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Analytical solutions for a uniformly loaded circular plate with clamped edges*

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Abstract: A bi-harmonic potential function was constructed in this study. Love solution was employed to obtain analytical solutions of uniformly loaded plates with two different types of clamped edges. The treatment of clamped boundary conditions was the same as that adopted by Timoshenko and Goodier (1970). The analytical solution for the first type of clamped boundary condition is identical with that obtained by Luo *et al*.(2004), and the solutions for both types were compared with the FEM results and the calculations of thin plate theory.

Key words: Three-dimensional analytical solution, Circular plates with clamped edges, Bi-harmonic functions, Axisymmetric deformation

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INTRODUCTION

The problem of circular plates subjected to a uniform load is one of the classical problems in elasticity theory, which also is encountered frequently in practice. Timoshenko and Goodier (1970) presented a three-dimensional solution for uniformly loaded isotropic circular plates with simply supported edges. Analytical solutions of isotropic circular plates with either clamped or simply supported edges subjected to a uniform load, which were derived based on thin plate theory with Kirchhoff hypothesis, can be found in (Timoshenko and Woinowsky-Krieger, 1959). Lekhnitskii (1968) investigated problems of anisotropic circular plates with clamped and simply supported boundaries. Ding *et al*.(1999) obtained three-dimensional exact solutions of elastic circular plates subjected to axisymmetric deformation under two specific boundary conditions. Luo *et al*.(2004) derived a three-dimensional analytical solution of clamped circular plates by taking displacements as the basic variables; however, their analysis involves a step to solve ordinary differential equations, and hence is very cumbersome. Furthermore, Luo *et al*.(2004) took only one type of boundary condition into consideration. Recently, Ding *et al*.(2005) obtained two analytical solutions of plane elasticity problem of a fixed-fixed isotropic beam subjected to a uniform load. Both solutions have not been reported in literature and have an important theoretical value.

This article, following the work of Timoshenko and Goodier (1970), presents three-dimensional analytical solutions for a uniformly loaded circular plate with clamped edges by making use of Love potential functions. It can be shown that the present analysis is very simple and clear. Numerical comparison is made with FEM and the thin plate theory, and some interesting observations are obtained.

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BASIC EQUATIONS FOR AXISYMMETRIC DEFORMATION PROBLEMS

In the case of axisymmetric deformation without the effect of body forces, the stress and displacement components, according to (Timoshenko and Goodier, 1970), can be expressed in terms of a potential function as

$$\tau_{rz}=(1-\nu)\frac{\partial}{\partial r}\nabla^2\phi-\frac{\partial^3\phi}{\partial r\partial z^2},\quad(1)$$

$$\sigma_z=(2-\nu)\frac{\partial}{\partial z}\nabla^2\phi-\frac{\partial^3\phi}{\partial z^3},\quad(2)$$

$$\sigma_r=\nu\frac{\partial}{\partial z}\nabla^2\phi-\frac{\partial^3\phi}{\partial z\partial r^2},\quad(3)$$

$$\sigma_\theta=\nu\frac{\partial}{\partial z}\nabla^2\phi-\frac{1}{r}\frac{\partial^2\phi}{\partial z\partial r},\quad(4)$$

$$2Gu=-\frac{\partial^2\phi}{\partial r\partial z},\quad(5)$$

$$2Gw=2(1-\nu)\nabla^2\phi-\frac{\partial^2\phi}{\partial z^2},\quad(6)$$

where τ_{rz}, σ_z, σ_r, and σ_θ are the stress components; u and w are the displacements in x- and z-directions, respectively; G is the shear modulus and ν Poisson's ratio. The potential function ϕ in Eqs.(1)~(6) satisfies the following bi-harmonic equation

$$\nabla^2\nabla^2\phi=0\quad(7)$$

where ∇^2 is the three-dimensional Laplace's operator, which reads in the axisymmetric case as

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\quad(8)$$

STRESS AND DISPLACEMENT

Consider a clamped circular plate with radius b and height h, subjected to a uniform load q as shown in Fig.1.

