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Dynamic stiffness for thin-walled structures by power series*

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Abstract: The dynamic stiffness method is introduced to analyze thin-walled structures including thin-walled straight beams and spatial twisted helix beam. A dynamic stiffness matrix is formed by using frequency dependent shape functions which are exact solutions of the governing differential equations. With the obtained thin-walled beam dynamic stiffness matrices, the thin-walled frame dynamic stiffness matrix can also be formulated by satisfying the required displacements compatibility and forces equilibrium, a method which is similar to the finite element method (FEM). Then the thin-walled structure natural frequencies can be found by equating the determinant of the system dynamic stiffness matrix to zero. By this way, just one element and several elements can exactly predict many modes of a thin-walled beam and a spatial thin-walled frame, respectively. Several cases are studied and the results are compared with the existing solutions of other methods. The natural frequencies and buckling loads of these thin-walled structures are computed.

Key words: Dynamic stiffness method, Thin-wall structures, Power series, Buckling
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INTRODUCTION

Thin-walled structures have considerable technological importance in many situations of engineering practice. The rotary and warping inertia terms should be considered in the analyses of thin-walled structures (Alwis and Wang, 1996), although it is difficult to obtain the analytical solutions of governing differential equations, especially of thin-walled frame and spatial twisted structure. The finite element method (FEM) is widely used for vibration and stability analyses of structures, including thin-walled beams and columns (Gao *et al*., 2005; Kubiak, 2005; Saade *et al*., 2004). Since the assumed shape functions are used in FEM, many finite elements should be combined to obtain satisfactory solutions. Instead of the assumed shape functions in the conventional finite element analyses, the dynamic stiffness method uses the solutions of the governing equations as shape functions in vibration analysis. One element can predict many modes exactly in the classical sense. The method has been applied with success to many eigen-value problems (Leung *et al*., 2001a; 2001b) employing power series to solve the equations with high accuracy. It has also been applied to other applications such as damage identification (Lee and Shin, 2002) and analysis of vehicle-bridge interaction (Yang *et al*., 2004). The method can be applied to frames by using the finite element concept and frequency dependent shape functions. By this way, the compatibility and equilibrium requirements at the common nodes of the constituent members are easily satisfied.

In this paper, the exact solution by the dynamic stiffness method for the free vibration of thin-walled beam, frame and the spatial twisted helix beam is given. The accuracy and efficiency of the present method are illustrated by the numerical example of a thin-walled beam for comparison with existing solutions of other methods. A thin-walled frame and a twisted helix beam are also analyzed in this study.

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SELF-ADJOINT GOVERNING EQUATION

It is well known that the equilibrium of a structural member is governed by a partial differential equation, which may be reduced to a system of ordinary differential equations depending on one spatial parameter alone, by means of the Kantorovich method (Libai and Simmonds, 1998), due to a certain regularity of the member. Consider a system of uniform beams or one-dimensional structures with arbitrary cross-section subjected to external loads including static or dynamic excitation. The governing equation can eventually be written in the spatial domain in the general form

$$(\boldsymbol{A}_0 + \boldsymbol{A}_1 D^{(1)} + \boldsymbol{A}_2 D^{(2)} + \ldots + \boldsymbol{A}_n D^{(n)})\boldsymbol{u}(\xi) = \boldsymbol{f}(\xi) \quad (1)$$

with boundary conditions

$$\begin{cases} D^{(i)}\{\boldsymbol{u}(\xi)\}\big|_{\xi=0} = D^{(i)}\boldsymbol{u}(0), \\ D^{(i)}\{\boldsymbol{u}(\xi)\}\big|_{\xi=1} = D^{(i)}\boldsymbol{u}(1), \end{cases} \quad (2)$$
$$i = 0, 1, \ldots, n/2+1,$$

where $\xi=x/L$, L is the length of the element; $D^{(i)}(\cdot)$ denotes derivatives with respect to the position variable ξ; $\boldsymbol{u}(\xi)$ and $\boldsymbol{f}(\xi)$ are the responses vector and the excitation respectively; $\boldsymbol{A}_0, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_n$ are real square matrices of order m. Eq.(2) is self-adjoint with the condition that $\boldsymbol{A}_i$ is symmetrical or skew-symmetrical when i is even or odd, respectively. The highest differential order n is assumed to be even.

