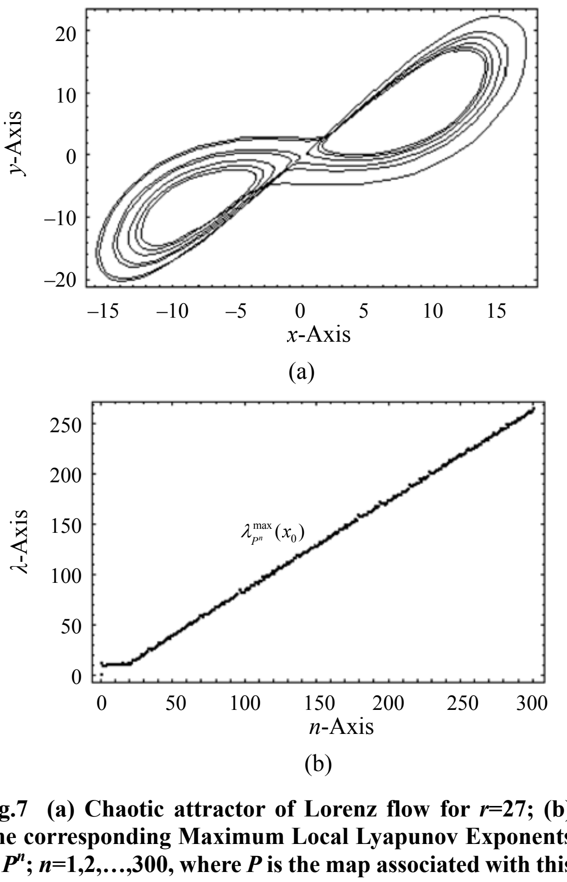

Fig.7 (a) Chaotic attractor of Lorenz flow for r=27; (b) The corresponding Maximum Local Lyapunov Exponents of P^n; n=1,2,…,300, where P is the map associated with this flow

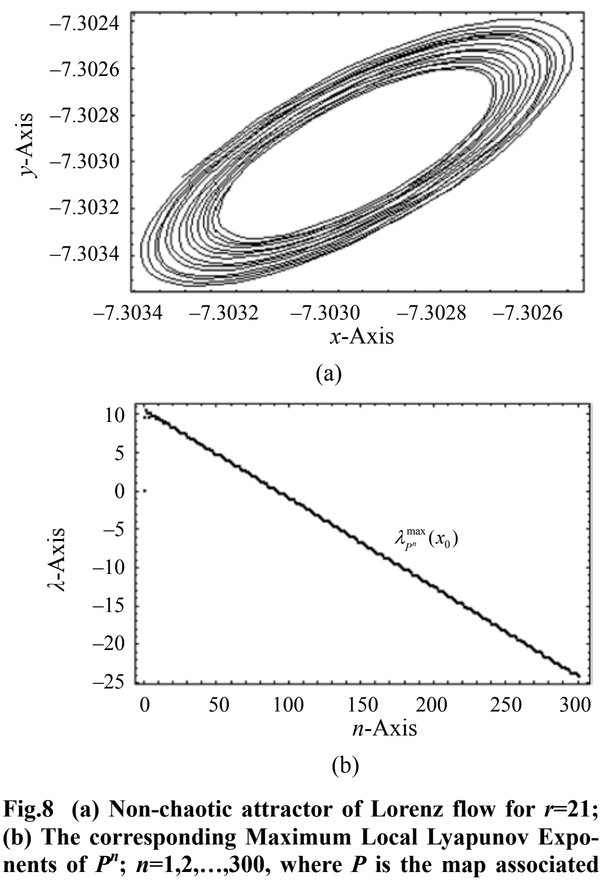

Fig.8 (a) Non-chaotic attractor of Lorenz flow for r=21; (b) The corresponding Maximum Local Lyapunov Exponents of P^n; n=1,2,…,300, where P is the map associated with this flow

this attractor. This is a contradiction which leads to the establishment of the proof.

Lemma 3 If a fixed point x^* of a map P is embedded within a chaotic attractor A of the same map, then there exist two direction unit vectors $\boldsymbol{E}_1,\boldsymbol{E}_2\in\mathrm{R}^N$; such that $\lambda_P^{\boldsymbol{E}_1}(x^*)\cdot\lambda_P^{\boldsymbol{E}_2}(x^*)<0$. Moreover, this condition is equivalent to $\lambda_P^{\max}(x^*)\cdot\lambda_P^{\min}(x^*)<0$.

Proof Using corollary Eq.(2), this fixed point is not stable, therefore there exists a direction unit vector $\boldsymbol{E}_1$; such that $\lambda_P^{\boldsymbol{E}_1}(x^*)>0$.

