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A generalization of exotic options pricing formulae*

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Abstract: Exotic options, or "path-dependent" options are options whose payoff depends on the behavior of the price of the underlying between 0 and the maturity, rather than merely on the final price of the underlying, such as compound options, reset options and so on. In this paper, a generalization of the Geske formula for compound call options is obtained in the case of time-dependent volatility and time-dependent interest rate by applying martingale methods and the change of numeraire or the change of probability measure. An analytic formula for the reset call options with predetermined dates is also derived in the case by using the same approach. In contrast to partial differential equation (PDE) approach, our approach is simpler.

Key words: Risk-neutral measure, Compound options, Change of probability measure, Numeraire, Girsanov's theorem

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INTRODUCTION

The valuation and hedging of the ever increasing number of exotic options, is a topic that interests many practitioners seeking to answer their customer's need to hedge risk (in particular in the foreign exchange markets). Exotic options, or "path-dependent" options, such as compound options, reset options and Barrier options and the like, are options whose payoff depends on the behavior of the price of the underlying between 0 and the maturity (assumed to be fixed), rather than merely on the final price of the underlying. In this paper, we consider pricing compound and reset options with predetermined dates when the spot rate and the volatility of the underlying are all time-varying.

Compound option is a merely option of option with value based on the price of the underlying options. Compound options have been extensively used in corporate finance, where corporate investment opportunities are viewed as options. In practice it is more useful to obtain pricing formula for compound option. In the corporate finance setting, the underlying asset is the total value of the firm's asset, the various corporate securities, such that equity, warrants and convertible bonds, can be valued as contingent claim on the asset, therefore the options written on the corporate securities are examples of compound options. The valuation of compound options has potential applications for the value of several compound real opportunities where earlier investments are prerequisites for others to follow (Carr, 1988; Brealey and Myers, 1991; Geske and Johnson, 1984; Trigeorgis, 1996).

Since (Black and Scholes, 1973) an analytical formula for the price of compound option was obtained by Geske (1979) within the Black-Scholes framework, namely the stock price follows geometrical Brownian motion with constant volatility and constant interest rate when the market is complete. Basically, the pricing procedure consists of the following two steps: at first, the underlying option is priced according to the Black-Scholes method; then, the compound option is valued as a contingent claim on the option whose price has been found in the first

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step.

In (Elettra and Rossella, 2003), a generalization of the Geske formula for compound call option was derived when the volatility of risky asset and short-term interest rate are all time-dependent. They solved a partial differential equation with the terminal condition to extend the Geske formula and obtained a generalization for the compound option, which is also derived by a different approach—martingale theory and the change of probability measure in Section 2 in this paper. In contrast to partial differential equation (PDE) approach, our approach is simpler.

Reset options with predetermined dates are contracts whose strike price can be reset at predetermined dates. We assume that there are n reset dates, which are t_i (i=1,2,…,n), respectively. K is the strike price at issue of the option and T is maturity of the option. When the price of the underlying $S(t_1)$ is less than the strike price K at predetermined date t_1, the strike price is reset as $S(t_1)$, or else still K. Analogously the strike price is reset at dates t_i (i=2,3,…,n) like at t_1. We know that the reset strike price at T is $\min\{S(t_1),S(t_2),\ldots,S(t_n),K\}$. Reset options have been studied in past decades (Cheng and Zhang, 2000; Gray and Whaley, 1997; 1999; Nelken, 1998). In practice these studies were considered in B-S framework. Until now, the result on the reset options is still little.

However, in the above-mentioned application, the assumptions of constant volatility and constant interest rate seem to be inadequate. In this paper we derive an identical generalization of the Geske formula for compound call option in the case of time-dependent volatility and time-dependent interest rate by martingale method and the change of probability measure in Section 2. Section 3 extends the pricing formula of reset option in this situation. We conclude the paper in Section 4.

