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Shape modification of Bézier curves by constrained optimization*

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Abstract: The Bézier curve is one of the most commonly used parametric curves in CAGD and Computer Graphics and has many good properties for shape design. Developing more convenient techniques for designing and modifying Bézier curve is an important problem, and is also an important research issue in CAD/CAM and NC technology fields. This work investigates the optimal shape modification of Bézier curves by geometric constraints. This paper presents a new method by constrained optimization based on changing the control points of the curves. By this method, the authors modify control points of the original Bézier curves to satisfy the given constraints and modify the shape of the curves optimally. Practical examples are also given.

Key words: Shape modification, Bézier curve, Constrained optimization
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INTRODUCTION

The Bézier curve is widely used in Computer Aided Geometric Design (CAGD) and Computer Graphics, and have many properties which are helpful for shape design. Developing more convenient techniques for designing and modifying Bézier curve is an important problem, and is also an important research issue in CAD/CAM and NC technology fields. When Bézier curves are created, we often need to modify them to satisfy our design requirement.

The problem of shape modification of curves by constrained optimization was proposed recently. Piegl (1989) proposed two methods to vary the shape of NURBS curves and surfaces: control-point-based modification and weight-based modification. Fowler and Bartels (1993) gave a shape operator to force a curve or surface to assume the specified derivatives at selected parameter values. Au and Yuen (1995) and Sánchez (1997) presented an approach for modifying the shape of NURBS curves by altering weights and the control points simultaneously. Developing more convenient solution for shape modification of Bézier curves is an important goal. Hu *et al*.(1999; 2001) developed a new method for shape modification of NURBS curves and surfaces with geometric constraint. Xu *et al*.(2002) discussed the way of shape modification of Bézier curves by contrained optimization based on the discrete coefficient norm. Opfer and Oberle (1988) proposed the derivation of cubic splines with obstacles by methods of optimization and optimal control. Meek *et al*.(2003) proposed constrained interpolation with rational cubic. Inspired by these results, we proposed a new method to modify the shape of Bézier curves by minimizing the changes of the shape in sense of least area. Shape modification of Bézier curve with added end point and tangent constraints are also discussed.

The paper is arranged as follows. Problem statement is given in Section 1. Section 2 presents the single point constrained optimization method. Section 3 presents the Multi-Target point constraints optimization method. Practical examples and conclusion are given in Section 4.

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PROBLEM STATEMENT

A Bézier curve of degree n can be defined as:

$$C(t)=\sum_{i=0}^{n} p_i B_{i,n}(t), \quad 0\le t\le 1.$$

where p_i are the control points, $B_{i,n}(t)$ is Bernstein function of degree n which can be defined as:

$$B_{i,n}(t)=\frac{n!}{i!(n-i)!}t^i(1-t)^{n-i}=C_n^i t^i(1-t)^{n-i}.$$

As shown in Fig.1, T is a target point and S is a start point in curve $C(t)$ with parameter t_S. In order to let the curve pass through the target point T, we need to modify the curve.

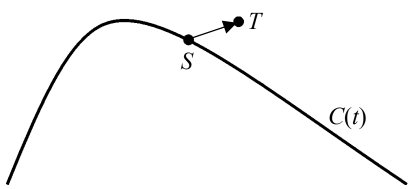


Fig.1 Illustration of local shape modification

CONSTRAINED OPTIMIZATION SOLUTION FOR SINGLE POINT CONSTRAINT

Single target point constraint

Then we consider the problem upward. The shape of the curve must be modified as the point S has changed. Surely the shape of the Bézier surface can be determined automatically by control points, so the solution is to make sure how many control points should be adjusted and how to adjust them. We suppose the perturbation of the control point p_i is $\varepsilon_i=(\varepsilon_i^x,\varepsilon_i^y,\varepsilon_i^z)$, so that the modified Bézier curve is:

$$\tilde{C}(t)=\sum_{i=0}^{n}(p_i+\varepsilon_i)B_{i,n}(t), \ 0\le t\le 1.$$

Here we define the distance $d(C(t),\tilde{C}(t))$ between $C(t)$ and $\tilde{C}(t)$ as:

$$d(C(t),\tilde{C}(t))=\int_0^1 (C(t)-\tilde{C}(t))^2\mathrm{d}t.$$

Because $\tilde{C}(t)$ passes through target point T, the curve satisfies the following equation:

$$T=\tilde{C}(t_S)=\sum_{i=0}^{n}(p_i+\varepsilon_i)B_{i,n}(t_S).$$

We determine ε_i by constrained optimization method, such that:

$$\int_0^1 (C(t)-\tilde{C}(t))^2\mathrm{d}t=\int_0^1\left(\sum_{i=0}^{n}\varepsilon_i B_{i,n}(t)\right)^2\mathrm{d}t=\text{Min}$$

