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Monotone routing in multirate rearrangeable $\log_d(N,m,p)$ network

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Abstract: The construction of multirate rearrangeable network has long been an interesting problem. Of many results published, all were achieved on 3-stage Clos network. The monotone routing algorithm proposed by Hu *et al.*(2001) was also first applied to 3-stage Clos network. In this work, we adopt this algorithm and apply it to $\log_d(N,m,p)$ networks. We first analyze the properties of $\log_d(N,m,p)$ networks. Then we use monotone algorithm in $\log_d(N,0,p)$ network. Furthermore we extend the result to construct multirate rearrangeable networks based on $\log_d(N,m,p)$ network ($1\leq m\leq n-1$).

Key words: Rearrangeable, Multirate, Monotone routing, $\log_d(N,m,p)$
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INTRODUCTION

Switching networks have been widely used in telecommunication, data communication, satellite communication, optical fiber network, etc. It was first proposed to meet the need to interconnect pairs of telephone users. When a telephone user wants to call another one, a request is generated. In order to satisfy all the requests, people first try to fully connect the callers (inputs) and receivers (outputs) who want to communicate. Later Clos (1953) showed that through some clever design supported by mathematical principles, there exist nonblocking networks with significantly less hardware than a network with dedicated lines. In traditional telephone usage, an input (output) is engaged in only one request at a given time, and this is called the classical model. With the emergence of the new technology, it becomes beneficial to integrate different types of networks such as audio, data and video into one switching network. Then each request is associated with a weight (rate, or bandwidth requirement) while an input (output) can be engaged in many requests as long as the sum of weights is within the capacity of a link, which is usually normalized to be 1. This is called the multirate model. Before introducing the definitions of nonblocking, we first present the concept of network state.

A network state is a set of paths connecting a set of requests $\{(i_x, o_y, w)\}$ such that no link carries a load exceeding 1, where i_x is an input, o_y is an output and w is the associated weight. Given a state, a new request (i, o, w) must satisfy the condition that i has not generated and o has not received requests whose total weights are more than $1-w$.

A network is strictly nonblocking if at any state a new request can always be connected without any link carrying a load exceeding 1. A weaker nonblocking property is called rearrangeable in which any set of requests can be routed in an empty network. Wide-sense nonblocking means the connection of the current request is assured only when all connections are routed according to a given algorithm. More detailed description can be found in Hwang (1998)'s book.

The current research of rearrangeable nonblocking is focused on multirate model, which was first introduced by Melen and Turner (1989). From then on, many researches have been conducted on the

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3-stage Clos network.

Chung and Ross (1991) conjectured that $C(n, 2n-1, r)$ is multirate rearrangeable for restricted traffic; that is, all the requests have weights selected from a finite set $p_1, p_2,\ldots, p_k$ where $1\geq p_1>p_2>\ldots>p_k>0$ and p_i is an integer multiple of p_k for $1\leq i\leq k-1$.

Du *et al*.(1999) obtained an upper bound and a lower bound for $m(n,r)$, which is the number of middle stage switches and is sufficient to guarantee the multirate rearrangeability of $C(n, m, r)$. They proved that

$$\lceil 11n/9 \rceil \leq m(n,r) \leq 41n/16 + O(1).$$

Lin *et al*.(1999) showed that the Chung-Ross conjecture holds given a restricted discrete bandwidth where the rates satisfy the semi-nested condition, i.e. $w_1>w_2>\ldots>w_{i-1}>1/2\geq w_i>w_{i+1}>\ldots>w_k$ and w_j is an integer multiple of w_{j+1} for $i\leq j\leq k-1$.

Hu *et al*.(2001) studied the monotone routing algorithm and proved that using this strategy

$$m(n,r)\leq 2n+1,\ n=2,3,4$$
$$m(n,r)\leq 2n+3,\ n=5,6$$

Ngo (2003) proposed the grouping algorithm and obtained the following result

$$m(n,r) \leq \left\lfloor 2n + r - \frac{n+r}{2^{k-1}} \right\rfloor.$$

Ngo and Vu (2003) improve further both the lower bound and upper bound. They obtained

$$\lceil 5n/4 \rceil \leq m(n,r) \leq 2n - 1 + \lceil (r-1)/2 \rceil.$$

Clos network is one of the most widely used switching networks, although there are also many other multistage interconnecting networks that have been universally applied, such as $BY_d^{-1}(n,m)$, $\log_d(N,m,p)$, etc. In this paper we make an attempt to extend the monotone routing algorithm proposed by Hu *et al*.(2001) to $\log_d(N,m,p)$ network. In the second section, we show some properties of $\log_d(N,m,p)$ network, and present the main results in the third section.

