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Study on applicability of modal analysis of thin finite length cylindrical shells using wave propagation approach*

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Abstract: Donnell's thin shell theory and basic equations based on the wave propagation method discussed in detail here, is used to investigate the natural frequencies of thin finite length circular cylindrical shells under various boundary conditions. Mode shapes are drawn to explain the circumferential mode number n and axial mode number m, and the natural frequencies are calculated numerically and compared with those of FEM (finite element method) to confirm the reliability of the analytical solution. The effects of relevant parameters on natural frequencies are discussed thoroughly. It is shown that for long thin shells the method is simple, accurate and effective.

Keywords: Wave propagation, Natural frequency, Mode shape, Cylindrical shell
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INTRODUCTION

Vibrations of cylindrical shells are of considerable importance as they are extensively used in industry, flight structures and marine crafts. The natural frequencies and mode shapes are important sources of information for understanding and controlling the vibration of these structures, so many papers on the prediction of the natural frequencies of cylindrical shells have been published over the past years.

Many shell theories have been developed and various solution methods have been proposed. Sharma (1974) investigated the natural frequencies of fixed-free circular cylindrical shells, and gave a detailed analysis for the case of the axial mode being approximated by characteristic beam functions with appropriate end conditions; Soedel (1980) used a formula to calculate the natural frequencies and modes of circular cylindrical shells in which transverse deflections dominate; Chung (1981) used Stokes' transformation technique to obtain the natural frequencies of circular cylindrical shells with different boundary conditions; Chakravorty and Bandyopadhyay (1995), Bouabdallah and Batoz (1996), and Guo *et al.*(2002) used a finite element method (FEM) to obtain the natural frequencies of the cylindrical shells; Callahan and Baruh (1999) presented a systematic procedure using the computational power of existing commercial software packages for obtaining the closed-form eigensolution for thin circular cylindrical shell vibrations; Zhang *et al.*(2001a) used wave propagation method to evaluate the natural frequencies of finite cylindrical shells.

This paper focuses mainly on Zhang *et al.*(2001a)'s method using an interesting technique combining an exact frequency wavenumber characteristics formula with appropriate beam functions in the axial direction to give relatively more accurate predictions of circular cylindrical shells' natural frequencies. Although they highlighted the advantage of

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the method, its applicability is still to be explored. In this paper, the effects of relevant parameters on natural frequencies are thoroughly discussed. It is shown that for the long-thin finite cylindrical shells the method is more simple and effective than other methods.

THEORETICAL ANALYSIS

Equation of cylindrical shells motion

The shell under consideration is shown in Fig.1.

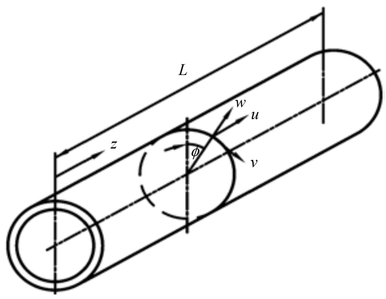

Fig.1 A circular cylindrical shell with relevant parameters

For this analysis we will use the equations of motion derived by Junger and Feit (1986). The equations of motion for cylindrical shells can be written as:

$$\frac{\partial^2 u}{\partial z^2}+\frac{1-\mu}{2a^2}\frac{\partial^2 u}{\partial\phi^2}+\frac{1+\mu}{2a}\frac{\partial^2 v}{\partial z\partial\phi}+\frac{\mu}{a}\frac{\partial w}{\partial z}=\frac{1-\mu^2}{E}\rho\frac{\partial^2 u}{\partial t^2}$$
$$\frac{1+\mu}{2a}\frac{\partial^2 u}{\partial z\partial\phi}+\frac{1-\mu}{2}\frac{\partial^2 v}{\partial z^2}+\frac{1}{a^2}\frac{\partial^2 v}{\partial\phi^2}+\frac{1}{a^2}\frac{\partial w}{\partial\phi}=\frac{1-\mu^2}{E}\rho\frac{\partial^2 v}{\partial t^2}$$
$$\frac{\mu}{a}\frac{\partial u}{\partial z}+\frac{1}{a^2}\frac{\partial v}{\partial\phi}+\frac{w}{a^2}+\beta^2\left(a^2\frac{\partial^4 w}{\partial z^4}+2\frac{\partial^4 w}{\partial z^2\partial\phi^2}+\frac{1}{a^2}\frac{\partial^4 w}{\partial\phi^4}\right)=-\frac{1-\mu^2}{E}\rho\frac{\partial^2 w}{\partial t^2} \quad (1)$$

