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One-parameter quasi-filled function algorithm for nonlinear integer programming*

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Abstract: A definition of the quasi-filled function for nonlinear integer programming problem is given in this paper. A quasi-filled function satisfying our definition is presented. This function contains only one parameter. The properties of the proposed quasi-filled function and the method using this quasi-filled function to solve nonlinear integer programming problem are also discussed in this paper. Numerical results indicated the efficiency and reliability of the proposed quasi-filled function algorithm.

Key words: Integer programming, Local minimizer, Global minimizer, Filled function, Global optimization

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INTRODUCTION

We consider the following nonlinear integer programming problem

$$(\mathrm{P_I}) \quad \min f(x), \quad \text{s.t. } x \in X_I \tag{1}$$

where $X_I \subset I^n$ is a bounded and closed box set containing more than one point; I^n is the set of integer points in R^n.

Notice that the formulation in ($\mathrm{P_I}$) allows the set X_I to be defined by equality constraints as well as inequality constraints. Furthermore, when $f(x)$ is coercive, i.e., $f(x) \to +\infty$ as $\|x\| \to \infty$, there always exists a box which contains all discrete global minimizers of $f(x)$, thus constituting an unconstrained nonlinear integer programming problem

$$(\mathrm{UP_I}) \quad \min f(x), \quad \text{s.t. } x \in I^n \tag{2}$$

that can be reduced into an equivalent problem formulation in ($\mathrm{P_I}$). In other words, both unconstrained and constrained nonlinear integer programming problem can be considered in ($\mathrm{P_I}$).

PRELIMINARIES

Now, we recall some definitions involved in nonlinear integer programming problem.

Definition 1 (Zhu, 2000) For any $x \in I^n$, the neighborhood of x is defined by $N(x)=\{x, x \pm e_i: i=1, 2, \ldots, n\}$, where e_i is the n-dimensional vector with the ith component equal to one and other components equal to zero. Let $N^0(x)=N(x)\backslash\{x\}$.

Definition 2 (Zhu, 2000) An integer point $x_0 \in X_I$ is called a local minimizer of $f(x)$ over X_I if there exists a neighborhood $N(x_0)$ for any $x \in N(x_0) \cap X_I$, holds for $f(x) \geq f(x_0)$; an integer point $x_0 \in X_I$ is called a global minimizer of $f(x)$ over X_I if for any $x \in X_I$, holds for $f(x) \geq f(x_0)$. In addition, if $f(x) > f(x_0)$ for all $x \in N^0(x_0) \cap X_I$ ($x \in X_I \backslash \{x_0\}$), then x_0 is called a strictly local (global)

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minimizer of $f(x)$ over X_I.

Theorem 1 (Zhang *et al*., 1999) If $x_0 \in X_I$ is a global minimizer of $f(x)$ over X_I, then $x_0 \in X_I$ must be a local minimizer of $f(x)$ over X_I.

The local minimizer of $f(x)$ over X_I is obtained by using following Algorithm 1 (Zhu, 2000).

Algorithm 1 (Zhu, 2000)

Step 1: Choose any integer $x_0 \in X_I$.

Step 2: If x_0 is a local minimizer of $f(x)$ over X_I, then stop; otherwise, we can obtain a $x \in N(x_0) \cap X_I$, have $f(x) < f(x_0)$.

Step 3: Let $x_0 := x$, go to Step 2.