It is very difficult to find the closed-form solutions of Eq.(1) taking into account the effect of shear distortion, warping inertia and rotatory inertia. Power series are used to solve the equation with exact dynamic stiffness matrices. From the solutions by dynamic stiffness method, the frequency dependent shape functions of the structure can be derived, and the dynamic stiffness matrices can be constructed without difficulty.

VIBRATION ANALYSES OF THIN-WALLED BEAN AND FRAME

Dynamic stiffness matrix

A uniform beam with open thin-walled cross-section is considered (Fig.1). When the rotary and warping inertia terms are included, the following equation with exact dynamic stiffness matrices can be obtained by applying the Hamiltonian principle for harmonic free vibration with frequency ω (Leung, 1993a; Friberg, 1985)

$$(\boldsymbol{A}_0 + \boldsymbol{A}_2 D^{(2)} + \boldsymbol{A}_4 D^{(4)})\,\boldsymbol{u}(\xi) = \boldsymbol{0}, \quad (3)$$

where

$$\boldsymbol{A}_0 = -\rho A\omega^2 \begin{bmatrix} 1 & 0 & a_y \\ 0 & 1 & -a_x \\ a_y & -a_x & r_0^2 \end{bmatrix},$$

$$\boldsymbol{A}_2 = \frac{1}{L^2}\begin{bmatrix} \rho I_y\omega^2 + P & 0 & Pa_y - M_x \\ 0 & \rho I_x\omega^2 + P & -Pa_x - M_y \\ Pa_y - M_x & -Pa_x - M_y & \varDelta \end{bmatrix},$$

$$\boldsymbol{A}_4 = \frac{E}{L^4}\begin{bmatrix} I_y & & \\ & I_x & \\ & & I_w \end{bmatrix}, \quad \boldsymbol{u}(\xi) = [u(\xi), v(\xi), \theta(\xi)].$$

Here, $r_0^2 = a_x^2 + a_y^2 + (I_x + I_y)/A$; $\varDelta = \rho I_w\omega^2 + P r_0^2 - GJ + M_x\beta_y + M_y\beta_x$; $\beta_x = \frac{1}{I_y}\int_A (x^2+y^2)x\,\mathrm{d}A - 2a_x$; $\beta_y = \frac{1}{I_x}\int_A (x^2+y^2)y\,\mathrm{d}A - 2a_y$.

Denotations of symbols in the above equations are given as follows: ρ is material density of the beam; A area of the cross-section; a_x, a_y coordinates of the shear center with respect to the geometric center; P initial constant force; $M_x=Pe_y$; $M_y=Pe_x$; E Young's modulus; G shear modulus; I_x, I_y principal moments of inertia about x and y axes, respectively; I_w principal sectorial moment of inertia; J cross-sectional factor in torsion; u, v, θ vibration amplitudes.

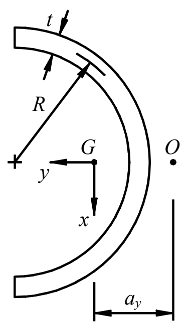

Fig.1 An open thin-walled beam cross-section. *G* is the geometric center and *O* is the shear center

The solutions of Eq.(3) can be assumed as a power series

$$\boldsymbol{u}(\xi)=\sum_{i=0}^{\infty}\boldsymbol{u}_i\xi^i. \tag{4}$$

Substituting Eq.(4) into Eq.(3) and collecting the coefficients of every power of ξ yields the following equations

$$\begin{bmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_2C_{4n}^{4n+2} & \boldsymbol{A}_4C_{4n}^{4n+4} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_0 & \boldsymbol{A}_2C_{4n+2}^{4n+4} & \boldsymbol{A}_4C_{4n+2}^{4n+6} \end{bmatrix} \times\left[\boldsymbol{u}_{4n},\boldsymbol{u}_{4n+2},\boldsymbol{u}_{4n+4},\boldsymbol{u}_{4n+6}\right]^{\mathrm{T}}=\boldsymbol{0},\quad n=0,1,..., \tag{5.1}$$

$$\begin{bmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_2C_{4n+1}^{4n+3} & \boldsymbol{A}_4C_{4n+1}^{4n+5} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_0 & \boldsymbol{A}_2C_{4n+3}^{4n+5} & \boldsymbol{A}_4C_{4n+3}^{4n+7} \end{bmatrix} \times\left[\boldsymbol{u}_{4n+1},\boldsymbol{u}_{4n+3},\boldsymbol{u}_{4n+5},\boldsymbol{u}_{4n+7}\right]^{\mathrm{T}}=\boldsymbol{0},\quad n=0,1,..., \tag{5.2}$$