and the Lagrange function is defined by

$$L=\int_0^1\left(\sum_{i=0}^{n}\varepsilon_i B_{i,n}(t)\right)^2\mathrm{d}t+\lambda\,(T-\tilde{C}(t_S)).$$

where $\lambda=(\lambda_1, \lambda_2, \lambda_3)$ is Lagrange multiplier. Let $\frac{\partial}{\partial\varepsilon_i^x}(L)=\frac{\partial}{\partial\varepsilon_i^y}(L)=\frac{\partial}{\partial\varepsilon_i^z}(L)=0$ for $i=0,1,\ldots,n$ and after writing the derived formula in vector form, the following equations can be obtained:

$$\begin{cases}\lambda B_{i,n}(t_S)=2\int_0^1\left(\sum_{j=0}^{n}\varepsilon_j B_{j,n}(t)\right)B_{i,n}(t)\mathrm{d}t, \quad i=0,1,\ldots,n.\\ T=S+\sum_{i=0}^{n}\varepsilon_i B_{i,n}(t_S).\end{cases}$$

By solving the above equation system, we can finally get the perturbation ε_i. We express the answer in matrix form. First we suppose

$$f(i,j)=\int_0^1 B_{i,n}(t)B_{j,n}(t)\mathrm{d}t,$$
$$\boldsymbol{x}=(\varepsilon_0,\varepsilon_1,\cdots,\varepsilon_n,\lambda)^{\mathrm{T}},$$
$$\boldsymbol{\beta}=(0,0,\ldots,0,T-S)^{\mathrm{T}},$$

then we get

$$\boldsymbol{A}=\begin{pmatrix} f(0,0) & f(0,1) & \cdots & f(0,n) & -B_{0,n}(t_S)/2\\ f(1,0) & f(1,1) & \cdots & f(1,n) & -B_{1,n}(t_S)/2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ f(n,0) & f(n,1) & \cdots & f(n,n) & -B_{n,n}(t_S)/2\\ B_{0,n}(t_S) & B_{1,n}(t_S) & \cdots & B_{n,n}(t_S) & 0\end{pmatrix}.$$

So the answer can be expressed in the form of $\boldsymbol{Ax}=\boldsymbol{\beta}$.

Based on the practical issue, we can get the solution for the system of linear equations with normal and numerical method. When $\boldsymbol{A}^{-1}$ is non-singular, $\boldsymbol{x}=\boldsymbol{A}^{-1}\boldsymbol{\beta}$.

WITH ADDED END POINT AND TANGENT CONSTRAINT

In many applications, we hope to keep the end points unchanged. So the perturbation of the control points p_0 and p_n should be kept to be zero. And we can get the similar function system to the function system upwards. Even more we hope to keep the tangent of the curves for GC^1 continuity between adjacent curve segments. So the control points p_0 and p_n should remain unchanged, and the new control points p_1' and p_{n-1}' should be on the side p_0p_1 and p_np_{n-1} respectively. So we suppose

$$\varepsilon_1=\xi_1\frac{p_1-p_0}{\|p_1-p_0\|},\qquad \varepsilon_{n-1}=\xi_2\frac{p_n-p_{n-1}}{\|p_n-p_{n-1}\|},$$

then the new modified curve can be defined as follows:

$$\tilde{C}(t)=p_0B_{0,n}(t)+\left(p_1+\xi_1\frac{p_1-p_0}{\|p_1-p_0\|}\right)B_{1,n}(t)+\sum_{i=2}^{n-2}(p_i+\varepsilon_i)B_{i,n}(t)+\left(p_{n-1}+\xi_2\frac{p_n-p_{n-1}}{\|p_n-p_{n-1}\|}\right)B_{n-1,n}(t)+p_nB_{n,n}(t)$$

and the Lagrange function can be defined as:

$$L=\int_0^1\left(\xi_1\frac{p_1-p_0}{\|p_1-p_0\|}B_{1,n}(t)+\sum_{i=2}^{n-2}\varepsilon_iB_{i,n}(t)+\xi_2\frac{p_n-p_{n-1}}{\|p_n-p_{n-1}\|}B_{n-1,n}\right)^2\mathrm{d}t+\lambda\,(T-\tilde{C}(t_S)).$$

So the following equations can be derived by constrained optimization method,

$$\begin{cases}T-S=\xi_1\dfrac{p_1-p_0}{\|p_1-p_0\|}B_{1,n}(t_S)+\displaystyle\sum_{i=2}^{n-2}\varepsilon_iB_{i,n}(t_S)+\xi_2\dfrac{p_n-p_{n-1}}{\|p_n-p_{n-1}\|}B_{n-1,n}(t_S),\\ \lambda B_{i,n}(t_S)=2\displaystyle\int_0^1\left(\xi_1\dfrac{p_1-p_0}{\|p_1-p_0\|}B_{1,n}(t)+\sum_{j=0}^{n}\varepsilon_jB_{j,n}(t)+\xi_2\dfrac{p_n-p_{n-1}}{\|p_n-p_{n-1}\|}B_{n-1,n}\right)B_{i,n}(t)\mathrm{d}t,\\ \varepsilon_1=\xi_1\dfrac{p_1-p_0}{\|p_1-p_0\|},\\ \varepsilon_{n-1}=\xi_2\dfrac{p_n-p_{n-1}}{\|p_n-p_{n-1}\|}.\end{cases}$$

Finally, the solution can be obtained by solving the equations above.