A QUASI-FILLED FUNCTION AND ITS PROPERTIES

In this section, we propose a quasi-filled function of $f(x)$ at a current local minimizer x_1^* and will discuss its properties. Let x_1^* be the current local minimizer of $f(x)$ (obtained by Algorithm 1 (Zhu, 2000)). Let

$$S_1 = \{x \in X_I : f(x) \geq f(x_1^*)\} \subset X_I$$
$$S_2 = \{x \in X_I : f(x) < f(x_1^*)\} \subset X_I .$$

Definition 3 $P_{x_1^*}(x)$ is called a quasi-filed function of $f(x)$ at a local minimizer x_1^* for nonlinear integer programming problem if $P_{x_1^*}(x)$ has the following properties:

(i) $P_{x_1^*}(x)$ has no local minimizer in the set $S_1 \backslash \{x_0\}$. The prefixed point x_0 is in the S_1;

(ii) If x_1^* is not a global minimizer of $f(x)$, then there exists a local minimizer x_1 of $P_{x_1^*}(x)$, such that $f(x_1) < f(x_1^*)$, that is, $x_1 \in S_2$.

Definition 3 is different from that of the filled function in (Ge, 1990; Ge and Qin, 1990; Zhu, 2000; Lucid and Piccialli, 2002); Definition 3 based on the discrete set in the Euclidean space and x_0 is not necessarily the local minimizer of $P_{x_1^*}(x)$.

Similar to (Zhu, 2003), we present a one-parameter quasi-filled function of $f(x)$ at local minimizer x_1^* as follows:

$$P_{A,x_1^*,x_0}(x) = \eta(\|x - x_0\|)$$
$$-\varphi(A \cdot (\exp([\min\{f(x) - f(x_1^*), 0\}]^2) - 1)) \quad (3)$$

where $A>0$ is a parameter, prefixed point $x_0 \in X_I$ satisfying $f(x_0) \geq f(x_1^*)$.

$\eta(t)$ and $\varphi(t)$ must satisfy the following conditions:

(a) $\eta(t)$ and $\varphi(t)$ are strictly monotonously increasing function for any $t \in [0, +\infty)$;

(b) $\eta(0)=0$ and $\varphi(0)=0$;

(c) $\varphi(t) \to C > B \geq \max_{x \in X_I} \eta(\|x - x_0\|)$ as $x \to +\infty$.

In the following we will prove that the above constructed function $P_{A,x_1^*,x_0}(x)$ satisfies the conditions (i) and (ii) of Definition 3, i.e., it is a quasi-filled function of $f(x)$ at a local minimizer x_1^* satisfying Definition 3. First, we give a Lemma 1 as follows:

Lemma 1 For any integer point $x \in X_I$, if $x \neq x_0$, then there exists a $d \in D = \{\pm e_i : i=1, 2, \ldots, n\}$ such that

$$\|x + d - x_0\| < \|x - x_0\| \quad (4)$$

Proof Since $x \neq x_0$, there exists an $i \in \{1, 2, \ldots, n\}$ such that $x_i \neq x_{0i}$. If $x_i > x_{0i}$, then $d = -e_i$. On the other hand, if $x_i < x_{0i}$, then $d = e_i$.

Theorem 2 $P_{A,x_1^*,x_0}(x)$ has no local minimizer in the integer set $S_1 \backslash \{x_0\}$ for any $A>0$.

Proof For any $x \in S_1$ and $x \neq x_0$, by using Lemma 1 we know there exists a $d \in D$, such that

$$\|x + d - x_0\| < \|x - x_0\|$$

Consider the following two cases:

(1) If $f(x_1^*) \leq f(x+d) \leq f(x)$ or $f(x_1^*) \leq f(x) \leq f(x+d)$, then

$$P_{A,x_1^*,x_0}(x+d) = \eta(\|x + d - x_0\|)$$
$$-\varphi(A \cdot (\exp([\min\{f(x+d) - f(x_1^*), 0\}]^2) - 1))$$
$$= \eta(\|x + d - x_0\|) < \eta(\|x - x_0\|) = P_{A,x_1^*,x_0}(x)$$

Therefore, x is not a local minimizer of function $P_{A,x_1^*,x_0}(x)$.