$\log_d(N,m,p)$ NETWORK

$\log_d(N,0,p)$ network was first proposed by Lea (1991). Then Shyy and Lea (1991) extended it to $\log_d(N,m,p)$ network. The $\log_d(N,m,p)$ network has an input (output) stage consisting of $N=d^n$ $1\times p$ $(p\times 1)$ crossbars, and p copies of d-nary m-extra-stage, $1\leq m\leq n-1$ inverse banyan network $BY_d^{-1}(n,m)$, where each input and output crossbar is connected to every copy of $BY_d^{-1}(n,m)$ (see Hwang (1998) for terminology). In Fig.1, $\log_2(8,1,3)$ is presented.

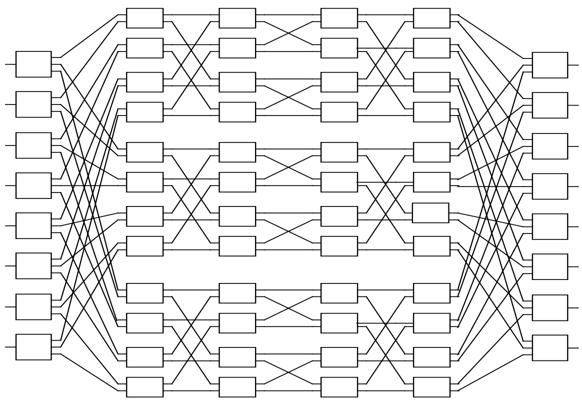

Fig.1 $\log_2(8,1,3)$

Consider a request from input i to output o in $\log_d(N,0,p)$. Then the (i,o) channel graph is simply the path from i to o consisting of $n+1$ links $L_0, L_1,\ldots, L_n$. A path from $i'\neq i$ to $o'\neq o$ is called a j-intersecting path if it contains L_j. Hence a j-intersecting path blocks the original (i,o) path. Note that a path can be both j-intersecting and j'-intersecting. An input is called a j-intersecting input if it can generate a j-intersecting path. Clearly, a j-intersecting input is also a j'-intersecting input for $j<j'\leq n-1$. Similarly, an output is j-intersecting if it can end a j-intersecting path. A j-intersecting output is also a j'-intersecting output for $1\leq j'<j$. Define

$|I_j|$: numbers of j-intersecting inputs,
$|O_j|$: numbers of j-intersecting outputs.

Then

$$|I_j|=d^j,\ |O_j|=d^{n-j}.$$

When considering the intersection of the j-link of the (i,o) path, we notice that both I_j and O_j can block it.

What is more, if the requests originated from I_j choose to block the j-link, they must arrive at O_j. Hence, when $j<\lceil (n-1)/2\rceil$, we just consider the I_j that will intersect rather than O_j, since $|I_j|\le|O_j|$; on the other hand, when $j>\lceil (n-1)/2\rceil$, we only focus on O_j that will intersect rather than I_j, since $|I_j|>|O_j|$. This property will be used in the next section.

When $\log_d(N,m,p)$ ($1\le m\le n-1$) is considered, the intersection will be more complicated. Consider a request from input i to output o in $\log_d(N,m,p)$. The (i,o) channel graph is a union of d^m paths from i to o each consisting of $n+m+1$ links L_0, L_1, …, L_{n+m}. We still use the concept of j-intersecting, and it is not difficult to find that there are d^k L_k (L_{n+m-k}) for $1\le k\le m$, d^m L_j for $m+1\le j\le n-1$. Thus, intersecting an L_k ($1\le k\le m$, $n\le k\le n+m-1$) will block $1/d^k$ middle copy, while intersecting an L_k ($m+1\le k\le n-1$) will block $1/d^m$ middle copy.

MAIN RESULTS

In this section, we first use monotone routing in $\log_d(N,0,p)$, and then extend it to $\log_d(N,m,p)$ ($1\le m\le n-1$).

Monotone routing: sort all weights in nonincreasing order and route the requests one by one whenever a connection can be found.