GENERALIZATION OF THE GESKE FORMULA FOR COMPOUND CALL OPTION

We now extend the pricing formula given by Geske (1979) for a compound option when the volatility of the underlying asset and the short rate are all time-dependent. In (Elettra and Rossella, 2003), the generalization of Geske formula was derived by solving a PDE with the terminal condition, which the price of compound option satisfies in this case. Here we make use of the martingale method and change of probability measure to derive it.

Let S denote the current value of the underlying asset, stock; $C(t, S)$ be the price at date t of a European call option on the stock, with strike price K_1 and exercise date T_1; $V(t)$ be the price of compound call option at date t on $C(t, S)$, with strike price K_2 and maturity $T_2<T_1$.

By martingale theory (Musiela and Rutkowski, 1997), we assume that the price of stock S under equivalent martingale measure Q follows the process

$$\frac{\mathrm{d}S(t)}{S(t)}=r(t)\mathrm{d}t+\sigma(t)\mathrm{d}W(t), \tag{1}$$

where $W(t)$ represents standard Brownian motion under Q, $\sigma(t)$ is the volatility assumed to be deterministic and time-dependent, $r(t)$ denotes the short rate assumed to be also time-dependent and deterministic.

As is well-known (Merton, 1973), the value of $C(t, S)$ is given by

$$C(t,S)=SN[k_2(t)]-\exp\left(-\int_t^{T_1}r(v)\mathrm{d}v\right)K_1N[k_1(t)],$$

where $N(k)$ represents the cumulative distribution function of normal random variable and

$$k_1(t)=\left[\ln\frac{S_t}{K_1}+\int_t^{T_1}\left(r(v)-\frac{1}{2}\sigma^2(v)\right)\mathrm{d}v\right]\Big/\left(\int_t^{T_1}\sigma^2(v)\mathrm{d}v\right)^{1/2}, \tag{2}$$

$$k_2(t)=\left[\ln\frac{S_t}{K_1}+\int_t^{T_1}\left(r(v)+\frac{1}{2}\sigma^2(v)\right)\mathrm{d}v\right]\Big/\left(\int_t^{T_1}\sigma^2(v)\mathrm{d}v\right)^{1/2}. \tag{3}$$

In what follows, we consider the compound call option written on the option $C(t, S)$. The payoff of the compound call option at maturity time T_2 is equal to

$$V(T_2)=(C(T_2,S)-K_2)^+,$$

where $x^+=\max(x, 0)$.

Since the price of call option $C(t, S)$ is a monotonically increasing function of stock price S, there exists a critical value S^* such that

$$C(T_2, S^*) = K_2 . \quad (4)$$

Eq.(4) implies that the compound call option is exercised at maturity T_2 when the stock price $S(T_2)$ is greater than S^* and the compound option is valueless at time T_2 when stock price $S(T_2)$ is less than S^*. Let $A_1=\{\omega\in\Omega|S(T_1)\geq K_1\}$ be the exercise set of the call option C; and $A_2=\{\omega\in\Omega|S(T_2)\geq S^*\}$ be the exercise set of the compound call option V.

By martingale theory (Pliska, 1997), we have

$$C(t,S) = B_t E_Q[B_{T_1}^{-1}(S(T_1)-K_1)^+ |\Im_t] \quad \text{for all } t\leq T_1, \quad (5)$$

$$V(t) = B_t E_Q[B_{T_2}^{-1}(C(T_2,S)-K_2)^+ \mid \Im_t] \quad \text{for all } t\leq T_2, (6)$$

where E_Q represents the expectation under measure Q, and B_t the saving market account given by

$$B_t = \exp\left(\int_0^t r(s)\mathrm{d}s\right).$$

From Eq.(6), we have that for all $t\leq T_2$

$$\begin{aligned} V(t) &= B_t E_Q[B_{T_2}^{-1}(C(T_2,S)-K_2)I_{A_2} \mid \Im_t] \\ &= B_t E_Q[B_{T_2}^{-1}C(T_2,S)I_{A_2} \mid \Im_t] \\ &\quad - K_2 \exp\left(-\int_t^{T_2} r(v)\mathrm{d}v\right) Q(A_2 \mid \Im_t), \end{aligned} \quad (7)$$