(2) If $f(x+d) < f(x_1^*) \le f(x)$, then

$$P_{A,x_1^*,x_0}(x+d) = \eta(\|x+d-x_0\|) - \varphi(A\cdot(\exp([\min\{f(x+d)-f(x_1^*),0\}]^2)-1))$$
$$= \eta(\|x+d-x_0\|) - \varphi(A\cdot(\exp([f(x+d)-f(x_1^*)]^2)-1))$$
$$\le \eta(\|x+d-x_0\|) < \eta(\|x-x_0\|) = P_{A,x_1^*,x_0}(x)$$

Therefore, it is shown that x is not a local minimizer of function $P_{A,x_1^*,x_0}(x)$.

By Theorem 2, we know that the constructed function $P_{A,x_1^*,x_0}(x)$ satisfies the first property of Definition 3 without any further assumption on the parameter A.

Since $X_I = S_1 \cup S_2$, Theorem 2 implies the following corollary.

Corollary 1 Any local minimizers of function $P_{A,x_1^*,x_0}(x)$ except x_0 must be in the integer set S_2.

However, if $A=0$, then $P_{A,x_1^*,x_0}(x) = \eta(\|x-x_0\|)$ has a unique local minimizer x_0 in the X_I. Since $f(x_0) \ge f(x_1^*)$, that is, $x_0 \in S_1$, $P_{A,x_1^*,x_0}(x)$ has no local minimizers in the set S_2 and $P_{A,x_1^*,x_0}(x)$ is not a quasi-filled function of $f(x)$ at a local minimizer x_1^*. So we have a question of how large the parameter A would be such that a local minimizer can be in the set S_2. In fact, we have the following theorem.

Theorem 3 $P_{A,x_1^*,x_0}(x)$ has local minimizer in the integer set S_2 if $S_2 \ne \varnothing$ and $A>0$ satisfies the following condition:

$$A > \frac{\varphi^{-1}(B)}{\exp([f(x^*)-f(x_1^*)]^2)-1} \tag{5}$$

where x^* is a global minimizer of $f(x)$.

Proof Since $S_2 \ne \varnothing$ and x^* is a global minimizer of $f(x)$, we have $f(x^*) < f(x_1^*)$, and

$$P_{A,x_1^*,x_0}(x^*) = \eta(\|x^*-x_0\|) - \varphi(A\cdot(\exp([\min\{f(x^*)-f(x_1^*),0\}]^2)-1))$$
$$= \eta(\|x^*-x_0\|) - \varphi(A\cdot(\exp([f(x^*)-f(x_1^*)]^2)-1))$$
$$\le B - \varphi(A\cdot(\exp([f(x^*)-f(x_1^*)]^2)-1))$$

when $A>0$ and satisfies Eq.(5), we have $P_{A,x_1^*,x_0}(x) < 0$.

On the other hand, for any $y \in S_1$, we have

$$P_{A,x_1^*,x_0}(y) = \eta(\|y-x_0\|) - \varphi(A\cdot(\exp([\min\{f(y)-f(x_1^*),0\}]^2)-1))$$
$$= \eta(\|y-x_0\|) \ge 0$$

Therefore, the global minimizer of $P_{A,x_1^*,x_0}(x)$ must exist in the set S_2. By Theorem 1 we know that $P_{A,x_1^*,x_0}(x)$ has local minimizer in the set S_2.

In summary, by Theorems 2 and 3, if parameter A is large enough then the constructed function $P_{A,x_1^*,x_0}(x)$ does satisfy the conditions of Definition 3. i.e., function $P_{A,x_1^*,x_0}(x)$ is a quasi-filled function.

However, we know the value of $f(x_1^*)$, and generally we do not know the global minimal value or global minimizer of $f(x)$, so it is difficult to find the lower bound of parameter A in Theorem 3.

But for practical consideration, problem (P_I) might be solved if we can find an $x \in X_I$ such that $f(x) < f(x^*) + \varepsilon$, where $f(x^*)$ is the global minimal value of problem (P_I), and ε is a given desired optimality tolerance. So we consider the case when the current local minimizer x_1^* satisfies $f(x_1^*) \ge f(x^*) + \varepsilon$. In the following Theorem 4 we develop a lower bound of parameter A which depends only on the given optimality tolerance ε.