where $\beta^2=h^2/(12a^2)$; a is the radius of the cylinder; ρ is the density of the material; h is the thickness of the shells; E is Young's modulus; and μ is Poisson's ratio. The vibration displacements in the three directions (r, θ, z) are not independent of each other.

Applied wave propagation approach in cylindrical shells

In the wave propagation approach, the solution of Eq.(1) can be expressed in the form of traveling wave form as:

$$\begin{aligned} u&=U_m\cos[n(\phi-\alpha)]\mathrm{e}^{\mathrm{j}(\omega t-k_z z)}\\ v&=V_m\sin[n(\phi-\alpha)]\mathrm{e}^{\mathrm{j}(\omega t-k_z z)}\\ w&=W_m\cos[n(\phi-\alpha)]\mathrm{e}^{\mathrm{j}(\omega t-k_z z)}\end{aligned} \quad (2)$$

where u, v and w are the displacement components in the axial, tangential and radial directions, respectively; the coefficients U_m, V_m and W_m in the equations are the displacement amplitudes; α is an arbitrary angle, to account for the fact that there is no preferential direction of the mode shape in the circumferential direction; n is circumferential mode parameter (where $2n$=the number of cross points in the radial displacement shape); m is axial mode parameter (where m=the number of cross points in the radial displacement shape along any axial generatrix); the meanings of m and n are illustrated in Fig.2; ω is the angular natural frequency for (m, n) vibration mode; k_z is the wavenumber in the axial direction.

For infinite length cylindrical shells, all the vibration displacements are symmetric about ϕ, so the wavenumber in the circumferential direction can be written as:

$$k_\phi=n/a,\ n\in N$$

Substituting Eq.(2) into Eq.(1) yields:

$$\begin{bmatrix}L_{11}&L_{12}&L_{13}\\L_{21}&L_{22}&L_{23}\\L_{31}&L_{32}&L_{33}\end{bmatrix}\begin{Bmatrix}U_m\\V_m\\W_m\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix} \quad (3)$$

The items of this matrix can be expressed as:

$$L_{11}=K\left(k_z^{\ 2}+\frac{1-\mu}{2}k_\phi^{\ 2}\right)-\rho h\omega^2,$$
$$L_{12}=L_{21}=\frac{K(1+\mu)}{2}k_z k_\phi,\quad L_{13}=L_{31}=\frac{K\mu}{a}k_z,$$
$$L_{22}=K\left(\frac{1-\mu}{2}k_z^{\ 2}+k_\phi^{\ 2}\right)-\rho h\omega^2,$$

