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# 3D couette flow of dusty fluid with transpiration cooling

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**Abstract:** The couette dusty flow between two horizontal parallel porous flat plates with transverse sinusoidal injection of the dusty fluid at the stationary plate and its corresponding removal by constant suction through the plate in uniform motion was analyzed. Due to this type of injection velocity the dusty flow becomes 3D. Perturbation method is used to obtain the expressions for the velocity and temperature fields of both the fluid and dust. It was found that the velocity profiles of both the fluid and dust in the main flow direction decrease with the increase of the mass concentration of the dust particles, and those in cross flow direction increase with an increase in the mass concentration of the dust particles up to the middle of the channel and thereafter decrease with increase in mass concentration of the dust particles. The skin friction components $T_x$ and $T_z$ in the main flow and transverse directions respectively increase with an increase in the mass concentration of the dust particles (or) injection parameter. The heat transfer coefficient decreases with the increase of the injection parameter and increases with the increase in the mass concentration of the dust particles.

**Key words:** Couette dusty flow, 3D, Transpiration cooling, Transverse sinusoidal, Injection/suction, Dusty fluid, Mass concentration, Relaxation time

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## INTRODUCTION

Transpiration cooling is a very effective process to protect certain structural elements like combustion chamber walls, exhaust nozzles, or gas turbine blades in turbojet and rocket engines, from the influence of hot gases. In view of this, Eckert (1958) obtained an exact solution of the plane couette flow with transpiration cooling. The problem remained 2D due to the uniform injection and suction applied at the porous plates. Flow and heat transfer along a plane wall with periodic suction velocity was studied by Gersten and Gross (1974). Effects of such a suction velocity on various flow and heat transfer problems along flat and vertical porous plates were also studied extensively (Singh *et al*., 1978a; 1978b; Singh, 1990; 1991; 1993). However, to the best of the authors' knowledge, the application of transverse sinusoidal injection or suction velocity in the problems of transpiration cooling using two-phase fluid has not yet appeared in the literature. Therefore, a transverse sinusoidal injection velocity at the stationary porous plate of the channel using two-phase fluid is proposed because in actual practice injection or suction may not always be uniform.

Nag *et al*.(1979) discussed the couette flow of a dusty gas between two parallel infinite plates for impulsive start as well as uniformly accelerated start of one of the plates. Saffman (1962) formulated the equation of motion of dust laden gas in a simplified form by making some assumptions on dust particles. Miller (1966) investigated various aspects of hydro-dynamic and hydro-magnetic two-phase fluid flows in a non-rotating system. Debnath and Ghosh (1988) analyzed unsteady hydro-magnetic flow of a dusty fluid between two oscillating plates. Zung (1969) investigated the flow induced in fluid particle suspension by an infinite rotating disk. Dalal (1992) analyzed the generalized couette flow of a dusty gas due to an impulsive pressure gradient as well as due to impulsive start of the lower plate. Ahmed and Sharma (1997) investigated 3D free convective flow and heat transfer through a porous medium. Raptis (1983) analyzed the unsteady free convective flow through a porous medium bounded by an infinite vertical plate. Raptis and Perdikis (1985) studied the oscillatory flow through a porous medium by the presence of free convective flow. Singh and Verma (1995) investigated the 3D oscillatory flow through a porous medium with periodic permeability. Ahmed and Ahmed (2004) studied 2D MHD (magneto hydrodynamic) oscillatory flow along a uniformly moving infinite vertical porous plate bounded by porous medium.

The present paper attempts to study the effect of dust parameters on 3D couette flow with transpiration cooling and is aimed at studying the effect of the injection/suction parameter in the 3D couette dusty flow with transpiration cooling using two-phase fluid. The aim of the section is to study the effect of injection parameter, Prandtl number ($Pr=\upsilon/\alpha$, where $\upsilon$ and $\alpha$ are kinematic viscosity and thermal conductivity and $\alpha=k/(\rho c_p)$), dust parameters on the velocity field, temperature field, skin friction and Nusselt number ($Nu$).

The present paper is an extension of (Singh, 1999) for dusty fluid.

## FORMULATION OF THE PROBLEM

We consider the couette flow of a viscous incompressible dusty fluid between two parallel flat porous plates. A co-ordinate system is introduced with its origin on the lower stationary plate lying horizontally on the $x^*$-$z^*$ plane. The upper plate in uniform motion $U$ along the $x^*$-axis is subjected to a constant suction $V_0$ and the lower to a transverse sinusoidal injection velocity distribution of the form:

$$V^*(z^*)=V_0[1+\varepsilon\cos(\pi z^*/d)], \tag{1}$$

where $\varepsilon$ is a positive quantity (<<1), $V^*$ is transverse sinusoidal injection velocity distribution.