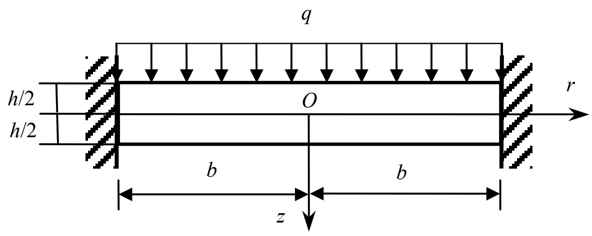

Fig.1 Clamped circular plate subjected to a uniform load

Take the potential function as the following bi-harmonic polynomial with 8 terms

$$\begin{aligned}\phi=&\frac{1}{3}a_6(16z^6-120z^4r^2+90z^2r^4-5r^6)\\&+b_6(8z^6-16z^4r^2-21z^2r^4+3r^6)\\&+a_4(8z^4-24r^2z^2+3r^4)\\&+b_4(2z^4+r^2z^2-r^4)\\&+a_3(2z^3-3r^2z)+b_3(z^3+r^2z)\\&+a_2(2z^2-r^2)+b_2(r^2+z^2),\end{aligned}\quad(9)$$

where a_i, b_i (i=2, 3, 4, 6) are unknown constants to be determined. The substitution of Eq.(9) into Eqs.(1)~(6) gives

$$\begin{aligned}\tau_{rz}=&a_6(960rz^2-240r^3)\\&+b_6[(-672+1056\nu)z^2r+(432-264\nu)r^3]\\&+96a_4r-2b_4(16-14\nu)r,\end{aligned}\quad(10)$$

$$\begin{aligned}\sigma_z=&a_6(-640z^3+960zr^2)-192a_4z\\&+b_6[(448-704\nu)z^3-(1728-1056\nu)zr^2]\\&+4b_4(16-14\nu)z-12a_3+(14-10\nu)b_3,\end{aligned}\quad(11)$$

$$\begin{aligned}\sigma_r=&a_6(320z^3-720zr^2)\\&+b_6[(128+704\nu)z^3+(504-1056\nu)zr^2]\\&+96a_4z+4b_4(14\nu-1)z+6a_3+(10\nu-2)b_3,\end{aligned}\quad(12)$$

$$\begin{aligned}\sigma_\theta=&a_6(320z^3-240zr^2)\\&+b_6[(128+704\nu)z^3+(168-1056\nu)zr^2]\\&+96a_4z+4b_4(14\nu-1)z+6a_3+(10\nu-2)b_3.\end{aligned}\quad(13)$$

$$\begin{aligned}2Gu=&a_6(320z^3r-240zr^3)\\&+b_6(128z^3r+168zr^3)\\&+96a_4rz-4b_4rz+6a_3r-2b_3r,\end{aligned}\quad(14)$$

$$\begin{aligned}2Gw=&-a_6(160z^4-480z^2r^2+60r^4)\\&+b_6[(174-132\nu)r^4-(864-1056\nu)z^2r^2\\&+(112-352\nu)z^4]-a_4(96z^2-48r^2)\\&+b_4[(32-56\nu)z^2-(30-28\nu)r^2]\\&-12a_3z+(14-20\nu)b_3z+c_2,\end{aligned}\quad(15)$$

where

$$c_2 = -4a_2 + (10-12\nu)b_2 \tag{16}$$

It can then be seen from Eqs.(10)~(15) that there are totally 7 unknown constants to be determined, including a_6, b_6, a_4, b_4, a_3, b_3 and c_2.

UNIFORMLY LOADED CIRCULAR PLATE WITH CLAMPED EDGES

Timoshenko and Goodier (1970) proposed two types of description of the clamped edge while investigating beam problems based on the plane elasticity. Both of them will be considered in this paper. The boundary conditions corresponding to the first type are:

$z=h/2$: $\sigma_z=0$;
$z=-h/2$: $\sigma_z=-q$;
$z=\pm h/2$: $\tau_{rz}=0$;
$z=0$, $r=b$: $u=v=0$, $\partial w/\partial r=0$.

