Journal of Zhejiang University SCIENCE A
ISSN 1009-3095 (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Pure bending of simply supported circular plate of transversely isotropic functionally graded material*

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Received Jan. 10, 2006; revision accepted Feb. 22, 2006

Abstract: This paper considers the pure bending problem of simply supported transversely isotropic circular plates with elastic compliance coefficients being arbitrary functions of the thickness coordinate. First, the partial differential equation, which is satisfied by the stress functions for the axisymmetric deformation problem is derived. Then, stress functions are obtained by proper manipulation. The analytical expressions of axial force, bending moment and displacements are then deduced through integration. And then, stress functions are employed to solve problems of transversely isotropic functionally graded circular plate, with the integral constants completely determined from boundary conditions. An elasticity solution for pure bending problem, which coincides with the available solution when degenerated into the elasticity solutions for homogenous circular plate, is thus obtained. A numerical example is finally presented to show the effect of material inhomogeneity on the elastic field in a simply supported circular plate of transversely isotropic functionally graded material (FGM).

Key words: Transversely isotropic, Functionally graded materials (FGMs), Pure bending problem, Simply supported circular plate, Axisymmetric deformation
doi:10.1631/jzus.2006.A1324 **Document code:** A **CLC number:** O343.1

INTRODUCTION

In recent years, functionally graded materials (FGMs) have attracted more and more attention. Due to their continuously varying material properties in space on the macroscopic scale, FGMs, therefore, are usually superior to conventional fiber-matrix materials in mechanical behavior, especially for performance under thermal loading. Now FGMs have been widely used in various fields including electronics, chemistry, optics, biomedicine, etc.

Heretofore, volumes of literatures have been published to investigate the mechanical performance of FGM structures (Chen *et al*., 2002; 2003; Wu and Tsai, 2004; Wu and Chen, 2006; Chi and Chung, 2006). Reddy *et al*.(1999) examined the axisymmetric bending of functionally graded circular and annular plates based on a first-order shear deformable plate theory. Exact relationships between the bending solutions of the classical plate theory (CPT) and the first-order shear theory (FST) were developed. Mian and Spencer (1998) established, in a simple manner, a large class of 3D thermo-elastic solutions for functionally graded plates with traction free surfaces from any solution of the classical thin plate equation. Three-dimensional elastic solution for functionally graded simply supported plate subject to transverse loading was also developed by Kashtalyan (2004). Recently, Bian (2005) comprehensively reviewed the state-of-the-art of study on functionally graded beams, plates and structures.

To the authors' knowledge, no literature on circular plates of transversely isotropic inhomogeneous materials has been published. In the present paper, two stress functions are introduced to seek the 3D elasticity analytical solution of pure bending problem of transversely isotropic functionally graded plate.

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* Project (Nos. 10472102 and 10432030) supported by the National Natural Science Foundation of China

Five elastic compliance coefficients involved can vary arbitrarily with the thickness coordinate, provided that positive-definiteness of strain-energy functions and some integrable conditions are satisfied. When all the elastic compliance coefficients are constant, the present solution degenerates to that for a homogeneous simply supported circular plate. Numerical results of a particular functionally graded circular plate, with only two elastic compliance coefficients being functions of the thickness coordinate, are given to clearly show the effect of material inhomogeneity parameter on the displacements and stresses in the plate.

BASIC EQUATIONS FOR AXISYMMETRIC DEFORMATION PROBLEM

The equations of equilibrium with body forces neglected and the strain-displacement relations, referred to a cylindrical coordinate system (r,θ,z), are listed as follows:

$$\left.\begin{aligned}&\sigma_{r,r}+\tau_{rz,z}+r^{-1}(\sigma_r-\sigma_\theta)=0,\\&\tau_{rz,r}+r^{-1}\tau_{rz}+\sigma_{z,z}=0,\end{aligned}\right\}\quad(1)$$