Without any loss of generality, the distance $d$ between the upper and lower plates is taken to be equal to the wave length of the injection velocity. The lower and upper plates are assumed to be at constant temperatures $T_0$ and $T_1$ respectively, where $T_1>T_0$. All physical quantities are independent of $x^*$ for this problem of fully developed laminar flow, however, the flow remains 3D due to the injection velocity Eq.(1).

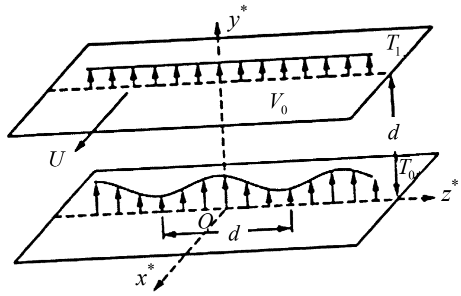


**Fig.1 Couette dusty flow with periodic injection and constant suction at the porous plates**

Denoting the velocity components of the fluid by $u$, $v$, $w$ in the $x$, $y$, $z$ directions and those of the dust particles by $u_p$, $v_p$, $w_p$ respectively, and the temperature of the fluid by $\theta$ and that of the dust by $\theta_p$, the problem is governed by the following non-dimensional equations.

### Equation of motion for fluid phase

$$\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{2}$$

$$v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\frac{1}{\lambda}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\frac{f}{\Lambda}(u_p-u), \tag{3}$$

$$v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{\partial p}{\partial y}+\frac{1}{\lambda}\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\frac{f}{\Lambda}(v_p-v), \tag{4}$$

$$v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\frac{1}{\lambda}\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\frac{f}{\Lambda}(w_p-w), \tag{5}$$

$$v\frac{\partial \theta}{\partial y}+w\frac{\partial \theta}{\partial z}=\frac{1}{\lambda Pr}\left(\frac{\partial^2 \theta}{\partial y^2}+\frac{\partial^2 \theta}{\partial z^2}\right)+\frac{2f}{3\Lambda Pr}(\theta_p-\theta), \tag{6}$$

where $\lambda$ is injection/suction parameter defined as $\lambda=V_0 d/\upsilon$, $\Lambda$ is relaxation time parameter of the dust particles, $f$ is mass concentration of the dust particles.

### Equation of motion for particle phase

$$\frac{\partial v_p}{\partial y}+\frac{\partial w_p}{\partial z}=0, \tag{7}$$

$$v_p\frac{\partial u_p}{\partial y}+w_p\frac{\partial u_p}{\partial z}=\frac{1}{\Lambda}(u-u_p), \tag{8}$$

$$v_{\mathrm{p}}\frac{\partial v_{\mathrm{p}}}{\partial y}+w_{\mathrm{p}}\frac{\partial v_{\mathrm{p}}}{\partial z}=\frac{1}{\varLambda}(v-v_{\mathrm{p}}),\tag{9}$$

$$v_{\mathrm{p}}\frac{\partial w_{\mathrm{p}}}{\partial y}+w_{\mathrm{p}}\frac{\partial w_{\mathrm{p}}}{\partial z}=\frac{1}{\varLambda}(w-w_{\mathrm{p}}),\tag{10}$$

$$v_{\mathrm{p}}\frac{\partial \theta_{\mathrm{p}}}{\partial y}+w_{\mathrm{p}}\frac{\partial \theta_{\mathrm{p}}}{\partial z}=\frac{-2(\theta_{\mathrm{p}}-\theta)}{3Pr\gamma\varLambda},\tag{11}$$

where

$$y=\frac{y^*}{d},\ z=\frac{z^*}{d},\ u=\frac{u^*}{U},\ v=\frac{v^*}{V_0},\ w=\frac{w^*}{V_0},$$

$$u_{\mathrm{p}}=\frac{u_{\mathrm{p}}^*}{U},\ v_{\mathrm{p}}=\frac{v_{\mathrm{p}}^*}{V_0},\ w_{\mathrm{p}}=\frac{w_{\mathrm{p}}^*}{V_0},\ p=\frac{p^*}{\rho V_0^2},$$

$$\theta=\frac{T^*-T_0}{T_1-T_0},\ \theta_{\mathrm{p}}=\frac{T_{\mathrm{p}}^*-T_0}{T_1-T_0},\ f=\frac{N_0 m}{\rho},\ \varLambda=\frac{\tau_{\mathrm{p}}V_0}{d},$$