Theorem 4 Suppose that ε is a small positive constant, and $A > \frac{\varphi^{-1}(B)}{\exp(\varepsilon^2)-1}$. Then for any current local minimizer x_1^* of $f(x)$ such that $f(x_1^*) \ge f(x^*) + \varepsilon$, quasi-filled function $P_{A,x_1^*,x_0}(x)$ has local minimizer in the set S_2, where x^* is a global minimizer of $f(x)$.

Proof Since $\exp(t)-1$ is a strictly monotonously increasing function for any $t \in [0, +\infty]$ and $f(x_1^*) -$

$f(x^*)\geq\varepsilon$, we have

$$\exp([f(x^*)-f(x_1^*)]^2)-1\geq\exp(\varepsilon^2)-1,$$

that is

$$\frac{\varphi^{-1}(B)}{\exp([f(x^*)-f(x_1^*)]^2)-1}\leq\frac{\varphi^{-1}(B)}{\exp(\varepsilon^2)-1}$$

Hence, if

$$A>\frac{\varphi^{-1}(B)}{\exp(\varepsilon^2)-1},$$

then

$$A>\frac{\varphi^{-1}(B)}{\exp([f(x^*)-f(x_1^*)]^2)-1},$$

and by Theorem 3, the conclusion of this Theorem holds.

About prefixed point $x_0\in S_1$, we have the following property.

Theorem 5 The prefixed point $x_0\in S_1$ is a local minimizer of $P_{A,x_1^*,x_0}(x)$ if $x_0\in S_1$ is a local minimizer of $f(x)$.

Proof Since $x_0\in S_1$ is a local minimizer of $f(x)$, there exists a neighborhood $N(x_0)$, for any $x\in N(x_0)\cap X_I$ we have $f(x)\geq f(x_0)\geq f(x_1^*)$, therefore

$$\begin{aligned}P_{A,x_1^*,x_0}(x)&=\eta(\|x-x_0\|)\\&\quad-\varphi(A\cdot(\exp([\min\{f(x)-f(x_1^*),0\}]^2)-1))\\&=\eta(\|x-x_0\|)\geq\eta(\|x_0-x_0\|)=P_{A,x_1^*,x_0}(x_0)\end{aligned}$$

holds for any $x\in N(x_0)\cap X_I$. That is, $x_0\in S_1$ is a local minimizer of $P_{A,x_1^*,x_0}(x)$.

We construct the following auxiliary nonlinear integer programming problem (AP_I) related to the problem (P_I):

$$(AP_I)\quad \min P_{A,x_1^*,x_0}(x),\quad \text{s.t. } x\in X_I \tag{6}$$

According to the above discussions, given any desired tolerance $\varepsilon>0$, if $A>\frac{\varphi^{-1}(B)}{\exp(\varepsilon^2)-1}$, then $P_{A,x_1^*,x_0}(x)$ is a quasi-filled function of $f(x)$ at its current local minimizer x_1^* which satisfies that $f(x_1^*)\geq f(x^*)+\varepsilon$. Thus if we use a local minimization method to solve problem (AP_I) from any initial point on X_I, then by the properties of quasi-filled function, it is obvious that the minimization sequence converges either to the prefixed point x_0 or to a point $x'\in X_I$ such that $f(x')<f(x_1^*)$. If we find such an x', then using a local minimization method to minimize $f(x)$ on X_I from initial point x', we can find a point $x''\in X_I$ such that $f(x'')<f(x')$ which is better than x_1^*. This is the main idea of the algorithm presented in the next section to find an approximate global minimal solution of problem (P_I).

ALGORITHM AND NUMERICAL RESULTS

Based on the theoretical results in the previous section and similar to (Zhu, 2003), a global optimization quasi-filled function algorithm over X_I is proposed as follows.