$$\varepsilon_r=u_{,r},\ \varepsilon_\theta=r^{-1}u,\ \varepsilon_z=w_{,z},\ \gamma_{rz}=u_{,z}+w_{,r},\quad(2)$$

where comma denotes differentiation with respect to the indicated variable. Stress-strain relations for transversely isotropic materials are

$$\left.\begin{aligned}&\varepsilon_r=s_{11}\sigma_r+s_{12}\sigma_\theta+s_{13}\sigma_z,\\&\varepsilon_\theta=s_{12}\sigma_r+s_{11}\sigma_\theta+s_{13}\sigma_z,\\&\varepsilon_z=s_{13}(\sigma_r+\sigma_\theta)+s_{33}\sigma_z,\\&\gamma_{rz}=s_{44}\tau_{rz},\end{aligned}\right\}\quad(3)$$

In this paper, we consider FGMs, whose elastic compliance coefficients in Eq.(3) are functions of z, i.e., $s_{ij}=s_{ij}(z)$ (i=1,4; j=1,2,3,4). For homogeneous materials, we simply have s_{ij}=const. (i=1,4; j=1,2,3,4).

In order to satisfy the equations of equilibrium, Eq.(1), two stress functions F and ψ were introduced in (Ding and Xu, 1988) as follows

$$\left.\begin{aligned}&\sigma_r=F_{,zz}+r^{-1}\psi_{,r},\ \sigma_\theta=F_{,zz}+\psi_{,rr},\\&\sigma_z=r^{-1}(rF_{,r})_{,r},\ \tau_{rz}=-F_{,rz}.\end{aligned}\right\}\quad(4)$$

The strain compatibility equations, as presented in (Lekhnitskii, 1981), can be derived from Eq.(2)

$$\varepsilon_r=(r\varepsilon_\theta)_{,r},\quad r\varepsilon_{\theta,zz}+\varepsilon_{z,r}=\gamma_{rz,z}.\quad(5)$$

STRESS FUNCTIONS

Take the stress functions to be

$$F=F_0(z),\ \psi=r^2\psi_2(z).\quad(6)$$

Substituting Eq.(6) into Eq.(4) yields

$$\sigma_r=\sigma_\theta=F_{0,zz}+2\psi_2,\ \sigma_z=\tau_{rz}=0.\quad(7)$$

Expressions of the strain components can be obtained by substituting Eq.(7) into Eq.(3)

$$\left.\begin{aligned}&\varepsilon_r=\varepsilon_\theta=(s_{11}+s_{12})(F_{0,zz}+2\psi_2),\\&\varepsilon_z=2s_{13}(F_{0,zz}+2\psi_2),\ \gamma_{rz}=0.\end{aligned}\right\}\quad(8)$$

Substituting Eq.(8) into Eq.(5) yields

$$[(s_{11}+s_{12})(F_{0,zz}+2\psi_2)]_{,zz}=0.\quad(9)$$

Integrating Eq.(9) twice from the lower limit $-h/2$ yields

$$(s_{11}+s_{12})(F_{0,zz}+2\psi_2)=a_0z+b_0.\quad(10)$$

STRESS, AXIAL FORCE, BENDING MOMENT AND DISPLACEMENTS

Substituting Eq.(10) into Eq.(7) leads to expressions for stress components

$$\sigma_r=\sigma_\theta=a_0S_1(z)+b_0S_0(z),\ \sigma_z=\tau_{rz}=0,\quad(11)$$

where

$$S_i(z)=z^i/(s_{11}+s_{12}),\ \ i=0,1.\quad(12)$$

It is easy to obtain expressions of the axial force N and the bending moment M from the expression of σ_r in Eq.(11)