$$\tau_{\mathrm{p}}=m/K,\quad \gamma=C_{\mathrm{p}}/C_{\mathrm{s}},$$

where '$^*$' stands for dimensional quantities, $\gamma$ is ratio of specific heat of dust to that of the fluid, $p$ is nondimensional pressure, $p^*$ is pressure, $\rho$ is density of the fluid, $N_0$ is number density of the dust particles assumed to be constant, $m$ is mass of the dust particles, $K$ is Stoke's drag constant which is $6\pi\mu r$ for spherical particles of radius $r$, $\tau_{\mathrm{p}}$ is the relaxation time of the particle phase which is the time taken for the dust particles to adjust to the velocity of the fluid, $C_{\mathrm{p}}$ and $C_{\mathrm{s}}$ is specific heat of the fluid and dust particles at constant pressure, respectively.

The boundary conditions for the problem in the dimensionless form are:

$$\begin{aligned}
y=0:&\ u=0,\ v(z)=1+\varepsilon\cos\pi z,\ w=0,\ \theta=0,\\
&\ u_{\mathrm{p}}=0,\ v_{\mathrm{p}}(z)=1+\varepsilon\cos\pi z,\ w_{\mathrm{p}}=0,\ \theta_{\mathrm{p}}=0;\\
y=1:&\ u=1,\ v=1,\ w=0,\ \theta=1,\\
&\ u_{\mathrm{p}}=1,\ v_{\mathrm{p}}=1,\ w_{\mathrm{p}}=0,\ \theta_{\mathrm{p}}=1,
\end{aligned}\tag{12}$$

where $\varepsilon$ is amplitude of the injection variation which is very small.

## SOLUTION OF THE PROBLEM

In order to solve these differential equations, we assume that the solution has the following form and that the amplitude of the injection velocity $\varepsilon$ (<<1) is very small:

$$f(y,z)=f_0(y)+\varepsilon f_1(y,z)+\varepsilon^2 f_2(y,z)+\ldots,\tag{13}$$

where $f$ stands for any of $u$, $u_{\mathrm{p}}$, $v$, $v_{\mathrm{p}}$, $w$, $w_{\mathrm{p}}$, $p$, $\theta$, $\theta_{\mathrm{p}}$. When $\varepsilon$=0 the problem reduces to 2D flow with constant injection and suction at both plates given by Eckert (1958). The solution of this 2D problem is

$$\begin{aligned}
&u_0(y)=A\mathrm{e}^{m_1 y}+B\mathrm{e}^{m_2 y}+C,\\
&u_{\mathrm{p}0}(y)=\frac{A\mathrm{e}^{m_1 y}}{\varLambda m_1+1}+\frac{B\mathrm{e}^{m_2 y}}{\varLambda m_2+1}+C+D\mathrm{e}^{-y/\varLambda},\\
&\theta_0(y)=A_1\mathrm{e}^{m_3 y}+B_1\mathrm{e}^{m_4 y}+C_1,\\
&\theta_{\mathrm{p}0}=D_1\mathrm{e}^{-b_1 y}+\frac{b_1 A_1\mathrm{e}^{m_3 y}}{m_1+b_1}+\frac{B_1\mathrm{e}^{m_4 y}b_1}{m_2+b_1}+C_1,\\
&w_0=0,\ w_{\mathrm{p}0}=0,\ v_0=1,\ v_{\mathrm{p}0}=1,\ p_0=\text{constant},\\
&m_1=\frac{\lambda(1+f)}{1-\lambda\varLambda},\ m_2=\lambda-\frac{1}{\varLambda}-m_1,\\
&m_3=\frac{(\lambda Pr-b_1)+\sqrt{(\lambda Pr-b_1)^2+4(d+\lambda Prb_1)}}{2},\\
&m_4=\frac{(\lambda Pr-b_1)-\sqrt{(\lambda Pr-b_1)^2+4(d+\lambda Prb_1)}}{2}.
\end{aligned}\tag{14}$$

When $\varepsilon\neq0$, substituting Eq.(13) in Eqs.(2)~(11) and comparing the coefficients of identical powers of $\varepsilon$, neglecting those of $\varepsilon^2$, $\varepsilon^3$, etc., the following first order equations are obtained with the help of Eq.(14). The expressions of $A$, $B$, $C$, $D$, $A_1$, $B_1$, $C_1$, $D_1$ are given in Appendix A.