where $C_j^i=i(i-1)...(i-j+1)$. From Eq.(5), two reiterative relationships can be found

$$[\boldsymbol{u}_{4n+4}\ \ \boldsymbol{u}_{4n+6}]^{\mathrm{T}} =-\begin{bmatrix} \boldsymbol{A}_4C_{4n}^{4n+4} & \boldsymbol{0} \\ \boldsymbol{A}_2C_{4n+2}^{4n+4} & \boldsymbol{A}_4C_{4n+2}^{4n+6} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_2C_{4n}^{4n+2} \\ \boldsymbol{0} & \boldsymbol{A}_0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_{4n} \\ \boldsymbol{u}_{4n+2} \end{bmatrix} =J_n(\omega)[\boldsymbol{u}_{4n}\ \ \boldsymbol{u}_{4n+2}]^{\mathrm{T}}, \tag{6.1}$$

$$[\boldsymbol{u}_{4n+5}\ \ \boldsymbol{u}_{4n+7}]^{\mathrm{T}} =-\begin{bmatrix} \boldsymbol{A}_4C_{4n+1}^{4n+5} & \boldsymbol{0} \\ \boldsymbol{A}_2C_{4n+3}^{4n+5} & \boldsymbol{A}_4C_{4n+3}^{4n+7} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_2C_{4n+1}^{4n+3} \\ \boldsymbol{0} & \boldsymbol{A}_0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_{4n+1} \\ \boldsymbol{u}_{4n+3} \end{bmatrix} =K_n(\omega)[\boldsymbol{u}_{4n+1}\ \ \boldsymbol{u}_{4n+3}]^{\mathrm{T}}. \tag{6.2}$$

Eq.(6) yields any coefficient $\boldsymbol{u}_i$ (i>3) expressed by the first four coefficients $\boldsymbol{u}_0$, $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ as

$$\begin{bmatrix} \boldsymbol{u}_{4n+4} \\ \boldsymbol{u}_{4n+6} \end{bmatrix}=(-1)^{n+1}\prod_{i=0}^{n}J_i(\omega)\cdot\begin{bmatrix} \boldsymbol{u}_0 \\ \boldsymbol{u}_2 \end{bmatrix} =\begin{bmatrix} JJ_n^1(\omega) & JJ_n^2(\omega) \\ JJ_n^3(\omega) & JJ_n^4(\omega) \end{bmatrix}\cdot\begin{bmatrix} \boldsymbol{u}_0 \\ \boldsymbol{u}_2 \end{bmatrix}, \tag{7.1}$$

$$\begin{bmatrix} \boldsymbol{u}_{4n+5} \\ \boldsymbol{u}_{4n+7} \end{bmatrix}=(-1)^{n+1}\prod_{i=0}^{n}K_i(\omega)\cdot\begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_3 \end{bmatrix} =\begin{bmatrix} KK_n^1(\omega) & KK_n^2(\omega) \\ KK_n^3(\omega) & KK_n^4(\omega) \end{bmatrix}\cdot\begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_3 \end{bmatrix}, \tag{7.2}$$

where $\boldsymbol{u}_0$, $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ are to be determined by imposing the natural boundary conditions. Using Eq.(7), the general solutions of Eq.(3) can be rewritten as

$$\boldsymbol{u}(\xi)=\sum_{i=0}^{\infty}\boldsymbol{N}_i(\omega)\cdot[\boldsymbol{u}_0,\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3]^{\mathrm{T}}\cdot\xi^i, \tag{8}$$

in which, $\boldsymbol{N}_0(\omega)$=[1, 0, 0, 0], $\boldsymbol{N}_1(\omega)$=[0, 1, 0, 0], $\boldsymbol{N}_2(\omega)$=[0, 0, 1, 0], $\boldsymbol{N}_3(\omega)$=[0, 0, 0, 1], $\boldsymbol{N}_{4n+4}(\omega)$= $[JJ_n^1(\omega),\ 0,\ JJ_n^2(\omega),\ 0]$, $\boldsymbol{N}_{4n+5}(\omega)$=$[0,\ KK_n^1(\omega),\ 0,\ KK_n^2(\omega)]$, $\boldsymbol{N}_{4n+6}(\omega)$=$[JJ_n^3(\omega), 0, JJ_n^4(\omega), 0]$, $\boldsymbol{N}_{4n+7}(\omega)$= $[0,\ KK_n^3(\omega),\ 0,\ KK_n^4(\omega)]$, n=0,1,... The nodal forces and displacements are