where I denotes the indicator function. Substituting Eq.(5) into the first term of Eq.(7) yields

$$\begin{aligned} & B_t E_Q[B_{T_2}^{-1}C(T_2,S)I_{A_2} \mid \Im_t] \\ &= B_t E_Q[E_Q(B_{T_1}^{-1}(S(T_1)-K_1)^+ \mid \Im_{T_2})I_{A_2} \mid \Im_t] \\ &= B_t E_Q[E_Q(I_{A_2}B_{T_1}^{-1}(S(T_1)-K_1)^+ \mid \Im_{T_2}) \mid \Im_t]. \end{aligned}$$

Since I_{A_2} is measurable with respect to the filtration $\Im_{T_2}$. The smoothing property of conditional expectation implies

$$\begin{aligned} & B_t E_Q[B_{T_2}^{-1}C(T_2,S)I_{A_2} \mid \Im_t] \\ &= B_t E_Q[I_{A_2}B_{T_1}^{-1}(S(T_1)-K_1)^+ \mid \Im_t] \\ &= B_t E_Q[B_{T_1}^{-1}S(T_1)I_{A_1\cap A_2} \mid \Im_t] \\ &\quad - K_1 \exp\left(-\int_t^{T_1} r(v)\mathrm{d}v\right) Q(A_1\cap A_2 \mid \Im_t) . \end{aligned} \quad (8)$$

We introduce the measure Q_S (Geman *et al.*, 1995), where the stock price $S(t)$ is taken as numeraire, defined by its Radon-Nikodym derivative of Q_S with respect to Q:

$$\frac{\mathrm{d}Q_S}{\mathrm{d}Q}\Big|\Im_{T_1} = \frac{S(T_1)}{B(T_1)S(0)} . \quad (9)$$

We obtain a general pricing formula of the compound call option.

Theorem 1 The price of the compound call option at time t is given by

$$\begin{aligned} V(t) &= S_t Q_S(A_1\cap A_2 \mid \Im_t) \\ &\quad - K_1 \exp\left(-\int_t^{T_1} r(v)\mathrm{d}v\right) Q_S(A_1\cap A_2 |\Im_t) \\ &\quad - K_2 \exp\left(-\int_t^{T_2} r(v)\mathrm{d}v\right) Q(A_1\cap A_2 |\Im_t), \end{aligned} \quad (10)$$

where $Q(A|\Im_t)$ is conditional probability of event A under Q with respect to the filtration $\Im_t$.

Proof The conditional expectation formula implies

$$E_Q\left[\frac{\mathrm{d}Q_S}{\mathrm{d}Q} I_{A_1\cap A_2} |\Im_t\right] = E_{Q_S}[I_{A_1\cap A_2} |\Im_t] E_Q\left[\frac{\mathrm{d}Q_S}{\mathrm{d}Q} |\Im_t\right].$$

By the martingale property of $B_{T_1}^{-1}S(T_1)$ under Q and Eq.(9), we have

$$E_Q\left[\frac{\mathrm{d}Q_S}{\mathrm{d}Q} |\Im_t\right] = \frac{S(t)}{B_t S(0)} .$$

Hence

$$\begin{aligned} B_t E_Q[B_{T_1}^{-1}S(T_1)I_{A_1\cap A_2} \mid \Im_t] &= S_t E_{Q_S}[I_{A_1\cap A_2} \mid \Im_t] \\ &= S_t Q_S(A_1\cap A_2 |\Im_t). \end{aligned}$$

Then the general pricing Eq.(10) is derived.

Eq.(11) below is the generalization of the Geske formula for compound call option pricing in the case of both time-dependent volatility and time-dependent spot rate.