Algorithm 2 (The quasi-filled function method)

Step 1: Given a constant $N_L>0$ as the tolerance parameter for terminating the minimization process of problem (P_I) and a small constant $\varepsilon>0$ as a desired optimality tolerance; choose any integer $x_0\in X_I$.

Step 2: Obtain a local minimizer x_1^* of $f(x)$ by implementing Algorithm 1 (Zhu, 2000) starting from x_0.

Step 3: Construct the quasi-filled function $P_{A,x_1^*,x_0}(x)$ as follows:

$$\begin{aligned}P_{A,x_1^*,x_0}(x)&=\eta(\|x-x_0\|)\\&\quad-\varphi(A\cdot(\exp([\min\{f(x)-f(x_1^*),0\}]^2)-1))\end{aligned}$$

where $A>0$ and satisfying condition Eq.(5) or $A>\frac{\varphi^{-1}(B)}{\exp(\varepsilon^2)-1}$. Let $N=0$.

Step 4: If $N>N_L$, then go to Step 7.

Step 5: Set $N=N+1$. Draw an initial point on the boundary of the X_I, and start from it to minimize $P_{A,x_1^*,x_0}(x)$ on X_I using any local minimization method. Suppose that x' is an obtained local minimizer of

$P_{A,x_1^*,x_0}(x)$. If $x'=x_0$, then go to Step 4, otherwise go to Step 6.

Step 6: Minimize $f(x)$ on the X_I from the initial point x', and obtain a local minimizer x_2^* of $f(x)$. Let $x_1^*=x_2^*$ and go to Step 3.

Step 7: Stop the algorithm, output x_1^* and $f(x_1^*)$ as an approximate global minimal solution and global minimal value of problem (P_I) respectively.

Although the focus of this paper is more theoretical than computational, we still test our algorithm on several global minimization problems to have an initial feeling of the practical value of the quasi-filled function algorithm.

Example

$$\min f(x)=(x_1-1)^2+(x_n-1)^2+n\sum_{i=1}^{n-1}(n-i)(x_i^2-x_{i+1})^2$$
$$\text{s.t. } |x_i|\le 5,\ x_i \text{ integer}, i=1, 2, \ldots, n.$$

This problem is a box constrained nonlinear integer programming problem. It has 11^n feasible points and many local minimizers (4, 6, 7, 10 and 12 local minimizers for n=2, 3, 4, 5 and 6, respectively), but only one global minimum solution: $x_{\text{global}}^*=(1,1,\ldots,1)$ with $f(x_{\text{global}}^*)=0$ for all n. We considered three cases of the problem: n=2, 3 and 5. There were about 1.21×10^2, 1.331×10^3, 1.611×10^5 feasible points, for n=2, 3, 5, respectively.

In the following, the proposed solution algorithm is programmed in MATLAB 6.5.1 Release for working on the Windows XP system with 900 MHz CPU. The MATLAB 6.5.1 subroutine is used as the local neighborhood search scheme to obtain local minimizers of $f(x)$ in Step 2 and the local minimizers of $P_{A,x_1^*,x_0}(x)$ in Step 5. We choose $\eta(t)=t$, $\varphi(t)=t$, so the function $P_{A,x_1^*,x_0}(x)$ is as follows:

$$P_{A,x_1^*,x_0}(x)=\|x-x_0\| -A\cdot(\exp([\min\{f(x)-f(x_1^*),0\}]^2)-1)$$

where let ε=0.05, and $A=\frac{B}{\exp(\varepsilon^2)-1}+1$, $B=\max_{x\in X_I}\eta(\|x-x_0\|)+1=10\sqrt{n}+1$, the tolerance parameter $N_L=10^n$. n is the variable number of $f(x)$.