$$\boldsymbol{Q}=[\boldsymbol{S}(0),\boldsymbol{R}(0),\boldsymbol{S}(1),\boldsymbol{R}(1)]^{\mathrm{T}}=\boldsymbol{Y}(\omega)\cdot[\boldsymbol{u}_0,\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3]^{\mathrm{T}}, \tag{9.1}$$

$$\boldsymbol{q}=[\boldsymbol{u}(0),\boldsymbol{u}'(0),\boldsymbol{u}(1),\boldsymbol{u}'(1)]^{\mathrm{T}}=\boldsymbol{X}(\omega)\cdot[\boldsymbol{u}_0,\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3]^{\mathrm{T}}, \tag{9.2}$$

where $\boldsymbol{R}=\boldsymbol{A}_4\boldsymbol{u}''$ and $\boldsymbol{S}=-\boldsymbol{A}_2\boldsymbol{u}'-\boldsymbol{A}_4\boldsymbol{u}'''$. Combining both equations of Eq.(9) yields

$$\boldsymbol{Q}=\boldsymbol{Y}\boldsymbol{X}^{-1}\boldsymbol{q}=\boldsymbol{D}(\omega)\boldsymbol{q}, \tag{10}$$

where $\boldsymbol{D}(\omega)$ is the required dynamic stiffness.

Matrix formulation for frame

For frame consisting of thin-walled beams, there is more than one element in the system. Then the element coordinate $\boldsymbol{q}_{\mathrm{e}}$ is related to the system global coordinate $\boldsymbol{q}_{\mathrm{m}}$ by the transformation $\boldsymbol{T}_{\mathrm{e}}$,

$$\boldsymbol{q}_{\mathrm{e}}=\boldsymbol{T}_{\mathrm{e}}\boldsymbol{q}_{\mathrm{m}}. \tag{11}$$

The system matrix equation is obtained by the requirement of displacements compatibility and forces equilibrium

$$\boldsymbol{Q}_{\mathrm{m}}=\boldsymbol{D}_{\mathrm{m}}\boldsymbol{q}_{\mathrm{m}}, \tag{12}$$

where $\boldsymbol{Q}_{\mathrm{m}}=\sum_e\boldsymbol{T}_{\mathrm{e}}^{\mathrm{T}}\boldsymbol{Q}_{\mathrm{e}}$ and $\boldsymbol{D}_{\mathrm{m}}=\sum_e\boldsymbol{T}_{\mathrm{e}}^{\mathrm{T}}\boldsymbol{D}_{\mathrm{e}}\boldsymbol{T}_{\mathrm{e}}$. For natural vibration,

$$\boldsymbol{D}_{\mathrm{m}}(\omega)\boldsymbol{q}_{\mathrm{m}}=0. \tag{13}$$

With system dynamic stiffness matrix $\boldsymbol{D}_{\mathrm{m}}(\omega)$ available, the vibration and stability problem can be solved by means of the Wittrick-Williams Algorithm or the Sturm number and inverse iteration method (Leung, 1993a; 1993b).

Numerical examples

An example of a uniform thin-walled beam (see Fig.1) in (Leung, 1993a; Friberg, 1985) was analyzed. The following parameters are used in the computation: R=0.0245 m, t=0.004 m, L=0.82 m, A=3.08×10^{-6} m^2, a_x=0 m, a_y=0.0155 m, ρ=2711 kg/m^3, I_x=1.77×10^{-10} m^4, I_y=9.26×10^{-10} m^4, I_w=1.52×10^{-12} m^6, J=1.64×10^{-9} m^4, E=6.89×10^{10} Pa, G=2.65×10^{10} Pa. The lowest four frequencies in the increasing number of power series terms are listed in Table 1 for this clamped-free beam with center static axial load P=1790 N applied at the free end. It can be observed that fast convergence is possible with the increase of power series terms, and that the present solutions are in excellent agreement with those of Leung (1993a) and Friberg (1985). Fig.2 describes effects of the center axial load P on the first four natural frequencies of the beam computed with n=20. The existence of the compressive load has the strongest effects on the first mode and then the fourth mode, which correspond to the first and second bending vibration modes in the yz-plane. When the center compressive load exceeds 4.47 kN, the first mode will vanish.