Substituting Eqs.(10), (11), (14) and (15) into the above boundary conditions leads to 7 algebraic equations, which in turn determines the 7 unknown constants as follows

$$\begin{aligned}
a_6 &= \frac{18-11\nu}{3520}\frac{q}{h^3},\\
b_6 &= \frac{1}{352}\frac{q}{h^3},\\
a_4 &= \frac{8-7\nu}{896}\frac{qb^2}{h^3} - \frac{15-14\nu}{896(1-\nu)}\frac{q}{h},\\
b_4 &= \frac{3}{112}\frac{qb^2}{h^3} - \frac{3}{112(1-\nu)}\frac{q}{h},\\
a_3 &= -\frac{q}{60(1-\nu)},\\
b_3 &= -\frac{q}{20(1-\nu)},\\
c_2 &= \frac{3(1-\nu)}{16}\frac{qb^4}{h^3}.
\end{aligned} \tag{17}$$

Substitution Eq.(17) into Eqs.(10)~(15) gives rise to expressions of stress and displacement components:

$$\sigma_z = q\left(-2\frac{z^3}{h^3} + \frac{3}{2}\frac{b^2 z}{h^3} - \frac{1}{2}\right) \tag{18}$$

$$\tau_{rz} = 3q\frac{r}{h}\left(\frac{z^2}{h^2} - \frac{1}{4}\right) \tag{19}$$

$$\sigma_r = q\left\{(2+\nu)\frac{z^3}{h^3} - \frac{3}{4}(3+\nu)\frac{zr^2}{h^3} + \frac{3}{4}\left[(1+\nu)\frac{b^2}{h^3} - \frac{2}{(1-\nu)}\frac{1}{h}\right]z - \frac{\nu}{2(1-\nu)}\right\} \tag{20}$$

$$\sigma_\theta = q\left\{(2+\nu)\frac{z^3}{h^3} - \frac{3}{4}(1+3\nu)\frac{zr^2}{h^3} + \frac{3}{4}\left[(1+\nu)\frac{b^2}{h^3} - \frac{2}{(1-\nu)}\frac{1}{h}\right]z - \frac{\nu}{2(1-\nu)}\right\} \tag{21}$$

$$u = \frac{q}{2G}\left\{(2-\nu)\frac{z^3 r}{h^3} - \frac{3(1-\nu)}{4}\frac{zr^3}{h^3} + \frac{3}{4}\left[(1-\nu)\frac{b^2}{h^3} - \frac{2}{h}\right]rz\right\} \tag{22}$$

$$w = \frac{q}{2G}\left\{-\frac{1}{2}(1+\nu)\frac{z^4}{h^3} + \frac{3\nu}{2}\frac{z^2r^2}{h^3} + \frac{3(1-\nu)}{16}\frac{r^4}{h^3} - \frac{3}{4}\left[\nu\frac{b^2}{h^3} - \frac{1}{1-\nu}\frac{1}{h}\right]z^2 \frac{3}{8}(1-\nu)\frac{b^2}{h^3}r^2 - \frac{1-2\nu}{2(1-\nu)}z + \frac{3(1-\nu)}{16}\frac{b^4}{h^3}\right\} \tag{23}$$

Eqs.(18)~(23) are identical to those obtained by Luo *et al*.(2004), who employed a different method.

From Eq.(23), it is easy to obtain the expression for the deflection of the neutral surface $z=0$ as

$$w_1(r,0) = w_1 = \frac{3q(1-\nu)}{32Gh^3}(b^2-r^2)^2 \tag{24}$$

which is identical with the two dimensional solution as presented in (Timoshenko and Woinowsky-Krieger, 1959).

The difference between two types of boundary conditions is only at the point $(r,z)=(b,0)$. For the second type, $\partial u/\partial z=0$ should be used to replace $\partial w/\partial r=0$ in the first type of boundary conditions. The stress and displacement components are then obtained as

$$\sigma_z = q\left(-2\frac{z^3}{h^3} + \frac{3}{2}\frac{b^2 z}{h^3} - \frac{1}{2}\right) \quad (25)$$

$$\tau_{rz} = 3q\frac{r}{h}\left(\frac{z^2}{h^2} - \frac{1}{4}\right) \quad (26)$$

$$\sigma_r = q\left\{(2+\nu)\frac{z^3}{h^3} - \frac{3}{4}(3+\nu)\frac{zr^2}{h^3} + \frac{3}{4}\left[(1+\nu)\frac{b^2}{h^3} + \frac{2}{(1-\nu)}\frac{1}{h}\right]z - \frac{\nu}{2(1-\nu)}\right\} \quad (27)$$

$$\sigma_\theta = q\left\{(2+\nu)\frac{z^3}{h^3} - \frac{3}{4}(1+3\nu)\frac{zr^2}{h^3} + \frac{3}{4}\left[(1+\nu)\frac{b^2}{h^3} + \frac{2}{(1-\nu)}\frac{1}{h}\right]z - \frac{\nu}{2(1-\nu)}\right\} \quad (28)$$