### Fluid phase

$$\frac{\partial v_1}{\partial y}+\frac{\partial w_1}{\partial z}=0,\tag{15}$$

$$v_1\frac{\partial u_0}{\partial y}+\frac{\partial u_1}{\partial y}=\frac{1}{\lambda}\left(\frac{\partial^2 u_1}{\partial y^2}+\frac{\partial^2 u_1}{\partial z^2}\right)+\frac{f}{\varLambda}(u_{\mathrm{p}1}-u_1),\tag{16}$$

$$\frac{\partial v_1}{\partial y}=-\frac{\partial p_1}{\partial y}+\frac{1}{\lambda}\left(\frac{\partial^2 v_1}{\partial y^2}+\frac{\partial^2 v_1}{\partial z^2}\right)+\frac{f}{\varLambda}(v_{\mathrm{p}1}-v_1),\tag{17}$$

$$\frac{\partial w_1}{\partial y}=-\frac{\partial p_1}{\partial z}+\frac{1}{\lambda}\left(\frac{\partial^2 w_1}{\partial y^2}+\frac{\partial^2 w_1}{\partial z^2}\right)+\frac{f}{\varLambda}(w_{\mathrm{p}1}-w_1),\tag{18}$$

$$v_1\frac{\partial \theta_0}{\partial y}+\frac{\partial \theta_1}{\partial y}=\frac{1}{\lambda Pr}\left(\frac{\partial^2 \theta_1}{\partial y^2}+\frac{\partial^2 \theta_1}{\partial z^2}\right)+\frac{2f}{3Pr\varLambda}(\theta_{\mathrm{p}1}-\theta_1).\tag{19}$$

**Particle phase**

$$\frac{\partial v_{p1}}{\partial y}+\frac{\partial w_{p1}}{\partial z}=0, \tag{20}$$

$$v_{p1}\frac{\partial u_{p0}}{\partial y}+\frac{\partial u_{p1}}{\partial y}=\frac{1}{\Lambda}(u_1-u_{p1}), \tag{21}$$

$$\frac{\partial v_{p1}}{\partial y}=\frac{1}{\Lambda}(v_1-v_{p1}), \tag{22}$$

$$\frac{\partial w_{p1}}{\partial y}=\frac{1}{\Lambda}(w_1-w_{p1}), \tag{23}$$

$$v_{p1}\frac{\partial \theta_{p0}}{\partial y}+\frac{\partial \theta_{p1}}{\partial y}=-b_1(\theta_{p1}-\theta_1), \tag{24}$$

The corresponding boundary conditions reduce to

$$\begin{aligned} y=0:&\ u_1=0,\ v_1=\cos\pi z,\ w_1=0,\ \theta_1=0,\\ &u_{p1}=0,\ v_{p1}=\cos\pi z,\ w_{p1}=0,\ \theta_{p1}=0;\\ y=1:&\ u_1=0,\ v_1=0,\ w_1=0,\ \theta_1=0,\\ &u_{p1}=0,\ v_{p1}=0,\ w_{p1}=0,\ \theta_{p1}=0. \end{aligned} \tag{25}$$

These are the linear partial differential equations describing the 3D flow. In order to solve these equations we shall first consider Eqs.(15), (17), (18), (20), (22) and (23) for cross flow, being independent of the main flow components $u_1$ and $u_{p1}$, and the temperature fields $\theta_1$ and $\theta_{p1}$.

We assume $v_1$, $v_{p1}$, $w_1$, $w_{p1}$, $p_1$ of the following form:

$$v_1(y,z)=v_{11}(y)\cos\pi z, \tag{26}$$

$$w_1(y,z)=(-1/\pi)\{v'_{11}(y)\sin\pi z\}, \tag{27}$$

$$v_{p1}(y,z)=v_{p11}(y)\cos\pi z, \tag{28}$$

$$w_{p1}(y,z)=(-1/\pi)\{v'_{p11}(y)\sin\pi z\}, \tag{29}$$

$$p_1(y,z)=p_{11}(y)\cos\pi z, \tag{30}$$

where a prime denotes differentiation with respect to $y$. Eqs.(26)~(29) have been so chosen that the continuity Eqs.(15) and (20) are satisfied. Substituting these equations into Eqs.(17), (18), (22) and (23) and applying the corresponding transformed boundary conditions, we get the solution of $v_1$, $v_{p1}$, $w_1$, $w_{p1}$ and $p_1$ as:

$$v_1(y,z)=\{c_3e^{r_1y}+c_4e^{r_2y}+c_5e^{r_3y}-d_1c_1e^{\pi y}-d_2c_2e^{-\pi y}\}\cos\pi z, \tag{31}$$

$$w_1(y,z)=(-1/\pi)\{c_3r_1e^{r_1y}+c_4r_2e^{r_2y}+c_5r_3e^{r_3y}-d_1c_1\pi e^{\pi y}+d_2c_2\pi e^{-\pi y}\}\sin\pi z, \tag{32}$$

$$p_1(y,z)=(c_1e^{\pi y}+c_2e^{-\pi y})\cos\pi z. \tag{33}$$

Now assuming $u_1$, $u_{p1}$, $\theta_1$, $\theta_{p1}$ in the form:

$$u_1(y,z)=u_{11}(y)\cos\pi z, \tag{34}$$

$$u_{p1}(y,z)=u_{p11}(y)\cos\pi z, \tag{35}$$

$$\theta_1(y,z)=\theta_{11}(y)\cos\pi z, \tag{36}$$

$$\theta_{p1}(y,z)=\theta_{p11}(y)\cos\pi z, \tag{37}$$

and substituting in Eqs.(16), (19), (21) and (24) we obtain the following equations:

$$[\Lambda D^3+(1-\lambda\Lambda)D^2-((1+f)\lambda+\Lambda\pi^2)D-\pi^2]u_{11}=\lambda\left[v_{11}u'_0+fu'_{p0}+\Lambda(v'_{11}u'_0+v_{11}u''_0)\right], \tag{38}$$

$$(D\Lambda+1)u_{p11}=u_{11}-\Lambda v_{p11}u'_{p0}, \tag{39}$$

$$[D^3+(b_1-\lambda Pr)D^2-D[\pi^2+2f\lambda/(3\Lambda)+\lambda Prb_1]-\pi^2b_1]\theta_{11}=\lambda Pr[v_{11}\theta''_0+v'_{11}\theta'_0+b_1v_{11}\theta'_0]+2f\lambda/(3\Lambda)v_{p11}\theta'_{p0}, \tag{40}$$

$$(D+b_1)\theta_{p11}-b_1\theta_{11}=-v_{p11}\theta'_{p0}, \tag{41}$$

with corresponding boundary conditions

$$\begin{aligned} y=0:&\ u_{11}=0,\ \theta_{11}=0,\ u_{p11}=0,\ \theta_{p11}=0;\\ y=1:&\ u_{11}=0,\ \theta_{11}=0,\ u_{p11}=0,\ \theta_{p11}=0, \end{aligned} \tag{42}$$

where primes denote differentiation with respect to $y$. Solving Eqs.(38)~(41) under the boundary conditions Eq.(42) and using Eqs.(34)~(37) we get

$$\begin{aligned} u_1(y,z)=[&Le^{r_1y}+Me^{r_2y}+Ne^{r_3y}+K_1e^{(r_1+m_1)y}+K_2e^{(r_2+m_1)y}\\ &+K_3e^{(r_3+m_1)y}+K_4e^{(\pi+m_1)y}+K_5e^{(m_2-\pi)y}+K_6e^{(r_1+m_2)y}\\ &+K_7e^{(r_2+m_2)y}+K_8e^{(r_3+m_2)y}+K_9e^{(m_2+\pi)y}+K_{10}e^{(m_2-\pi)y}\\ &+K_{11}e^{(m_1-1/\Lambda)y}+K_{12}e^{(m_2-1/\Lambda)y}+K_{13}e^{-2y/\Lambda}\\ &+K_{14}e^{(r_1-1/\Lambda)y}+K_{15}e^{(r_2-1/\Lambda)y}+K_{16}e^{(r_3-1/\Lambda)y}\\ &+K_{17}e^{(\pi-1/\Lambda)y}+K_{18}e^{-(\pi+1/\Lambda)y}]\cos\pi z, \end{aligned} \tag{43}$$

where $K_1$, $K_2$, …, $K_{18}$ are known constants which are not presented for the sake of brevity. But these constants were taken into account while drawing the velocity profiles of the dusty fluid $u$ and dust $u_p$.