Define a linear system $I(d^{\lceil \frac{n-1}{2}\rceil},k)$ with $k\ge d^{\lceil \frac{n-1}{2}\rceil}+1$, consisting of $d^{\lceil \frac{n-1}{2}\rceil}+k$ inequalities as follows,

$$\left\{x_1^j+x_2^j+\ldots+x^j_{d^{\lceil \frac{n-1}{2}\rceil}}+x_1^0>1 \text{ for } j=1,2,\ldots,k. \right.\quad (1)$$

$$\begin{cases} x_1^0+x_1^1+x_1^2+\ldots+x_1^{d^{\lceil \frac{n-1}{2}\rceil}}+\ldots+x_1^k\le 1 \\ x_i^1+x_i^2+\ldots+x_i^{d^{\lceil \frac{n-1}{2}\rceil}}+\ldots+x_i^k\le 1 \\ \qquad \text{for}\quad i=2,3,\ldots,d^{\lceil \frac{n-1}{2}\rceil}. \end{cases}\quad (2)$$

where

$$x_i^j(x_i^j-x_1^0)\ge 0 \text{ for } 1\le i\le d^{\lceil \frac{n-1}{2}\rceil},\ 1\le j\le k. \quad (3)$$

Theorem 1 $\log_d(N,0,2k-1)$ is multirate rearrangeable under monotone routing if and only if $I(d^{\lceil \frac{n-1}{2}\rceil},k)$ has no solution.

Proof By contradiction, suppose $\log_d(N,0,2k-1)$ is not multirate rearrangeable under monotone routing, and that the first request that could not be routed is from input I to output J with weight w. A link carrying a load larger than 1 is called saturated.

We know that in the $\log_d(N,0,2k-1)$ network, there exists only one path for the connection I–J in every middle copy and thus we have $2k-1$ paths in total. Divide the single path in every copy into two parts: the left part from L_1 to $L_{\lceil \frac{n-1}{2}\rceil}$ and the right part from $L_{\lceil \frac{n-1}{2}\rceil+1}$ to L_{n-1}. If the path is blocked in a copy, then there must either be a link in the left part or in the right part that is saturated.

Since there are $2k-1$ copies in total, there are either k copies each having a saturated link in the left part or k copies having a saturated link in the right part. Without loss of generality we assume the first case happens. Then, as we have discussed in the preceding section, only $I_{\lceil \frac{n-1}{2}\rceil}$ is concerned. That is, $d^{\lceil \frac{n-1}{2}\rceil}$ inputs should be considered. We label them from 1 to $d^{\lceil \frac{n-1}{2}\rceil}$ and assume that (I,J,w) is from the first input to the first output, and that its weight is x_0^1. Let x_i^j be the weight of the request from the ith input routing through the jth center copy. Then, we have

$$\sum_{i=1}^{d^{\lceil \frac{n-1}{2}\rceil}} x_i^j>1-w \quad \text{for}\quad j=1,2,\ldots,k.$$

That is Eq.(1). From the restriction on the capacity we have

$$\begin{cases} x_1^0+x_1^1+x_1^2+\ldots+x_1^{d^{\lceil \frac{n-1}{2}\rceil}}+\ldots+x_1^k\le 1 \\ x_i^1+x_i^2+\ldots+x_i^{d^{\lceil \frac{n-1}{2}\rceil}}+\ldots+x_i^k\le 1 \\ \qquad \text{for}\quad i=2,3,\ldots,d^{\lceil \frac{n-1}{2}\rceil}. \end{cases}$$

That is Eq.(2). In addition, Eq.(3) are satisfied, since $x_i^j \ge x_1^0$. Therefore, contradictions will happen if and only if system $I(d^{\lceil\frac{n-1}{2}\rceil},k)$ has a solution.

Lemma 1 If $I(d^{\lceil\frac{n-1}{2}\rceil},k)$ has a solution, then $\frac{1}{3} \ge x_1^0 > \frac{k-d^{\lceil\frac{n-1}{2}\rceil}}{k-1}$.

Proof Summing all the inequalities in Eqs.(1) and (2), respectively, we will obtain $\sum_{i=1}^{d^{\lceil\frac{n-1}{2}\rceil}}\sum_{j=1}^{k} x_i^j + kx_1^0 > k$, and $\sum_{j=1}^{k}\sum_{i=1}^{d^{\lceil\frac{n-1}{2}\rceil}} x_i^j + x_1^0 \le d^{\lceil\frac{n-1}{2}\rceil}$.