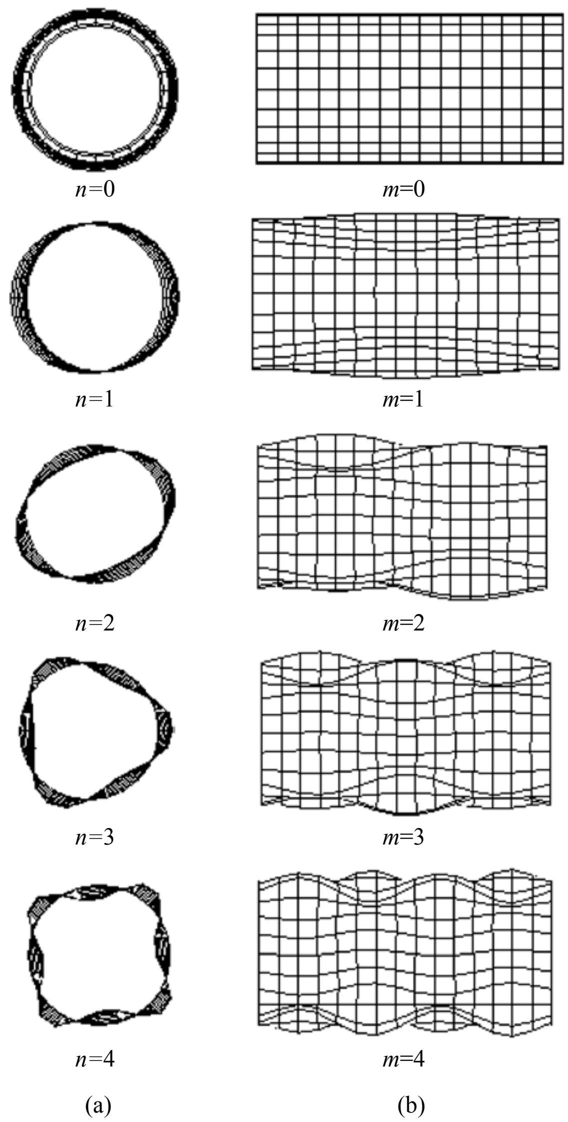

Fig.2 Illustration of parameter *n* and *m*
(a) Circumferential nodal pattern; (b) Axial nodal pattern

$$L_{23}=L_{32}=\frac{K}{a}k_{\phi},\quad L_{33}=Dk^4+\frac{K}{a^2}-\rho h\omega^2.$$

where $k=\sqrt{k_z^{\ 2}+k_{\phi}^{\ 2}}$, $K=\frac{Eh}{1-\mu^2}$, $D=\frac{Eh^3}{12(1-\mu^2)}$.

To obtain the non-trivial solution of Eq.(3), the determinant of the characteristic matrix in Eq.(3) must be zero:

$$\begin{vmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{vmatrix}=0 \tag{4}$$

Expanding Eq.(4) yields the following polynomial for the natural frequencies:

$$\omega^6+a_1\omega^4+a_2\omega^2+a_3=0 \tag{5}$$

where

$$a_1=-\frac{1}{\rho h}[c_1+c_2+c_3],$$

$$a_2=\frac{1}{(\rho h)^2}\begin{bmatrix} c_1c_3+c_2c_3+c_1c_2 \\ -L_{12}^{\ 2}-L_{23}^{\ 2}-L_{13}^{\ 2} \end{bmatrix},$$

$$a_3=\frac{1}{(\rho h)^3}\begin{bmatrix} c_1L_{23}^{\ 2}+c_2L_{13}^{\ 2}+c_3L_{12}^{\ 2} \\ -c_1c_2c_3-2L_{12}L_{13}L_{23} \end{bmatrix},$$

$$c_1=L_{11}+\rho h\omega^2,\ c_2=L_{22}+\rho h\omega^2,\ c_3=L_{33}+\rho h\omega^2.$$

In this paper, the software ANSYS was used to perform the finite element analysis.

Fig.2 and Fig.3 were calculated by ANSYS. Fig.2 presents circumferential nodal pattern and axial nodal pattern respectively. The results can be used to explain the parameter *n* and parameter *m*. Fig.3 shows some typical combined mode shapes.

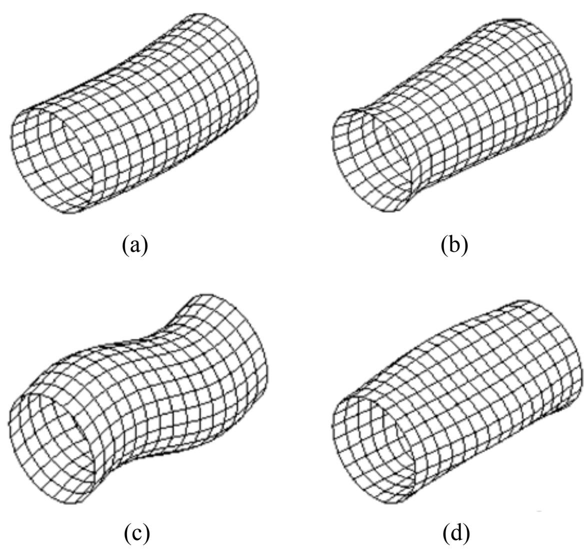

Fig.3 Some typical combined mode shapes
(a) n=1, m=1; (b) n=0, m=2; (c) n=1, m=2; (d) n=3, m=1

From the figures, the typical modes can be described as follows:

When n=0, the circumferential nodal pattern is a circle, indicating that this mode is an extensional mode referred to as breathing type mode.