$$\begin{aligned}\theta_1(y,z)=[&R_1 e^{s_1 y}+R_2 e^{s_2 y}+R_3 e^{s_3 y}+t_1 e^{(r_1+m_1)y}+t_2 e^{(r_2+m_1)y}\\&+t_3 e^{(r_3+m_1)y}+t_4 e^{(\pi+m_1)y}+t_5 e^{(m_1-\pi)y}+t_6 e^{(r_1+m_2)y}\\&+t_7 e^{(r_2+m_2)y}+t_8 e^{(r_3+m_2)y}+t_9 e^{(m_2+\pi)y}+t_{10} e^{(m_2-\pi)y}\\&-t_{11} e^{-(b_1+1/\Lambda)y}-t_{12} e^{(r_1-b_1)y}-t_{13} e^{(r_2-b_1)y}\\&-t_{14} e^{(r_3-b_1)y}+t_{15} e^{(\pi-b_1)y}+t_{16} e^{-(\pi+b_1)y}\\&+t_{17} e^{(m_1-1/\Lambda)y}+t_{18} e^{(m_2-1/\Lambda)y}]\cos\pi z. \qquad (44)\end{aligned}$$

Now after knowing the velocity field we can calculate the skin-friction components $T_x$ and $T_z$ in the main flow and transverse directions, respectively as:

$$\begin{aligned}T_x=\frac{dT_x^*}{\mu U}=&\left(\frac{du_0}{dy}\right)_{y=0}+\varepsilon\left(\frac{du_{11}}{dy}\right)_{y=0}\cos\pi z\\=&(Am_1+Bm_2)+\varepsilon[Lr_1+Mr_2+Nr_3+K_1(m_1+r_1)\\&+K_2(m_1+r_2)+K_3(m_1+r_3)+K_4(m_1+\pi)+K_5(m_1-\pi)\\&+K_6(m_2+r_1)+K_7(m_2+r_2)+K_8(m_2+r_3)+K_9(m_2+\pi)\\&+K_{10}(m_2-\pi)+K_{11}(m_1-1/\Lambda)+K_{12}(m_2-1/\Lambda)\\&-2K_{13}/\Lambda+K_{14}(r_1-1/\Lambda)+K_{15}(r_2-1/\Lambda)+K_{16}(r_3-1/\Lambda)\\&+K_{17}(\pi-1/\Lambda)-K_{18}(\pi+1/\Lambda)]\cos\pi z, \qquad (45)\end{aligned}$$

$$\begin{aligned}T_z=\frac{dT_z^*}{\mu V_0}=&\,\varepsilon\left(\frac{dw_{11}}{dy}\right)_{y=0}\sin\pi z\\=&-\varepsilon(C_3 r_1^2+C_4 r_2^2-\pi^2 D-\pi^2 E)\sin\pi z. \qquad (46)\end{aligned}$$

From the temperature field we can obtain the heat transfer coefficient in terms of $Nu$ as:

$$\begin{aligned}Nu=\frac{dqw^*}{k(T_1-T_0)}=&\left(\frac{d\theta_0}{dy}\right)_{y=0}+\varepsilon\left(\frac{d\theta_{11}}{dy}\right)\cos\pi z\\=&(Am_1+Bm_2)+\varepsilon[R_1 s_1+R_2 s_2+R_3 s_3+t_1(m_1+r_1)\\&+t_2(m_1+r_2)+t_3(m_1+r_3)+t_4(m_1+\pi)+t_5(m_1-\pi)\\&+t_6(m_2+r_1)+t_7(m_2+r_2)+t_8(m_2+r_3)+t_9(m_2+\pi)\\&+t_{10}(m_2-\pi)+t_{11}(b_1+1/\Lambda)-t_{12}(r_1-b_1)\\&-t_{13}(r_2-b_1)+t_{14}(r_3-b_1)+t_{15}(\pi+b_1)\\&-t_{16}(\pi+b_1)+t_{17}(m_1-1/\Lambda)+t_{18}(m_2-1/\Lambda)]\cos\pi z. \qquad (47)\end{aligned}$$

From Eqs.(45) and (47) for the main flow skin friction and heat transfer coefficient it can be shown easily that $Nu=T_x$ for $Pr$=1. This means that the Reynolds analogy holds when $Pr$=1.

RESULTS AND DISCUSSION

**Main flow velocity profiles of the fluid phase**

From Fig.2a and Table 1 we see that irrespective of any value of $\lambda$, the velocity profiles decrease with an increase in the mass concentration of the dust particles. Also the profiles decrease with an increase in the injection parameter for any value of the dust particles mass concentration. The velocity profiles maintain an increasing trend near the lower plate and attain its maximum value very near the lower plate and thereafter it decreases steadily and reaches the value 1 at the other plate.