Let $\delta>0$ be an arbitrarily small real number and since x^* is embedded within the chaotic attractor A, there exists a point $y^*\in A$; such that

$$\left\| x^* - y^* \right\| < \frac{\delta}{2} \tag{10}$$

Now, for any unit vector $\boldsymbol{E}\in\mathrm{R}^N$ and since A is chaotic, the orbit initiated at $x^*+\delta\boldsymbol{E}$ should wander in the whole area containing A and will get arbitrarily closed to any point on A. Thus, there exists an arbitrarily large integer $n_0>1$; such that

$$\left\| y^* - P^{n_0}(x^*+\delta\boldsymbol{E}) \right\| < \frac{\delta}{2} \tag{11}$$

Relations Eqs.(10) and (11) imply that

$$\lambda_{P^{n_0}}^{\boldsymbol{E}}(x^*) = \log\frac{\left\| P^{n_0}(x^*) - P^{n_0}(x^*+\delta\boldsymbol{E}) \right\|}{\delta} < 0 .$$

Since x^* is a fixed point, then $\Im_{P^{n_0}}(x^*) = (\Im_P(x^*))^{n_0}$. Using relation Eq.(2), we get

$$\log\left\|\Im_P(x^*)\cdot(((\Im_P(x^*))^{n_0-1}\boldsymbol{E})\right\| < 0 .$$

Hence, $\lambda_P^{\boldsymbol{E}_2}(x^*)>0$; where $\boldsymbol{E}_2=((\Im_P(x^*))^{n_0-1}\boldsymbol{E}$. This proves the first part of this lemma. The proof of the second part is trivial.

Note Let x^* be a periodic point of period n_T of a map P, then x^* is a fixed point of the composite map P^{n_T}. Therefore, if x^* is embedded within a chaotic attractor A of the same map, there exist two direction unit vectors $\boldsymbol{E}_1,\boldsymbol{E}_2\in\mathrm{R}^N$; such that $\lambda_{P^{n_T}}^{\boldsymbol{E}_1}(x^*)\cdot\lambda_{P^{n_T}}^{\boldsymbol{E}_2}(x^*)<0$. Moreover, this condition is equivalent to $\lambda_{P^{n_T}}^{\max}(x^*)\cdot\lambda_{P^{n_T}}^{\min}(x^*)<0$.

Corollary 3 Since there is only one direction unit vector in the phase space of one-dimensional map, no fixed points or periodic orbits can be embedded within a chaotic attractor of any one-dimensional map.

CONCLUSION

The approach triggered by Lemma 1 for calculating the Largest Lyapunov Exponent λ_1, by calculating the slope of the approximately straight line in the $(n\text{–}\lambda)$ plane around which the values of $\lambda_{P^n}^{\max}(x_0)$; for n=1,2,3,… are distributed, makes λ_1 more flexible for use in ordinary calculations.

However, an appropriate choice of the initial point makes the integer k (proposed in Lemma 2) smaller. Practically, we noticed during the calculation of $\lambda_{P^n}^{\max}(x_0)$ that this integer k is bounded for a dynamical system. This makes it possible to construct a graph, like the one in Fig.9, to analyze the behavior of a dynamical system for a range of parameter values. This figure shows the values of $\lambda_{P^{21}}^{\max}(x_0)$ for Henon map corresponding to different parameter's values in the interval $\alpha\in[0.5,1.5]$. The corresponding values of the Largest Lyapunov Exponent λ_1 of the map can be approximated later using the formula $\lambda_1\approx\lambda_{P^{21}}^{\max}(x_0)/21$. The positive (negative) values of $\lambda_{P^{21}}^{\max}(x_0)$ correspond to the values of the parameter α for which the attractor is chaotic (non-chaotic).

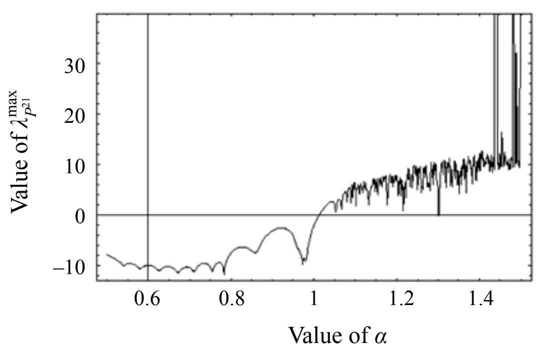

Fig.9 Estimation of the largest Lyapunov exponent for Henon map at the fiftieth iteration for a range of its parameter values $\alpha\in$[0.5,1.5]

By finding the characteristics of fixed and periodic points embedded within a chaotic attractor of a map, we established necessary conditions for such points to be embedded within a chaotic attractor. However, efforts should be directed towards finding sufficient conditions to guarantee the existence of a chaotic attractor in which a fixed or periodic point is embedded.

In most of the recent stabilization techniques like OGY technique (Ott *et al*., 1990), the existence of fixed points and periodic orbits embedded within a chaotic attractor is the basic idea for stabilization and chaos elimination. Therefore, the problem of stabilizing a chaotic system where no fixed points or periodic orbits are embedded within its chaotic attractor remains open and efforts should be made to modify those techniques accordingly.

Finally, as we did not encounter bad conditioned matrix product during the calculation of maximum local Lyapunov exponents of Lorenz flow, we did not investigate this issue in this paper, but will target this issue in our forthcoming papers.

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