Table 1 Natural frequencies of a thin-walled beam

Method		Natural frequencies (Hz)			
		Mode 1	Mode 2	Mode 3	Mode 4
Present	n=11	25.01	61.64	139.5	192.4
	n=12	25.01	61.29	136.1	192.4
	n=15	25.01	61.28	136.0	192.4
	n=20	25.01	61.28	136.0	192.4
(Leung, 1993a)		25.01	61.28	136.0	192.4
(Friberg, 1985)		25.01	61.28	136.0	192.4

Fig.3 plots the lowest four frequencies of the beam with a fixed axial load P=4000 N and various values of eccentricity e_x of the axial load. The applied moment softens the flexural modes but, on the other hand, hardens the torsional modes. The flexural modes in the yz-plane (first and fourth modes) decrease with increasing values of load eccentricity. The first mode vanishes when the value of the load eccentricity approaches 0.04 m since the applied moment decreases the critical buckling load of the beam.

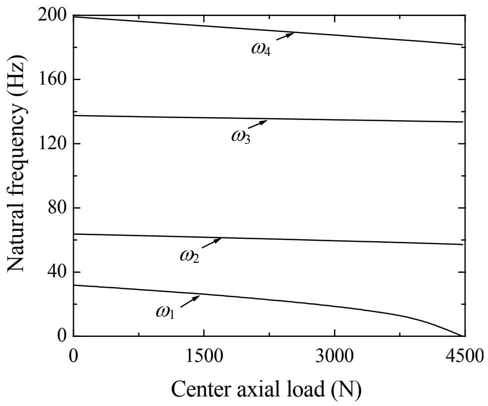

Fig.2 Effects of center static axial load on natural frequencies of beam

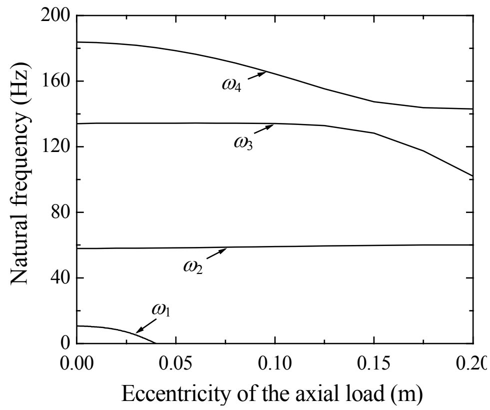

Fig.3 Effects of eccentricity of axial load on natural frequencies of beam

The second mode is the combination of the torsional modes and the flexural modes in the xz-plane. Since the eccentricity e_x has no effect on the flexural modes in the xz-plane, this mode increases with increasing values of e_x. But the characteristic curve of the third mode combining the torsional modes and the flexural modes in the yz-plane is more complicated than that of the other three modes.

With the formulation of Eq.(13), a thin-walled frame can be analyzed without difficulty. A frame (Fig.4) consisting of four thin-walled beams is shown in Fig.1. In order to investigate the effect of the central axial load applied on the top of the frame upon the natural frequencies, the variation of the lowest seven frequencies is plotted against the axial load P in Fig.5. The first and the seventh modes are the flexural modes in the yz-plane and the second mode is the flexural mode out of the yz-plane. The presence of the horizontal beam increases the natural frequencies and the critical buckling load of the frame.

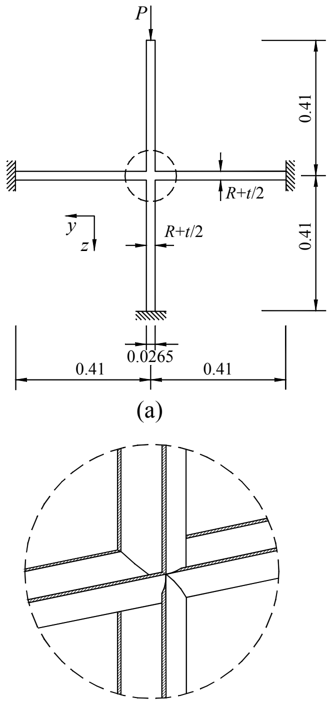

Fig.4 A frame consisting of two thin-walled beams. (a) Geometric size of frame; (b) 3D projection of the node of frame