The mode is a pure radial mode when m=0. Here, the cylinder retains a constant cross-sectional shape along its length.

When m, n are both equal to one, the mode is a circumferential mode. When n=1 and m≠1, the mode is an axial bending mode and the mode is radial motion with shearing mode when m=1 and n≠1.

Beam functions

In this paper, the wave propagation method was used in conjunction with beam functions. The natural frequencies of finite length cylindrical shell with different boundary conditions can be obtained.

Zhang *et al*.(2001b)'s wavenumbers for different boundary conditions of beams are listed in Table 1. Substitution of beam functions into motion Eq.(5), yields different approximate mode shapes and natural frequencies.

Table 1 Wavenumbers for different boundary conditions

Boundary conditions	Wave numbers
Clamped-free	$kL=(2m-1)\pi/2$
Free-simply supported	$kL=(4m+1)\pi/4$
Simply supported-simply supported	$kL=m\pi$
Clamped-simply supported	$kL=(4m+1)\pi/4$
Clamped-clamped	$kL=(2m+1)\pi/2$
Sliding-simply supported	$kL=(2m-1)\pi/2$
Free-free	$kL=(2m+1)\pi/2$

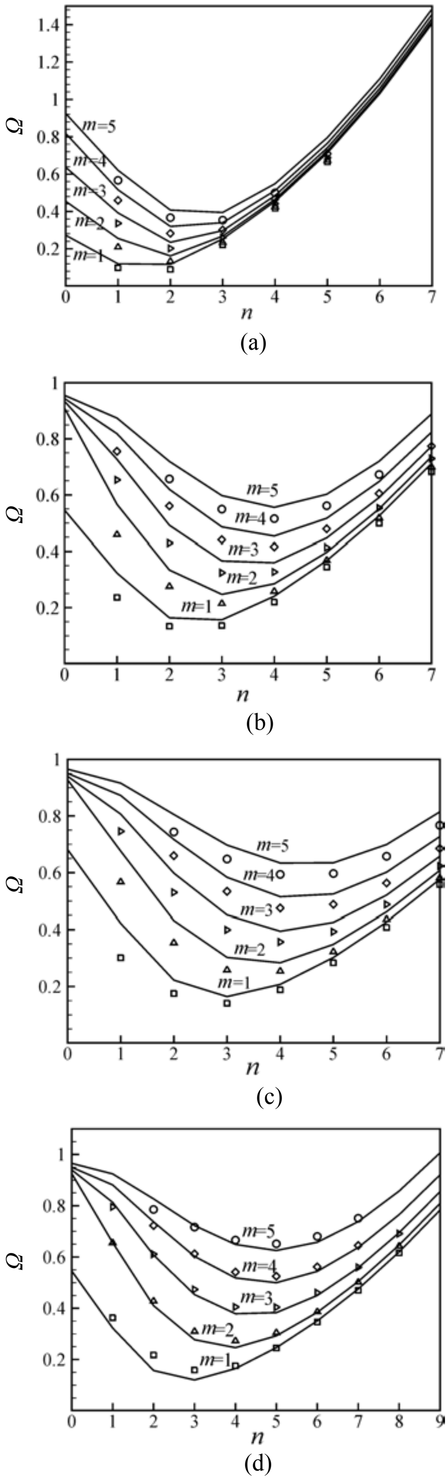

Fig.4 Variation of the non-dimensional frequencies with the parameter *n* for different parameter *m*
(a) a/h=10; (b) a/h=20; (c) a/h=25; (d) a/h=30

RESULTS AND DISCUSSION

To check the validity of the present analysis, the results were compared with those calculated by FEA.

The non-dimensional frequency parameter Ω was used to make the conclusions more widely applicable. Here define $\Omega=\omega/\omega_r$, where $\omega_r=\frac{1}{a}\sqrt{\frac{E}{\rho(1-\mu^2)}}$.