$$N=\int_{-h/2}^{h/2}\sigma_r \mathrm{d}z=a_0L_{01}+b_0L_{00}, \tag{13}$$

$$M=\int_{-h/2}^{h/2}z\sigma_r \mathrm{d}z=a_0L_{11}+b_0L_{10}, \tag{14}$$

where

$$L_{mi}=\int_{-h/2}^{h/2}z^m S_i(z)\mathrm{d}z,\ \ m=0,1;\ i=0,1. \tag{15}$$

Substituting Eq.(11) into Eq.(3) yields the expressions of displacements

$$\left.\begin{aligned}&u=b_0r+a_0rz,\\&w=a_0[2S_1^1(z)-r^2/2]+2b_0S_0^1(z)+w_0,\end{aligned}\right\} \tag{16}$$

where w_0 is an integral constant, and

$$S_i^1(z)=\int_{-h/2}^{z}s_{13}(\xi)S_i(\xi)\mathrm{d}\xi,\quad i=0,\ 1. \tag{17}$$

From Eqs.(11) and (16), we can see that the stress components are dependent upon the sum of s_{11} and s_{12} only, while the displacements are related to (s_{11}+s_{12}) and s_{13}, since s_{33} and s_{44} are absent in both expressions of stresses and displacements.

PURE BENDING PROBLEM OF SIMPLY SUPPORTED CIRCULAR PLATE

Consider a simply supported circular plate with radius a and height h. The boundary conditions at $r=a$ are

$$N(a)=0,\ M(a)=\bar{M},\ w(a,0)=0, \tag{18}$$

where $\bar{M}$ is the prescribed moment applied along the plate edge.

Substituting Eqs.(13) and (14) into the first two expressions of Eq.(18) leads to the expressions of a_0 and b_0 as follows

$$a_0=\bar{M}L_{00}/D,\ b_0=-\bar{M}L_{01}/D, \tag{19}$$

where

$$D=L_{11}L_{00}-L_{01}L_{10}. \tag{20}$$

Substitution of the expression of w in Eq.(16) into the last boundary condition in Eq.(18) and making use of Eq.(19), we obtain the expression of w_0

$$w_0=\bar{M}\left\{L_{00}[a^2/2-2S_1^1(0)]+2L_{01}S_0^1(0)\right\}\Big/D. \tag{21}$$

We determine the stress components by substituting Eq.(19) into Eq.(11)

$$\left.\begin{aligned}&\sigma_z=\tau_{rz}=0,\\&\sigma_r=\sigma_\theta=\bar{M}\left[L_{00}S_1(z)-L_{01}S_0(z)\right]\Big/D.\end{aligned}\right\} \tag{22}$$

Then the displacements can be deduced from Eqs.(16), (19) and (21) without any extra difficulty

$$\left.\begin{aligned}&u=r\bar{M}(L_{00}z-L_{01})\big/D,\\&w=\bar{M}\left\{L_{00}[2S_1^1(z)-r^2/2]-2L_{01}S_0^1(z)\right\}\Big/D\\&\quad+\bar{M}\left\{L_{00}[a^2/2-2S_1^1(0)]+2L_{01}S_0^1(0)\right\}\Big/D.\end{aligned}\right\} \tag{23}$$

For homogeneous materials, i.e., when s_{ij}=const. (i=1,4; j=1,2,3,4), calculation according to Eqs.(15), (17) and (20) yields

$$\left.\begin{aligned}&L_{00}=\frac{h}{s_{11}+s_{12}},\ L_{11}=\frac{h^3}{12(s_{11}+s_{12})},\\&L_{01}=0,\ L_{10}=0,\ D=\frac{h^4}{12(s_{11}+s_{12})^2},\end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned}&S_0^1(z)=s_{13}(z+h/2)\big/(s_{11}+s_{12}),\\&S_1^1(z)=s_{13}(z^2-h^2/4)\big/[2(s_{11}+s_{12})].\end{aligned}\right\} \tag{25}$$

Substituting Eqs.(12) and (24) into Eq.(22) yields expressions of stress components for homogeneous materials

$$\sigma_r=\sigma_\theta=12\bar{M}z\big/h^3,\ \ \sigma_z=\tau_{rz}=0, \tag{26}$$

which are independent of the elastic compliance coefficients and coincide with those for isotropic materials as presented in (Timoshenko and Goodier, 1970). The expressions for displacements are also obtained as

$$\left.\begin{aligned}&u=12(s_{11}+s_{12})\bar{M}zr/h^3,\\&w=12s_{13}\bar{M}z^2/h^3+6(s_{11}+s_{12})\bar{M}(a^2-r^2)/h^3.\end{aligned}\right\} \tag{27}$$