Thus we have $(k-1)x_1^0 > k - d^{\lceil\frac{n-1}{2}\rceil}$. The reason why $1/3 \ge x_1^0$ is similar to the reason in Hu's paper. Suppose to the contrary that $x_1^0 > 1/3$. Without loss of generality, we suppose $x_1^1 > 0$ and $x_2^1 > 0$. Furthermore, we assume x_2^2, $x_3^2 > 0$, and $x_3^3 > 0$, $x_4^3 > 0$, …, and so on. Then, under the constraints Eq.(2) there are at most k−1 inequalities in Eq.(1) that can be satisfied at the same time, and a contradiction occurs.

Corollary 1 $I(d^{\lceil\frac{n-1}{2}\rceil},k)$ has no solution if $k > \frac{3}{2}d^{\lceil\frac{n-1}{2}\rceil} - \frac{1}{2}$.

Proof It can easily be induced from Lemma 1.

In the following, we extend the result to $\log_d(N,m,p)$ for general m>0. The three similar equation systems as Eqs.(1), (2), (3) presented above are called $I(d^{\lceil\frac{n+m-1}{2}\rceil},k)$.

$$\begin{cases} \frac{1}{d}(x_1^0 + x_1^j + x_2^j + \ldots + x_d^j) + \frac{1}{d^2}(x_{d+1}^j + x_{d+2}^j + \ldots + x_{d^2}^j) \\ + \ldots + \frac{1}{d^m}\left(x_{d^{m-1}+1}^j + x_{d^{m-1}+2}^j + \ldots + x_{d^m}^j\right) \\ + \frac{1}{d^m}\left(x_{d^m+1}^j + x_{d^m+2}^j + \ldots + x_{d^{\lceil\frac{n+m-1}{2}\rceil}}^j\right) > 1 \\ \qquad \text{for } j = 1,2,\ldots,k. \end{cases} \tag{4}$$

and

$$\begin{cases} x_1^0 + x_1^1 + x_1^2 + \ldots + x_1^{d^{\lceil\frac{n+m-1}{2}\rceil}} + \ldots + x_1^k \le 1 \\ x_i^1 + x_i^2 + \ldots + x_i^{d^{\lceil\frac{n+m-1}{2}\rceil}} + \ldots + x_i^k \le 1 \\ \text{for} \quad i = 2,3,\ldots,d^{\lceil\frac{n+m-1}{2}\rceil}, \end{cases} \tag{5}$$

where

$$x_i^j(x_i^j - x_1^0) \ge 0 \ \text{ for } 1 \le i \le d^{\lceil\frac{n+m-1}{2}\rceil},\ 1 \le j \le k \tag{6}$$

Theorem 2 If and only if $I(d^{\lceil\frac{n+m-1}{2}\rceil},k)$ has no solution, then $\log_d(N,m,2k-1)$ is multirate rearrangeable under monotone routing.

Proof The method of contradiction will be used as in Theorem 1. Notice that the major difference between the two theorems lies in Eqs.(1) and (4). In fact, the difference is caused by the distinct properties of $\log_d(N,0,p)$ and $\log_d(N,m,p)$ for m>0. As discussed in Section 2, in $\log_d(N,0,p)$, every middle copy has only one path from I to J, and blocking one link in the path will consume this copy, while in $\log_d(N,m,p)$ there are d^m paths in every middle copy, and blocking one L_i or L_{i+m-i} for $1 \le i \le m$ will block only $1/d^i$ copy, while blocking L_j for $m+1 \le j \le n-1$ will block only $1/d^m$ copy. Hence,

$$\frac{1}{d}x_1^0 + \sum_{p=1}^{m}\frac{1}{d^p}\sum_{i=d^{p-1}+1}^{d^p} x_i^j + \frac{1}{d^m}\sum_{i=d^m+1}^{d^{\lceil\frac{n+m-1}{2}\rceil}} x_i^j > 1, \quad \text{for } j = 1,2,\ldots,k.$$

And the other part of the proof is the same.

Summing the inequalities in Eqs.(4) and (5), respectively, we obtain

$$\frac{k}{d}x_1^0 + \sum_{p=1}^{m}\frac{1}{d^p}\sum_{j=1}^{k}\sum_{i=d^{p-1}+1}^{d^p} x_i^j + \frac{1}{d^m}\sum_{j=1}^{k}\sum_{i=d^m+1}^{d^{\lceil\frac{n+m-1}{2}\rceil}} x_i^j > k$$

and

$$\sum_{j=1}^{k}\sum_{i=1}^{d^{\lceil\frac{n+m-1}{2}\rceil}} x_i^j + x_1^0 \le d^{\lceil\frac{n+m-1}{2}\rceil}.$$