The partial main of the computational process for the numerical example are summarized in Tables 1, 2, and 3 for n=2, 3, 5, respectively. The symbols used are as follows:

n: the number of variables; T_S: the number of initial points to be chosen; k: the times for the local minimization process of the problem (P_I); x_{ini}^k: the initial point for the kth local minimization process of problem (P_I); $x_{f-l_0}^k$: the minimizer for the kth local minimization process of problem (P_I); $f(x_{f-l_0}^k)$: the minimum of the $x_{f-l_0}^k$; $x_{p-l_0}^k$: the minimizer for the kth local minimization process of problem (AP_I); $f(x_{p-l_0}^k)$: the minimum of the $x_{p-l_0}^k$; *QIN*: the iteration number for the kth local minimization process of problem (AP_I).

Table 1 Results of numerical example, n=2, ε=0.05, A=6.0503e+003, B=$10\sqrt{2}$ +1, N_L=10^2+1

T_S	k	x_{ini}^k	$x_{f-l_0}^k$	$f(x_{f-l_0}^k)$	$x_{p-l_0}^k$	$f(x_{p-l_0}^k)$	*QIN*
1	1	(4,3)	(2,3)	7	(1,2)	3	2
	2	(1,2)	(1,1)	0			≥10^2+1
2	1	(−5,−3)	(0,0)	2	(1,1)	0	0
	2	(1,1)	(1,1)	0			≥10^2+1
3	1	(−4,3)	(−2,3)	15	(1,1)	0	1
	2	(1,1)	(1,1)	0			≥10^2+1
4	1	(0,−2)	(0,0)	2	(1,1)	0	1
	2	(1,1)	(1,1)	0			≥10^2+1

Table 2 Results of numerical example, n=3, ε=0.05, A=7.3200e+003, B=$10\sqrt{3}+1$, N_L=10^3+1

T_S	k	x_{ini}^k	$x_{f-l_0}^k$	$f(x_{f-l_0}^k)$	$x_{p-l_0}^k$	$f(x_{p-l_0}^k)$	QIN
1	1	(3,3,3)	(1,2,3)	13	(1,1,1)	0	0
	2	(1,1,1)	(1,1,1)	0			≥10^3+1
2	1	(−4,0,4)	(−1,2,3)	17	(−1,1,1)	4	4
	2	(−1,1,1)	(0,0,0)	2	(1,1,1)	0	2
	3	(1,1,1)	(1,1,1)	0			≥10^3+1
3	1	(0,4,4)	(1,2,3)	13	(1,1,1)	0	1
	2	(1,1,1)	(1,1,1)	0			≥10^3+1
4	1	(−1,4,2)	(−1,1,1)	4	(1,1,1)	0	0
	2	(1,1,1)	(1,1,1)	0			≥10^3+1

Table 3 Results of numerical example, n=5, ε=0.05, A=9.3336e+003, B=$10\sqrt{5}+1$, N_L=10^5+1

T_S	k	x_{ini}^k	$x_{f-l_0}^k$	$f(x_{f-l_0}^k)$	$x_{p-l_0}^k$	$f(x_{p-l_0}^k)$	QIN
1	1	(0,0,2,0,2)	(0,0,0,0,0)	2	(1,1,1,1,1)	0	1
	2	(1,1,1,1,1)	(1,1,1,1,1)	0			≥10^5+1
2	1	(−2,2,0,1,1)	(−1,1,1,1,1)	4	(0,0,0,0,0)	2	8
	2	(0,0,0,0,0)	(0,0,0,0,0)	2	(1,1,1,1,1)	0	4
	3	(1,1,1,1,1)	(1,1,1,1,1)	0			≥10^5+1
3	1	(0,3,0,3,3)	(1,1,1,2,3)	19	(1,1,1,1,1)	0	0
	2	(1,1,1,1,1)	(1,1,1,1,1)	0			≥10^5+1

CONCLUSION

This paper gives a definition of the quasi-filled function for nonlinear integer programming problem, and presents a quasi-filled function which has only one parameter. A quasi-filled function algorithm based on the given quasi-filled function was designed. Numerical results indicated the efficiency and reliability of the proposed quasi-filled function algorithm.

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