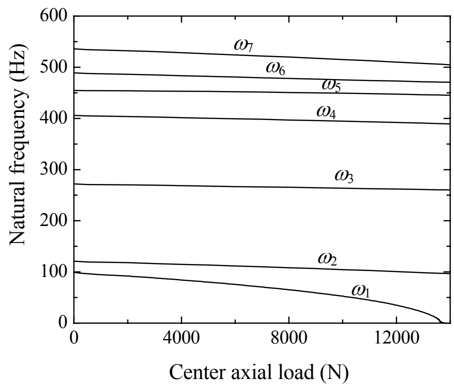

Fig.5 Effects of center static axial load on natural frequencies of frame

VIBRATION ANALYSES OF TWISTED HELIX BEAM

Exact stiffness matrix

A cylindrical helical rod with radius R and angle α is shown in Fig.6. Let the center line of the helix be measured by its arc length and the unit tangent, normal and binormal vectors along s be $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$,

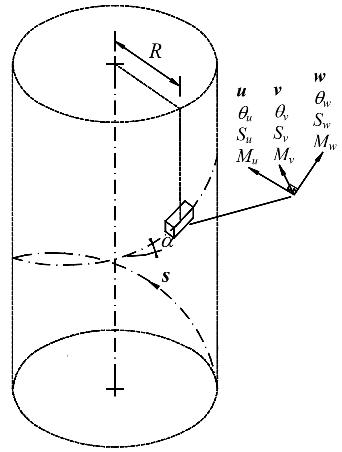

Fig.6 Definition of displacement, rotation, forces and moments

respectively. Then the Frenet-Serret formulae (Leung, 1993a) are

$$\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s}=k\boldsymbol{n},\ \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}s}=-k\boldsymbol{t}+\tau\boldsymbol{b},\ \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}s}=-\tau\boldsymbol{n}, \tag{14}$$

in which the curvature $k=\cos^2\alpha/R$ and the torsion $\tau=\sin\alpha\cos\alpha/R$. Let $\boldsymbol{u}(\xi)=[u(\xi), v(\xi), w(\xi), \theta(\xi), \phi(\xi), \psi(\xi)]^{\mathrm{T}}$ be defined as vector of the displacement and angular displacement along the local $\boldsymbol{n}$, $\boldsymbol{b}$ and $\boldsymbol{t}$ axes, in which $\xi=s/L$ with L being the beam length. Then the equilibrium equation, in the absence of internal forces, is given by

$$\begin{bmatrix} \boldsymbol{I}\frac{\mathrm{d}}{L\cdot\mathrm{d}\xi}-\boldsymbol{G} & \boldsymbol{0} \\ -\boldsymbol{J} & \boldsymbol{I}\frac{\mathrm{d}}{L\cdot\mathrm{d}\xi}-\boldsymbol{G} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_u & \\ & \boldsymbol{E}_\theta \end{bmatrix} \times \begin{bmatrix} \boldsymbol{I}\frac{\mathrm{d}}{L\cdot\mathrm{d}\xi}-\boldsymbol{G} & -\boldsymbol{J} \\ \boldsymbol{0} & \boldsymbol{I}\frac{\mathrm{d}}{L\cdot\mathrm{d}\xi}-\boldsymbol{G} \end{bmatrix} \boldsymbol{u}(\xi)=\boldsymbol{0}, \tag{15}$$

where $\boldsymbol{I}$ is the identity matrix, $\boldsymbol{E}_u=\mathrm{diag}[GA_x, GA_y, GA_z]$, $\boldsymbol{E}_\theta=\mathrm{diag}[EI_x, EI_y, EI_z]$, and

$$\boldsymbol{G}=\begin{bmatrix} 0 & \tau & -k \\ -\tau & 0 & 0 \\ k & 0 & 0 \end{bmatrix},\ \boldsymbol{J}=\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{16}$$

where A_z is the cross-sectional area of the beam; $A_x = A_z/\gamma_x$ and $A_y = A_z/\gamma_y$ are the effective shear areas; I_z is the torsional constant; I_x and I_y are the second moments of area about x and y axes respectively.