Theorem 2 The price of the compound call option at time t is given by

$$\begin{aligned} V(t) &= S_t N[h_2(t), k_2(t), \rho(t, T_2, T_1)] \\ &- K_1 \exp\left(-\int_t^{T_1} r(v)\mathrm{d}v\right) N[h_1(t), k_1(t), \rho(t, T_2, T_1)] \\ &- K_2 \exp\left(-\int_t^{T_2} r(v)\mathrm{d}v\right) N(h_1(t)), \end{aligned} \quad (11)$$

where $N(h)$ and $N(h, k, \rho)$ represent the cumulative distribution function of standard normal random variable and bivariate cumulative normal distribution function with the correlation coefficient ρ, respectively and

$$h_1(t) = \left[\ln\frac{S_t}{S^*} + \int_t^{T_2} r(v)\mathrm{d}v - \frac{1}{2}\int_t^{T_2}\sigma^2(v)\mathrm{d}v\right] \bigg/ \left(\int_t^{T_2}\sigma^2(v)\mathrm{d}v\right)^{1/2}, \quad (12)$$

$$h_2(t) = \left[\ln\frac{S_t}{S^*} + \int_t^{T_2} r(v)\mathrm{d}v + \frac{1}{2}\int_t^{T_2}\sigma^2(v)\mathrm{d}v\right] \bigg/ \left(\int_t^{T_2}\sigma^2(v)\mathrm{d}v\right)^{1/2}, \quad (13)$$

$$\rho(t, s, u) = \left(\int_t^s \sigma^2(v)\mathrm{d}v\right)^{1/2} \bigg/ \left(\int_t^u \sigma^2(v)\mathrm{d}v\right)^{1/2}, \quad (14)$$

and $k_1(t)$ refers to Eq.(2) and $k_2(t)$ to Eq.(3).

Proof Since $S(t)$ satisfies the process Eq.(1), we obtain by Ito's formula

$$S(T_2) = S_t \exp\left(\int_t^{T_2} r(v)\mathrm{d}v - \frac{1}{2}\int_t^{T_2}\sigma^2(v)\mathrm{d}v + \int_t^{T_2}\sigma(v)\mathrm{d}W_v\right)$$

under Q. The exercise set of the compound call option A_2 is equivalent to

$$-\int_t^{T_2} r(v)\mathrm{d}v + \frac{1}{2}\int_t^{T_2}\sigma^2(v)\mathrm{d}v - \int_t^{T_2}\sigma(v)\mathrm{d}W_v \le \ln\frac{S_t}{S^*}.$$

By the independence of increment property of Brownian motion and the assumption of volatility $\sigma(t)$, we have

$$Q(A_2|\Im_t) = N(h_1(t)). \quad (15)$$

Similarly, $A_1 \cap A_2$ is equivalent to

$$-\int_t^{T_1} r(v)\mathrm{d}v + \frac{1}{2}\int_t^{T_1}\sigma^2(v)\mathrm{d}v - \int_t^{T_1}\sigma(v)\mathrm{d}W_v \le \ln(S_t/K_1),$$

$$-\int_t^{T_2} r(v)\mathrm{d}v + \frac{1}{2}\int_t^{T_2}\sigma^2(v)\mathrm{d}v - \int_t^{T_2}\sigma(v)\mathrm{d}W_v \le \ln(S_t/S^*).$$

Again applying the independence of increment property of Brownian motion and the assumption of volatility $\sigma(t)$, we obtain

$$Q(A_1 \cap A_2 \mid \Im_t) = N[h_1(t), k_1(t), \rho(t, T_2, T_1)]. \quad (16)$$

By Eq.(9) and Ito's formula, we have

$$\frac{\mathrm{d}Q_S}{\mathrm{d}Q}\Big|\Im_{T_1} = \exp\left(-\frac{1}{2}\int_0^{T_1}\sigma^2(v)\mathrm{d}v + \int_0^{T_1}\sigma(v)\mathrm{d}W_v\right).$$