Relax the first inequality and rewrite the second one and we will obtain

$$\sum_{j=1}^{k}\sum_{i=1}^{d^{\lceil\frac{n+m-1}{2}\rceil}} x_i^j > dk - kx_1^0$$

and

$$\sum_{j=1}^{k}\sum_{i=1}^{d^{\lceil\frac{n+m-1}{2}\rceil}} x_i^j \le d^{\lceil\frac{n+m-1}{2}\rceil} - x_1^0.$$

Then, it is easy to prove that

$$x_1^0 > d\,\frac{k - d^{\lceil\frac{n+m-1}{2}\rceil - 1}}{k-1}.$$

Lemma 2 If $I(d^{\lceil\frac{n+m-1}{2}\rceil}, k)$ has a solution, then $1 \ge x_1^0 > d\,\frac{k - d^{\lceil\frac{n+m-1}{2}\rceil - 1}}{k-1}$.

Though we attempt to get a better upper bound of x_i^j than 1, which is trivial, the method used in Lemma 1 cannot be applied here since the coefficients of x_i^j are not 1. It can easily be seen that if the lower bound of x_1^0 is larger than 1, $I(d^{\lceil\frac{n+m-1}{2}\rceil}, k)$ surely has no solution, and that is the following corollary.

Corollary 2 $I(d^{\lceil\frac{n+m-1}{2}\rceil}, k)$ has no solution, if $k > \frac{d^{\lceil\frac{n+m-1}{2}\rceil} - 1}{d-1}$.

From what has been discussed before, we can finally draw the conclusion.

Theorem 3 $\log_d(N,0,p)$ is multirate rearrangeable under monotone routing when $p = 2\left\lfloor \frac{3}{2} d^{\lceil\frac{n-1}{2}\rceil} - \frac{1}{2} \right\rfloor + 1$; $\log_d(N,m,p)$ $(1 \le m \le n-1)$ is multirate rearrangeable under monotone routing when $p = 2\left\lfloor \frac{d^{\lceil\frac{n+m-1}{2}\rceil} - 1}{d-1} \right\rfloor + 1.$

CONCLUSION

Monotone routing algorithm was first used by Hu *et al.*(2001) to study the multirate rearrangeability of 3-stage Clos network. In this paper, we construct a multirate rearrangeable network based on $\log_d(N,m,p)$ network by using this algorithm. We should also notice that there is still a lot of space for further improvement for the $\log_d(N,m,p)$ $(1 \le m \le n-1)$ network since the analysis in Lemma 2 is not very sharp and the upper bound of x_1^0 is taken as 1.

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References

Chung, S.P., Ross, K.W., 1991. On nonblocking interconnecting networks. *SIAM J Comput*, **20**(4):726-736.

Clos, C., 1953. A study of non-blocking switching networks. *Bell Syst Tech J*, **32**(1953):406-424.

Du, D.Z., Gao, B., Hwang, F.K., Kim, J.H., 1999. On multirate rearrangeable Clos networks. *SIAM J Comput*, **28**(2): 467-471 (electronic).

Hu, X.D., Jia, X.H., Du, D.Z, Hwang, F.K, 2001. Monotone routing in multirate rearrangeable Clos networks. *J Parallel Distrib Comput*, **61**(9):1382-1388.

Hwang, F.K., 1998. The Mathematical Theory of Nonblocking Switching Networks. World Scientific.

Lea, C.T., 1991. Multi-$\log_2 N$ networks and their applications in high speed electronic and photonic switching systems. *IEEE Trans Commun*, **38**(10):1740-1749.

Lin, G.H., Du, D.Z., Hu, X.D., Xue, G., 1999. On rearrangeability of multirate Clos networks. *SIAM J Comput*, **28**(4):1225-1231 (electronic).

Melen, R., Turner, J.S., 1989. Nonblocking multirate networks. *SIAM J Comput*, **18**(2):301-313.

Ngo, H.Q., 2003. A new routing algorithm for multirate rearrangeable Clos networks. *Theoretical Computer Science*, **290**(3):2157-2167.

Ngo, H.Q., Vu, V.H., 2003. Multirate rearrangeable Clos networks and a general edge coloring problem on bipartite graphs. *SIAM J Comput*, **32**(4):1040-1049.

Shyy, D.J., Lea, C.T., 1991. $\log_2(N, m, p)$ strictly nonblocking networks. *IEEE Trans Commun*, **39**(10):1502-1510.