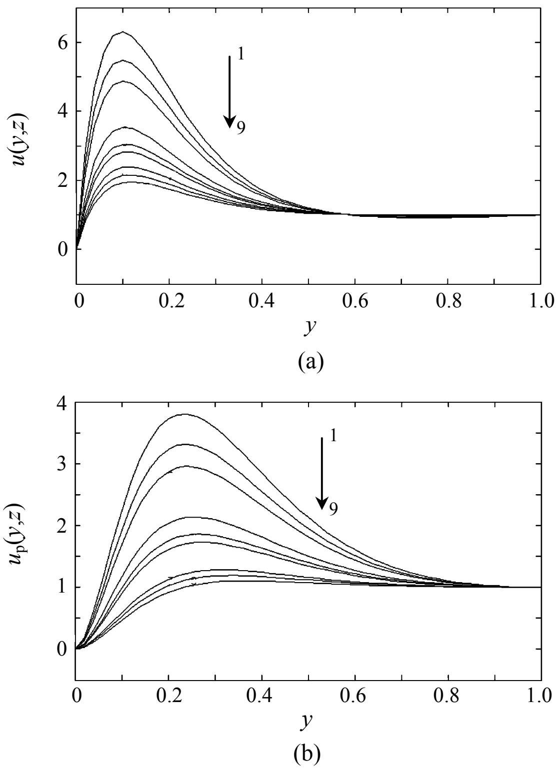


**Fig.2 Main flow velocity profiles of the fluid (a) $u(y, z)$ and (b) $u_p(y, z)$ for $z$=0, $\Lambda$=0.15 plotted against $y$ for various values of $f$ and $\lambda$**

**Table 1 Main flow velocity profiles of fluid and dust for various values of the injection parameter and concentration parameter**

| Curve | $f$ | $\lambda$ | Curve | $f$ | $\lambda$ |
|---|---|---|---|---|---|
| 1 | 0.2 | 0 | 6 | 0.6 | 0.5 |
| 2 | 0.4 | 0 | 7 | 0.2 | 1.0 |
| 3 | 0.6 | 0 | 8 | 0.4 | 1.0 |
| 4 | 0.2 | 0.5 | 9 | 0.6 | 1.0 |
| 5 | 0.4 | 0.5 | – | – | – |

**Main flow velocity profiles of the particle phase**

From Fig.2b we conclude that the velocity profiles of the dust particles decrease with an increase of either the mass concentration of the dust particles (or) injection parameter. The velocity profiles increase steadily near the lower plate and reach the maximum value a little away from the lower plate and thereafter decrease steadily and reach 1 at the other plate.

Both the fluid and dust particles behave in the same manner. But the profiles of the dust are at a lower height as compared with the fluid.

In the case of clean fluid the velocity profiles increase steadily whereas in the case of the dusty fluid the profiles increase steadily near the lower plate and a decreasing trend is seen as it approaches the other plate and attains the value one there. This phenomenon can be attributed to the presence of dust. So we can say that the presence of dust has an influence of accelerating the motion of the fluid.

**Cross flow velocity profiles of the fluid phase**

The cross flow velocity component $w$ is due to the transverse sinusoidal injection velocity distribution applied through the porous plate at rest. This secondary flow component is shown in Fig.3 and Table 2. It is interesting to note from Fig.3b that the cross flow velocity increases with an increase in the concentration of the dust particles at a point which is located a little away from the mid-way between the 2 plates and thereafter reverses its trend.

All the profiles increase steadily near the lower plate and reach the maximum value at a point a little away from the lower plate and thereafter a reverse trend occurred. From Fig.3a it is clear that the profiles increase with an increase in the injection parameter up to a point located a little away from the mid-way between the 2 plates and thereafter the profiles reverse their trend. All the profiles increase steadily near the lower plate, reach the maximum value very near it and decrease steadily and become zero at the other plate, because there is injection at the stationary plate and suction at the other plate, which is in motion. These two are exactly opposite processes.

**Cross flow velocity profiles of the particle phase**

From Fig.4 irrespective of any value of $\lambda$, the velocity profiles of the dust particles in the cross flow direction increase with an increase in the mass con-