By Girsanov's theorem, we know that

$$\tilde{W}(t) = W(t) - \int_0^t \sigma(v)\mathrm{d}v$$

is standard Brownian motion under Q_S. Therefore, under the measure Q_S, the stock price $S(t)$ satisfies

$$\frac{\mathrm{d}S_t}{S_t} = [r(t) + \sigma^2(t)]\mathrm{d}t + \sigma(t)\mathrm{d}\tilde{W}_t. \quad (17)$$

Again applying Ito's lemma yields

$$S(T_1) = S_t \exp\left(\int_t^{T_1} r(v)\mathrm{d}v + \frac{1}{2}\int_t^{T_1}\sigma^2(v)\mathrm{d}v + \int_t^{T_1}\sigma(v)\mathrm{d}\tilde{W}_v\right),$$

$$S(T_2) = S_t \exp\left(\int_t^{T_2} r(v)\mathrm{d}v + \frac{1}{2}\int_t^{T_2}\sigma^2(v)\mathrm{d}v + \int_t^{T_2}\sigma(v)\mathrm{d}\tilde{W}_v\right).$$

Under measure Q_S, the event $A_1 \cap A_2$ is equivalent to

$$-\int_t^{T_1} r(v)\mathrm{d}v - \frac{1}{2}\int_t^{T_1}\sigma^2(v)\mathrm{d}v - \int_t^{T_1}\sigma(v)\mathrm{d}\tilde{W}_v \le \ln\frac{S_t}{K_1},$$

$$-\int_t^{T_2} r(v)\mathrm{d}v - \frac{1}{2}\int_t^{T_2}\sigma^2(v)\mathrm{d}v - \int_t^{T_2}\sigma(v)\mathrm{d}\tilde{W}_v \le \ln\frac{S_t}{S^*}.$$

Similarly, we obtain

$$Q_S(A_1 \cap A_2 \mid \Im_t) = N[h_2(t), k_2(t), \rho(t, T_2, T_1)]. \quad (18)$$

Substituting Eqs.(15), (16) and (18) into Eq.(10) in Theorem 1, we obtain Eq.(11), which is given in (Eletta and Rossella, 2003) by solving a PDE with complex terminal condition.

PRICING RESET OPTION WITH PREDETERMINED DATES

For expositional simplification, we consider reset option with a predetermined date t_1 as its reset date and strike price K and maturity T. This means, if $S(t_1)<K$ at reset date t_1, the strike price of option is reset as $S(t_1)$ instead of K, while the strike price of option is the same as K if $S(t_1)\geq K$. We know that the payoff of the call option at maturity T is equal to

$$RC(T)=(S(T)-K)^{+}I_{(S(t_1)\geq K)}+(S(T)-S(t_1))^{+}I_{(S(t_1)<K)}. \tag{19}$$

In terms of the martingale theory, we have that the price of the call at time t equals

$$RC(t)=B_tE_Q[B_T^{-1}RC(T)\,|\,\Im_t]. \tag{20}$$

Substituting Eq.(19) into Eq.(20) yields

$$\begin{aligned}RC(t)=&B_tE_Q[B_T^{-1}\{S(T)-K\}^{+}I_{(S(t_1)\geq K)}\,|\,\Im_t]\\&+B_tE_Q[B_T^{-1}\{S(T)-S(t_1)\}^{+}I_{(S(t_1)<K)}\,|\,\Im_t].\end{aligned} \tag{21}$$

Theorem 3 The price at time t of the reset call option with one reset date t_1 is equal to

(1) For all $t\leq t_1$,

$$\begin{aligned}RC(t)=&S_t\{Q_S(A\,|\,\Im_t)+Q_S(D\,|\,\Im_t)\}\\&-K\exp\left(-\int_t^T r(v)\mathrm{d}v\right)Q(A\,|\,\Im_t)\\&-S_t\exp\left(-\int_{t_1}^T r(v)\mathrm{d}v\right)Q_S(D\,|\,\Im_t),\end{aligned} \tag{22}$$

where $A=\{\omega\in\Omega\,|\,S_T\geq K, S_{t_1}\geq K\}$,

$D=\{\omega\in\Omega\,|\,S_T\geq S_{t_1}, S_{t_1}<K\}$.