$$u = \frac{q}{2G}\left\{(2-\nu)\frac{z^3 r}{h^3} - \frac{3(1-\nu)}{4}\frac{zr^3}{h^3} + \frac{3(1-\nu)}{4}\frac{b^2}{h^3}rz\right\} \quad (29)$$

$$w = \frac{q}{2G}\left\{-\frac{1}{2}(1+\nu)\frac{z^4}{h^3} + \frac{3\nu}{2}\frac{z^2 r^2}{h^3} + \frac{3(1-\nu)}{16}\frac{r^4}{h^3} - \frac{3}{4}\left[\nu\frac{b^2}{h^3} - \frac{1-2\nu}{1-\nu}\frac{1}{h}\right]z^2 - \frac{3}{8}\left[(1-\nu)\frac{b^2}{h^3}r^2 + 2\frac{1}{h}\right]r^2 - \frac{1-2\nu}{2(1-\nu)}z + \frac{3}{4}\frac{b^2}{h} + \frac{3(1-\nu)}{16}\frac{b^4}{h^3}\right\}. \quad (30)$$

From Eq.(30), the deflection of the neutral surface $z=0$ is determined as

$$w_2(r,0) = w_2 = \frac{3(1-\nu)q}{32Gh^3}(b^2 - r^2)^2 + \frac{3q}{8Gh}(b^2 - r^2) \quad (31)$$

By comparison, it can be seen that the stress components σ_z and τ_{rz} keep unchanged for the two types of boundary conditions, in other words, the different boundary conditions exert no influence on them. The situation changes for σ_r, σ_θ, u and w, all are slightly different for the two types of boundary conditions. According to Eqs.(24) and (31), the difference between the deflections w under the two boundary conditions can be obtained as

$$w_2 - w_1 = \frac{3q}{8Gh}(b^2 - r^2) \quad (32)$$

which indicates that the deflection for the boundary conditions of the second type is always larger than that for the first type. Here and after, a subscript 1 denotes the quantity corresponding to the first type of boundary conditions, while a subscript 2 denotes the one corresponding to the second type.

Subtracting Eq.(20) from Eq.(27) results in

$$\sigma_{r2} - \sigma_{r1} = \frac{3qz}{(1-\nu)h} \quad (33)$$

which implies that $\sigma_{r2} > \sigma_{r1}$ when $z>0$, while $\sigma_{r2} < \sigma_{r1}$ when $z<0$.

From Eqs.(22) and (29), we get

$$u_2 - u_1 = \frac{3qrz}{4Gh} \quad (34)$$

from which, conclusion can be drawn that, $u_2 > u_1$ when $z>0$, while $u_2 < u_1$ when $z<0$.

NUMERICAL RESULTS AND DISCUSSION

In this paper, for the purpose of comparison, the commercial FEM code ANSYS is utilized to study the uniformly loaded plate with a clamped edge which is modeled using plane elasticity theory, i.e. $u=w=0$ at $r=b$, $-h/2 \le z \le h/2$. Young's modulus and Poisson's ratio of the plate are taken as $E=2\times10^{11}$ N/m^2 and $\nu=0.3$, respectively, and the load intensity is $q=6\times10^6$ N/m^2. In the calculation, axisymmetric elements with 8 nodes are employed, and for the sake of convenience, the thickness of the plate is taken to be 1 m, and then the meridian plane is discretized evenly by $10b\times20$ (b in meter) rectangular elements.

The dimensionless deflections $w(r,0)/h$ of the neutral plane are plotted in Figs.2 and 3 for two diameter-to-height ratios, i.e. $2b/h=10$ and $2b/h=4$, respectively. The numerical values of the dimensionless central deflection of the plates are listed in Table 1 for various different diameter-to-height ratios, while those of the dimensionless radial stresses σ_r/q at the point $(r,z)=(0,h/2)$ are given in Table 2. When using the present analysis, two different types of boundary conditions can be taken into consideration, so that two results can be obtained for each case, which are denoted as Type 1 and Type 2, respectively in these tables and figures. Moreover, calculation based on the classical thin plate theory with Kirchhoff hypothesis is also performed, and the associated results are indicated by TPT as in Table 2.

As seen in Table 1, the dimensionless deflection obtained by FEM lies between the two results for two different types of clamped boundaries. This is further demonstrated in Figs.2 and 3. From Figs.2 and 3, we can draw two conclusions: (1) the FEM calculation is closer to the analytical solution for the second type of boundary conditions; (2) the larger the diameter-to-height ratio, the closer the two analytical solutions.