Relationship between the natural frequencies and the circumferential mode number *n* with different radius-thickness ratio *a*/*h*

In the computation of Fig.4, the material used is aluminum with mass density ρ of 2700 kg/m^3, the Poisson ratio μ is equal to 0.33, and Young's modulus $E=7.1\times10^{10}$ N/m^3. The boundary condition considered is clamped-clamped.

Fig.4 shows non-dimensional frequency parameter Ω versus circumferential wave number n with different radius-thickness ratio a/h. The solid curves correspond to the results obtained by using the present method and curves marked "Δ", "□" and "○" correspond to those from FEM.

These shells all have the same thickness h=0.01 and the same length L=1.

From these plots, the following observations can be made:

1. We can find that, for the same thickness h, the smaller the shell radius-to-thickness ratio a/h is, the larger is the difference between the results from FEM and those from the present method. Figs.4a, 4b and 4c show the results from FEM are lower than those from the present method. This can be attributed to the fact that the effects of shear deflection and rotary inertia of the shell (which would reduce the natural frequencies) should not be neglected for small a/h (Soedel, 1982). As the above effects are not taken into account by Eq.(5), it may be expected that Eq.(5) can be applied to thin finite cylindrical shells.

2. The relative error decreases with decreasing axial mode number m. It means that the results are more exact when the shells are longer. This indicates that, for the long-thin shell, the effect of the boundary conditions are small, the wave propagation in the cylindrical shell trends to the form of an approaching wave.

3. When the ratio a/h=30, the results from the method are more accurate than the results when ratio a/h<30. And we also can find that higher order modes led to more accurate results. All these phenomena indicate that although the coupling of the vibration between the axial and circumferential direction was neglected in this method, the effects of such coupling were less important for long-thin shells and higher order modes.

4. The lowest frequency does not occur at the lowest values of n, and for different values of m, the lowest frequency occurs at different mode. For example, at a/h=30 and m=1, the lowest frequency occurs at the mode when n=3, this phenomenon can be explained by considering the strain energy of the middle surface under both bending and stretching (Kraus, 1967).

5. For the larger circumferential mode number n, the curves change dramatically, in other words, the natural frequency is sensitive to the geometric sizes when the ratio a/h is small. It indicates that the method is more effective for thin cylindrical shells.

From this section, we know that the present method can be used to evaluate the natural frequencies of thin cylindrical shells.

Relationship between the natural frequencies and the axial mode number *m* with different length-thickness ratio *L*/*a*

The results are plotted in Fig.5 to study the effect of the axial length on the modes of the radius.

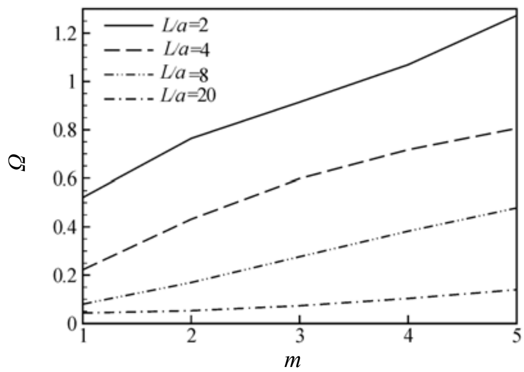

Fig.5 Variation of the non-dimensional frequencies with the parameter *m* for different length-thickness ratio *L*/*a* while *n*=2

It was found that, when L/a becomes larger, the natural frequencies of shells become smaller for the same m, because larger L/a ratio leads to smaller shell rigidity. The curves' variations level off with increasing L/a ratio, that is to say, the natural frequency is more sensitive to the geometric sizes when the cylindrical shell is short. It indicates that the method is more effective for long cylindrical shells.

CONCLUSION

Donnell's shell theory and wave propagation method were applied to analyze the free vibration characteristics of long-thin finite circle cylindrical shells. The curves of the relation between the parameter n and the shell radius-to-thickness ratio a/h, as well as between the parameter m and the shell length-to-radius ratio L/a are numerically presented and some important conclusions can be obtained from them.

The results from the present paper compared with the solutions obtained from FEM showed that the method is effective for long thin cylindrical shells. As far as the applications are concerned, the results obtained can commendably satisfy the criterion of precision. The method can be extended to long thin ring-stiffened cylindrical shells and some long-thin shell structures with complicated boundary conditions.

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