Stresses and displacements for isotropic homogeneous materials can be easily determined by letting

$$\left.\begin{aligned}&s_{12}=s_{13}=-\nu/E,\ s_{11}=s_{33}=1/E,\\&s_{44}=2(s_{11}-s_{12})=2(1+\nu)/E,\end{aligned}\right\}\tag{28}$$

where E and ν are Young's modulus and Poisson's ratio, respectively.

NUMERICAL EXAMPLE AND DISCUSSION

Assume the elastic compliance coefficients to be of the form

$$\left.\begin{aligned}&s_{11}=s_{11}^0\mathrm{e}^{\lambda(\xi+1/2)},\ s_{12}=s_{12}^0\mathrm{e}^{\lambda(\xi+1/2)},\\&s_{13}=s_{13}^0,\ s_{33}=s_{33}^0,\ s_{44}=s_{44}^0,\end{aligned}\right\}\tag{29}$$

where s_{ij}^0 are elastic compliance coefficients at $z=-h/2$, $\xi=z/h$, $-1/2\le\xi\le1/2$, and λ is the gradient index. When $\lambda>0$, $s_{1j}(\xi)$ (j=1,2) are growing functions, i.e., they increase with ξ and hence the rigidity of the plate decreases. $s_{1j}(\xi)$ (j=1,2) are decaying functions when $\lambda<0$, indicating they decrease with ξ and hence the rigidity of the plate increases.

Expressions of stresses and displacements in this case are obtained as follows

$$\left.\begin{aligned}&\sigma_z=\tau_{rz}=0,\\&\sigma_r=\sigma_\theta=\bar{M}\mathrm{e}^{-\lambda(\xi+1/2)}\{\lambda^2[\lambda(1+\mathrm{e}^{-\lambda})-2(1-\mathrm{e}^{-\lambda})]\\&\quad+2\lambda^3\xi(1-\mathrm{e}^{-\lambda})\}/\{2h^2[(1-\mathrm{e}^{-\lambda})^2-\lambda^2\mathrm{e}^{-\lambda}]\},\end{aligned}\right\}\tag{30}$$

$$\left.\begin{aligned}&u=r\bar{M}(s_{11}^0+s_{12}^0)\lambda^2\{2\lambda(1-\mathrm{e}^{-\lambda})\xi+[\lambda(1+\mathrm{e}^{-\lambda})\\&\quad-2(1-\mathrm{e}^{-\lambda})]\}/\{2h^2[(1-\mathrm{e}^{-\lambda})^2-\lambda^2\mathrm{e}^{-\lambda}]\},\\&w=\bar{M}\lambda\{\lambda^2(s_{11}^0+s_{12}^0)(1-\mathrm{e}^{-\lambda})a^2(1-\eta^2)+2h^2s_{13}^0\mathrm{e}^{-\lambda/2}\\&\quad\times[\lambda(1+\mathrm{e}^{-\lambda})-2(1-\mathrm{e}^{-\lambda})][1-\mathrm{e}^{-\lambda\xi}]+4h^2s_{13}^0(1-\mathrm{e}^{-\lambda})\mathrm{e}^{-\lambda/2}\\&\quad\times(1-\mathrm{e}^{-\lambda\xi}-\lambda\xi\mathrm{e}^{-\lambda\xi})\}\{2h^3[(1-\mathrm{e}^{-\lambda})^2-\lambda^2\mathrm{e}^{-\lambda}]\}^{-1},\end{aligned}\right\}\tag{31}$$

where $\eta=r/a$, $0\le\eta\le1$. Applying L'Hospital's rule to Eqs.(30) and (31), when $\lambda\to0$, results in Eqs.(26) and (27), respectively, i.e., we obtain the stresses and displacements for a homogeneous material. From Eq.(30), we know that the radial and circumferential stresses are independent of r, bearing the same function of a single variable ξ.