Expanding Eq.(15) yields the following form

$$(\boldsymbol{A}_0 + \boldsymbol{A}_1 D^{(1)} + \boldsymbol{A}_2 D^{(2)})\,\boldsymbol{u}(\xi) = \boldsymbol{0}. \tag{17}$$

Substituting Eq.(4) into Eq.(17) and collecting the coefficients of every power of ξ yields the following equations

$$\begin{bmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_1 C_{2n}^{2n+1} & \boldsymbol{A}_2 C_{2n}^{2n+2} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_0 & \boldsymbol{A}_1 C_{2n+1}^{2n+2} & \boldsymbol{A}_2 C_{2n+1}^{2n+3} \end{bmatrix} \times \left[\boldsymbol{u}_{2n}, \boldsymbol{u}_{2n+1}, \boldsymbol{u}_{2n+2}, \boldsymbol{u}_{2n+3}\right]^{\mathrm{T}} = \boldsymbol{0}, \quad n = 0, 1, \ldots \tag{18}$$

Then a reiterative relationship can be found

$$\begin{aligned} &[\boldsymbol{u}_{2n+2} \quad \boldsymbol{u}_{2n+3}]^{\mathrm{T}} \\ &= -\begin{bmatrix} \boldsymbol{A}_2 C_{2n}^{2n+2} & \boldsymbol{0} \\ \boldsymbol{A}_1 C_{2n+1}^{2n+2} & \boldsymbol{A}_2 C_{2n+1}^{2n+3} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_1 C_{2n}^{2n+1} \\ \boldsymbol{0} & \boldsymbol{A}_0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_{2n} \\ \boldsymbol{u}_{2n+1} \end{bmatrix} \\ &= L_n(\omega)[\boldsymbol{u}_{2n} \quad \boldsymbol{u}_{2n+1}]^{\mathrm{T}}. \end{aligned} \tag{19}$$

Just the same as for the thin-walled beam, Eq.(19) yields any coefficient $\boldsymbol{u}_i$ expressed by the first two coefficients $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ as

$$\begin{aligned} \begin{bmatrix} \boldsymbol{u}_{2n+2} \\ \boldsymbol{u}_{2n+3} \end{bmatrix} &= (-1)^{n+1} \prod_{i=0}^{n} L_i(\omega) \cdot \begin{bmatrix} \boldsymbol{u}_0 \\ \boldsymbol{u}_1 \end{bmatrix} \\ &= \begin{bmatrix} LL_n^1(\omega) & LL_n^2(\omega) \\ LL_n^3(\omega) & LL_n^4(\omega) \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{u}_0 \\ \boldsymbol{u}_1 \end{bmatrix}, \end{aligned} \tag{20}$$

where $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ are to be determined by imposing the natural boundary conditions. Eq.(20) can be used to rewrite the general solutions of Eq.(17) as

$$\boldsymbol{u}(\xi) = \sum_{i=0}^{\infty} N_i(\omega) \cdot \left[\boldsymbol{u}_0 \quad \boldsymbol{u}_1\right]^{\mathrm{T}} \cdot \xi^i, \tag{21}$$

in which, $\boldsymbol{N}_0(\omega)=[1, 0]$, $\boldsymbol{N}_1(\omega)=[0, 1]$, $\boldsymbol{N}_{2n+2}(\omega)=[LL_n^1(\omega), LL_n^2(\omega)]$, $\boldsymbol{N}_{2n+3}(\omega)=[LL_n^3(\omega), LL_n^4(\omega)]$, n=0, 1,... The local shear vector $\boldsymbol{V}=[V_x, V_y, V_z]^{\mathrm{T}}$ and moment $\boldsymbol{M}=[M_x, M_y, M_z]^{\mathrm{T}}$ are

$$\begin{bmatrix} \boldsymbol{V} \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} \boldsymbol{E}_u & \\ & \boldsymbol{E}_\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{I}\dfrac{\mathrm{d}}{L\cdot\mathrm{d}\xi} - \boldsymbol{G} & -\boldsymbol{J} \\ \boldsymbol{0} & \boldsymbol{I}\dfrac{\mathrm{d}}{L\cdot\mathrm{d}\xi} - \boldsymbol{G} \end{bmatrix} \boldsymbol{u}(\xi). \tag{22}$$