(2) For all $t>t_1$,

$$RC(t)=I_{(S(t_1)\geq K)}BS(t,S,K,T)+I_{(S(t_1)<K)}BS(t,S,S_{t_1},T), \tag{23}$$

where $BS(t, S, K, T)$ represents the B-S price of the European call option written on the underlying $S(t)$ with strike price K and maturity T at time t.

Proof (1) From Eq.(21), we obtain that for all $t\leq t_1$,

$$\begin{aligned}RC(t)=&B_tE_Q[B_T^{-1}S_TI_A\,|\,\Im_t]-KB_tE_Q[B_T^{-1}I_A\,|\,\Im_t]\\&+B_tE_Q[B_T^{-1}S_TI_D\,|\,\Im_t]-B_tE_Q[B_T^{-1}S_{t_1}I_D\,|\,\Im_t].\end{aligned}$$

The last term is equal to

$$B_tE_Q[B_T^{-1}S_{t_1}I_D\,|\,\Im_t]=\exp\left(-\int_{t_1}^T r(v)\mathrm{d}v\right)B_tE_Q[B_{t_1}^{-1}S_{t_1}I_D\,|\,\Im_t].$$

By the formula for conditional expectations under change of measure, we obtain the desired pricing Eq.(22).

(2) For all $t>t_1$, we know that $I_{(S(t_1)\geq K)}$ and $I_{(S(t_1)<K)}$ are measurable with respect to the filtration $\Im_t$ since $S(t_1)$ is adapted with respect to $\Im_{t_1}$ and $\Im_{t_1}\subseteq\Im_t$. Again from Eq.(21), we have

$$\begin{aligned}RC(t)=&I_{(S(t_1)\geq K)}B_tE_Q[B_T^{-1}\left(S_T-K\right)^{+}\,|\,\Im_t]\\&+I_{(S(t_1)<K)}B_tE_Q[B_T^{-1}(S_T-S_{t_1})^{+}\,|\,\Im_t].\end{aligned}$$

This is Eq.(23).

In order to obtain the explicit price formula at time $t\leq t_1$, we need to calculate the probability of the event A under measures Q and Q_S conditional on the filtration $\Im_t$, respectively and the probability of event D under Q_S conditional on the filtration $\Im_t$. In what follows, we use Girsanov's theorem to obtain the explicit pricing formula of the reset call option with one reset date t_1.

Theorem 4 The price at time $t\leq t_1$ of the reset call option with one reset date t_1 is equal to

$$\begin{aligned}RC(t)=&S_tN[d_1(t,T),d_1(t,t_1),\rho(t,t_1,T)]\\&-K\exp\left(-\int_t^T r(v)\mathrm{d}v\right)N[d_2(t,T),d_2(t,t_1),\rho(t,t_1,T)]\\&-S_t\exp\left(-\int_t^{t_1} r(v)\mathrm{d}v\right)N(d_3(t_1,T))N(-d_1(t,t_1))\\&-S_tN(d_3(t_1,T))N(-d_1(t,t_1)),\end{aligned} \tag{24}$$

where

$$d_1(t,s)=d(t,s)+\sigma(t,s)/2, \tag{25}$$

$$d_2(t,s)=d(t,s)-\sigma(t,s)/2, \tag{26}$$

$$d_3(t,s)=\int_t^s r(v)\mathrm{d}v+\sigma(t,s)/2, \tag{27}$$

$$d(t,s)=\left(\ln(S_t/K)+\int_t^s r(v)\mathrm{d}v\right)/\sigma(t,s),$$

$$\sigma^2(t,s)=\int_t^s \sigma^2(v)\mathrm{d}v.$$

and $\rho(t, t_1, T)$ refers to Eq.(14).