For numerical calculation, we take the applied moment $\bar{M}=6000$ N·m/m, the radius of the plate a=0.005 m, the height h=0.001 m, and the material properties are: s_{11}^0 = 2.3189079×10^{-12} $\mathrm{m^2/N}$; s_{12}^0 = −0.68229254×10^{-12} $\mathrm{m^2/N}$; s_{13}^0 = −0.364385961×10^{-12} $\mathrm{m^2/N}$; s_{33}^0 = 2.1698872×10^{-12} $\mathrm{m^2/N}$; s_{44}^0 = 6.7842605×10^{-12} $\mathrm{m^2/N}$.

The dimensionless displacement w/h on the middle surface z=0 is shown in Fig.1, where $\eta=r/a$. It is seen that the dimensionless deflection of the plate increases with λ. At the point (r, z)=(0, 0), the displacement w=0.024478804h when λ=1, and w=0.00900525h when λ=−1. The former is about 2.717 times that of the latter.

The curves of $\sigma_r h^2/\bar{M}$ vs ξ for different λ's are shown in Fig.2 showing that the curve for λ=−1 is concave, the curve for λ=0 is a straight line, while that for λ=1 is convex.

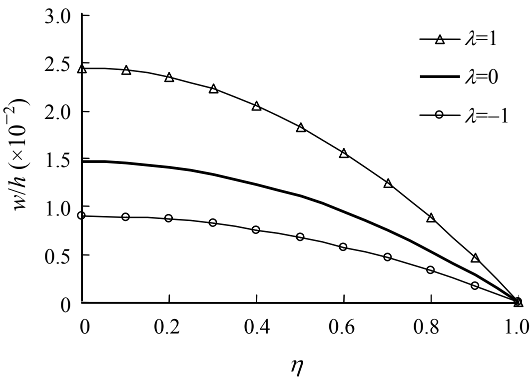

Fig.1 Dimensionless deflection curves when a/h=5

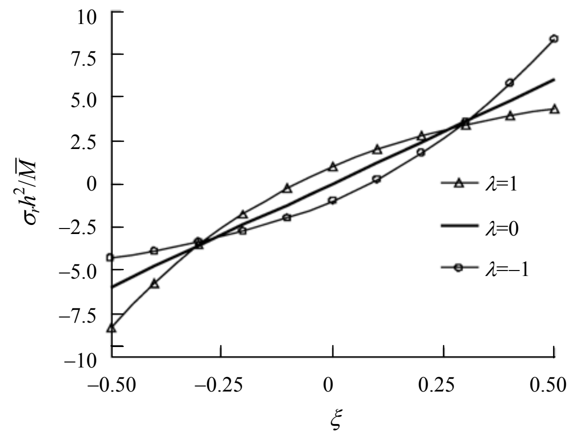

Fig.2 Dimensionless radial stresses curves

CONCLUSION

(1) An elasticity solution was derived based on stress function formulation for the axisymmetric problem of transversely isotropic circular plates of functionally graded materials. The material constants can vary along the thickness in an arbitrary way.

(2) Numerical results showed that the material inhomogeneity has important effect on the elastic deformation of and stresses in the plate. Thus, the deflection as well as the distribution of stress along the thickness can be easily controlled by changing the material gradient index. For practical purpose, however, optimizing object should be pre-determined case by case for the sake of easy choice of this index.

(3) Since the solution presented in this paper was derived based on the elasticity theory without introducing any simplifying assumptions on deformation and stress distribution as usually made in plate theories, it definitely serves as a benchmark for clarifying any approximate analysis or numerical method. The importance of such a benchmark solution is highlighted if we notice the fact that no numerical method performs well while keeping an economic computation cost, especially when the material properties of the plate vary rapidly through the thickness.

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