Then the dynamic stiffness matrix $\boldsymbol{D}(\omega)$ can also be obtained similar to that of the case for thin-walled beam mentioned in Section 3, but the expressions of nodal forces and displacements are a little different as shown below

$$\boldsymbol{Q}=[\boldsymbol{V}(0), \boldsymbol{M}(0), \boldsymbol{V}(1), \boldsymbol{M}(1)]^{\mathrm{T}}=Y(\omega)\cdot[\boldsymbol{u}_0, \boldsymbol{u}_1]^{\mathrm{T}}, \tag{23.1}$$
$$\boldsymbol{q}=[\boldsymbol{u}(0), \boldsymbol{u}'(0), \boldsymbol{u}(1), \boldsymbol{u}'(1)]^{\mathrm{T}}=X(\omega)\cdot[\boldsymbol{u}_0, \boldsymbol{u}_1]^{\mathrm{T}}. \tag{23.2}$$

Numerical examples

A twisted stairway with both ends clamped is shown in Fig.7, and it assumed as a uniform circularly cylindrical helix with rectangular cross-section of b=1.2 m and h=0.3 m. The other properties are γ_x=1.2; γ_y=1.2; $E=3\times10^{10}$ Pa; $G=E/2.5$; ρ=2500 kg/m^3; height of the beam H=3.6 m; $\alpha=1/\tan[H/(2\pi R)]$; I_z=0.0108 m^{-4}. With various radii R and the common value of n=60, the first six frequencies of the beam are plotted in Fig.8. The increasing values of R 'soften' the beams. With a fixed value of radius R=1.6 m, the mode shapes of the first six modes for the central line of the beam are shown in Fig.9.

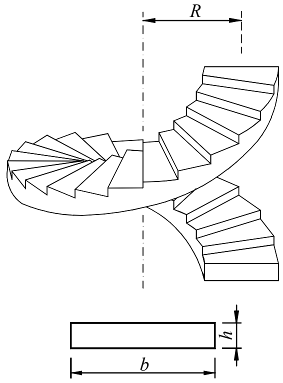

Fig.7 3D projection of a twisted stairway

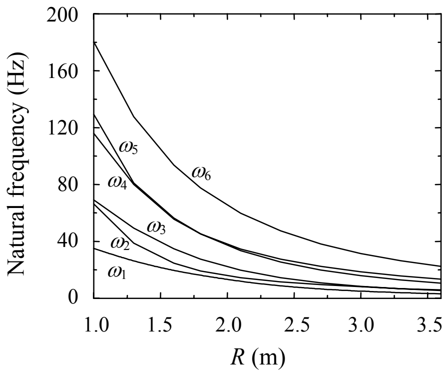

Fig.8 Effects of centroidal radii on natural frequencies of twisted stairway

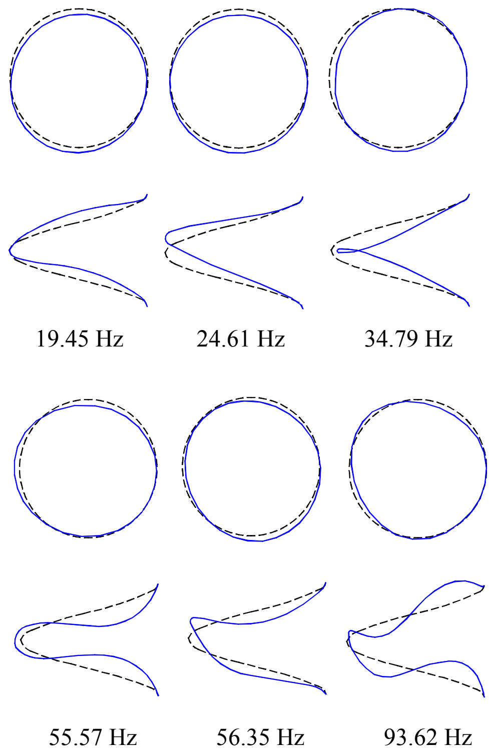

Fig.9 Top view and front view of mode shapes for central line of beam (solid lines: displaced shapes; dashed lines: undisplaced shapes)

CONCLUSION

The dynamic stiffness of thin-walled straight and twisted beams is derived explicitly in this paper. Comparison with the results of existing methods shows that the power series can provide very accurate solutions. Solution convergence rate increases with increasing number of power series terms. With the exact shape functions of each element, a structure consisting of thin-walled beams can be analyzed without difficulty.

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