Proof Since the price $S(t)$ satisfies Eq.(1), we have that the event A is equivalent to

$$\int_t^T r(v)\mathrm{d}v-\frac{1}{2}\int_t^T \sigma^2(v)\mathrm{d}v+\int_t^T \sigma(v)\mathrm{d}W_v \ge \ln(K/S_t),$$

$$\int_t^{t_1} r(v)\mathrm{d}v-\frac{1}{2}\int_t^{t_1} \sigma^2(v)\mathrm{d}v+\int_t^{t_1} \sigma(v)\mathrm{d}W_v \ge \ln(K/S_t).$$

By using the independence of increment property of Brownian motion, we obtain

$$Q(A|\Im_t)=N[d_2(t,T),d_2(t,t_1),\rho(t,t_1,T)]. \tag{28}$$

From previous argument, we know that

$$\tilde{W}(t)=W(t)-\int_0^t \sigma(s)\mathrm{d}s$$

is standard Brownian motion under Q_S and that the price of the underlying satisfies

$$\frac{\mathrm{d}S_t}{S_t}=(r(t)+\sigma^2(t))\mathrm{d}t+\sigma(t)\mathrm{d}\tilde{W}_t$$

under the measure Q_S. Similarly, under Q_S, the event A is equivalent to

$$\int_t^T r(v)\mathrm{d}v+\frac{1}{2}\int_t^T \sigma^2(v)\mathrm{d}v+\int_t^T \sigma(v)\mathrm{d}\tilde{W}_v \ge \ln(K/S_t),$$

$$\int_t^{t_1} r(v)\mathrm{d}v+\frac{1}{2}\int_t^{t_1} \sigma^2(v)\mathrm{d}v+\int_t^{t_1} \sigma(v)\mathrm{d}\tilde{W}_v \ge \ln(K/S_t).$$

Straightforward calculation yields

$$Q_S(A|\Im_t)=N[d_1(t,T),d_1(t,t_1),\rho(t,t_1T)]. \tag{29}$$

At the same time, the independence of increment property of Brownian motion implies that

$$\begin{aligned}Q_S(D|\Im_t)&=Q_S\{S(T)>S(t_1)\,|\,\Im_t\}Q_S\{S(t_1)<K\,|\,\Im_t\}\\&=N(d_3(t_1,T))N(-d_1(t,t_1)).\end{aligned} \tag{30}$$

Substituting Eqs.(28), (29) and (30) into Eq.(22) yields the pricing Eq.(24).

When there is more than one reset date for a reset option, the pricing approach is similar to that used to price the reset option with a predetermined date. Its pricing formula is more complicated involving multivariate cumulative normal distribution. We omit it in this paper.

CONCLUSION

Compound options are merely options of options. Since (Geske, 1979) an analytical formula for the price of compound options was obtained within the Black-Scholes framework, that is, the stock price follows geometrical Brownian motion with constant volatility and constant interest rate when the market is complete. Until now, Elettra and Rossella (2003) obtained a generalization of the Geske formula for compound options when the volatility of the underlying asset and interest rate are all time-dependent. The formula of compound option in (Elettra and Rossella, 2003) was obtained by solving a PDE with a complex terminal condition. Reset options with predetermined dates are options with strike price reset at predetermined dates. Since (Gray and Whaley, 1999) an analytical formula for the price of reset put options was also obtained within the Black-Scholes framework.

In this paper, we derive the generalization of the Geske formula obtained in (Elettra and Rossella, 2003) by different approach—martingale theory and the change of probability measure. Our approach is more explicit and simple than the approach they adopted; a price formula for reset call option with one predetermined date is also obtained in this case by making use of the same approach.

In our approach, the change of measure is very important for obtaining the price formulae of exotic options. Girsanov's theorem is an essential tool. The choice of appropriate numeraire will provide the easiest calculation for obtaining relevant pricing formulae.

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