MULTI-TARGET POINTS CONSTRAINTS

In many cases, there is more than one target point; how to adjust control points so that the modified curve passes through all those target points?

Suppose there are r+1 target points T_l, l=0,1,…,r. By projecting point T_l to curve $C(t)$, the corresponding parameters t_l can be obtained. Then we still choose perturbation $\varepsilon_i=(\varepsilon_i^x,\varepsilon_i^y,\varepsilon_i^z)$ for every control point (except p_0 and p_n), so that the modified curve:

$$\tilde{C}(t)=p_0B_{0,n}(t)+\sum_{i=1}^{n-1}(p_i+\varepsilon_i)B_{i,n}(t)+p_nB_{n,n}(t)$$

satisfies the requirement:

$$T_l=\tilde{C}(t_l)=p_0B_{0,n}(t_l)+\sum_{i=1}^{n-1}(p_i+\varepsilon_i)B_{i,n}(t_l)+p_nB_{n,n}(t_l),\quad l=0,1,\ldots,r.$$

We determine ε_i (i=1,…,n−1) by constrained optimization method, such that

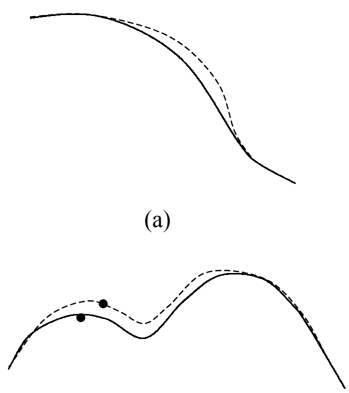
$$\int_0^1(C(t)-\tilde{C}(t))^2\mathrm{d}t=\int_0^1\left(\sum_{i=1}^{n-1}\varepsilon_iB_{i,n}(t)\right)^2\mathrm{d}t=\text{Min}$$

and Lagrange function is defined as:

$$L=\int_0^1\left(\sum_{i=1}^{n-1}\varepsilon_i B_{i,n}(t)\right)^2 \mathrm{d}t+\sum_{l=0}^{r}\lambda_l(T_l-\tilde{C}(t_l)) .$$

The following equation system can be derived:

$$\begin{cases} T_l=\sum_{\mathrm{i}=0}^{\mathrm{n}} p_i B_{i,n}(t_l)\ +\sum_{i=1}^{n-1}\varepsilon_i B_{i,n}(t_l) \quad l=0,1,\dots,r. \\ \sum_{l=0}^{r}\lambda_l B_{i,n}(t_l)=2\int_0^1\left(\sum_{j=0}^{n}\varepsilon_j B_{j,n}(t)\right)B_{i,n}(t)\mathrm{d}t. \end{cases}$$



(a)

(b)

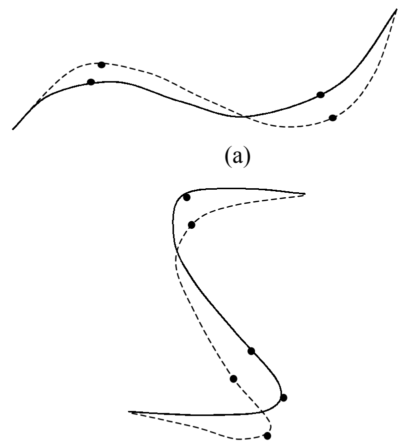
Fig.2 Shape modification of Bézier curve with single target point

(a) Cubic Bézier curve; (b) Quartic Bézier curve

The constrained optimization solution can be obtained by solving the above equation system.

PRACTICAL EXAMPLES

We now give several examples to show the effects of the proposed method. In Fig.2 and Fig.3, original curves are shown as solid line, and modified curves are shown as dash line. We give examples for both single and multiple target points cases.



(a)

(b)

Fig.3 Shape modification of Bézier curve with multiple target points

(a) Quartic Bézier curve; (b) Cubic Bézier curve

CONCLUSION

The paper presents a method for shape modification of Bézier curve by minimizing squared difference integral norm. So we convert the problem into the mathematical model of the Lagrange function. Both single and multiple target points are considered.

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