Table 1 Dimensionless central deflection $w(0,0)/h$

$2b/h$	Type 1	FEM	Type 2
20	5.12×10^{-2}	5.33×10^{-2}	5.41×10^{-2}
10	3.20×10^{-3}	3.75×10^{-3}	3.93×10^{-3}
8	1.31×10^{-3}	1.67×10^{-3}	1.78×10^{-3}
6	4.15×10^{-4}	6.16×10^{-4}	6.78×10^{-4}
4	8.19×10^{-5}	1.71×10^{-4}	1.99×10^{-4}
2	5.12×10^{-6}	2.62×10^{-5}	3.44×10^{-5}

Table 2 Dimensionless radial stresses $\sigma_r(0,h/2)/q$

$2b/h$	Type 1	TPT	FEM	Type 2
20	47.395	48.750	48.806	49.538
10	10.832	12.188	12.285	12.975
8	6.445	7.800	7.909	8.588
6	3.032	4.388	4.508	5.175
4	0.595	1.950	2.084	2.738
2	–0.868	0.488	0.604	1.275

From Table 2, it is seen that the dimensionless radial stresses $\sigma_r(0,h/2)/q$ calculated by FEM are higher than those by the classical thin plate theory, but smaller than the stresses obtained by the analytical solution for the second type of boundary conditions. In particular, the classical thin plate theory gives stresses that are closer to the FEM results. When $2b/h=2$, we have $\sigma_r(0,h/2)/q<0$, which indicates that the first type of boundary conditions is not capable of simulating the FEM model as employed in our study. As for the second type of boundary conditions, although we have $\sigma_r(0,h/2)/q>0$, which still bears the same sign as that of the stress calculated by FEM, the error between the two arrives at as much as 105%. In fact, according to Eqs.(20) and (27), we can get

$$\frac{\sigma_{r1}(0,h/2)}{q}=\frac{3}{8}(1+\nu)\frac{b^2}{h^2}-\frac{(1+\nu)(4+\nu)}{8(1-\nu)} \tag{35a}$$

$$\frac{\sigma_{r2}(0,h/2)}{q}=\frac{3}{8}(1+\nu)\frac{b^2}{h^2}+\frac{8-5\nu-\nu^2}{8(1-\nu)} \tag{35b}$$

where the term $3(1+\nu)b^2/(8h^2)$ is just that predicted by the classical thin plate theory. It can be shown from Eq.(35a) that $\sigma_r(0,h/2)/q\leq0$ will hold provided that $b/h\leq\sqrt{(4+\nu)/[3(1-\nu)]}$. In a particular case of $\nu=0.3$, as long as $2b/h<2.862$, $\sigma_{r1}(0,h/2)$ becomes negative.

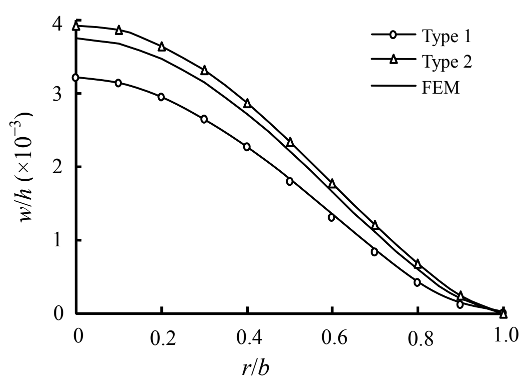

Fig.2 Dimensionless deflection curve when $2b/h=10$

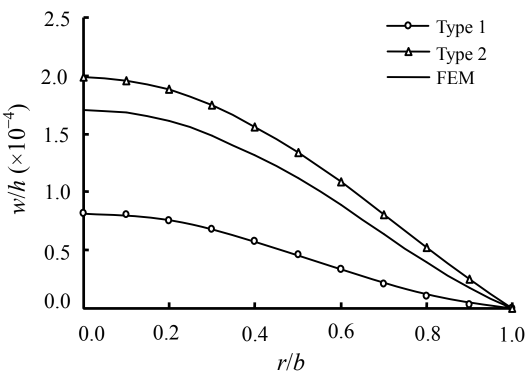

Fig.3 Dimensionless deflection curve when $2b/h=4$

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