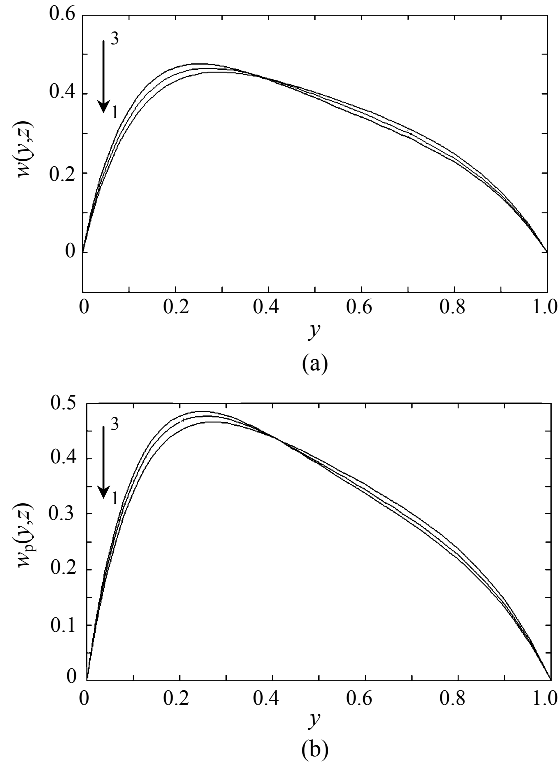


**Fig.3 Cross flow velocity profiles of the fluid (a) $w(y, z)$ and (b) $w_p(y, z)$ for $z$=0.5, $\Lambda$=0.15 and $f$=0.2 plotted against $y$ for various values of $\lambda$**

**Table 2 Cross flow velocity profiles of fluid and dust for various values of the injection parameter**

| Figures | Curve | $\lambda$ |
|---|---|---|
| Fig.3a | 1 | 0.2 |
| | 2 | 0.5 |
| | 3 | 1.0 |
| Fig.3b | 1 | 0.2 |
| | 2 | 0.4 |
| | 3 | 0.6 |

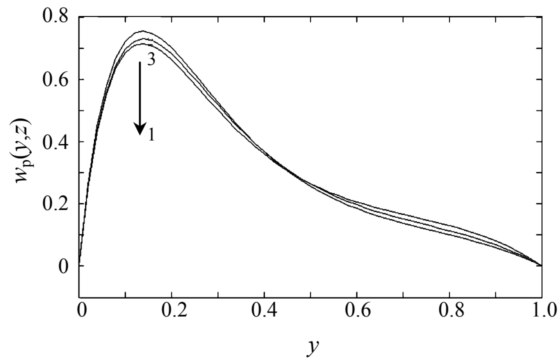


**Fig.4 Cross flow velocity profiles of the particle phase $w_p(y, z)$ for $z$=0.5, $\Lambda$=0.15, $\lambda$=0.2 plotted against $y$ for various values of $f$**

centration of the dust particles upto a point located a little away from the mid-way between the 2 plates and thereafter a reverse trend is seen. All the profiles increase steadily near the lower plate and reach the maximum value very near it and decrease steadily and become zero at the other plate.

**Skin friction $T_x$ and $T_z$**

The variations of the skin friction components $T_x$ and $T_z$ in the main flow and transverse directions, respectively, and *Nu* are shown in Tables 3 and 4.

From Tables 3 and 4 we conclude that for a given value of the mass concentration and relaxation time of the dust particles the skin friction $T_x$ increases with an increase in the injection parameter. Also for a given value of the injection parameter the skin friction $T_x$ increases with an increase in the mass concentration of the dust particles, whereas a reverse trend is seen when the relaxation time of the dust particle increases. We find an appreciable increase in $T_x$ value even for a slight increase in the relaxation time of the dust particles.

The skin friction component in transverse direction $T_z$ increases with an increase in the injection parameter for fixed values of the mass concentration and relaxation time of the dust particles. Also for a given value of the injection parameter the skin friction $T_z$ increases with an increase in the mass concentration of the dust particles. $T_z$ decreases with an increase in the relaxation time of the dust particles. Whereas for the clean fluid the skin friction $T_z$ decreases with an increase of the injection parameter.

**Nusselt number**

The Nusselt number *Nu* for both air ($Pr$=0.7) and water ($Pr$=7.0) decreases with an increase in the injection parameter for a given value of the mass concentration of the dust particles whereas *Nu* for both air and water increases with an increase of the mass concentration of the dust particles for any value of the injection parameter. *Nu* for both air and water decreases with an increase in the relaxation time of the dust particles for a fixed value of the mass concentration and injection parameters. It is to be noted that for clean fluid the heat transfer coefficient *Nu* decreases with increase of the injection parameter $\lambda$ for both air and water. It is also noted that the heat transfer coefficient is much lower in the case of water

**Table 3 Variations of $T_x$, $T_z$ and *Nu* for $Pr$=0.7 and $Pr$=7.0 with various values of $\lambda$**

| | $\lambda$ | $T_x$ | $T_z$ | *Nu* for $Pr$=0.7 | *Nu* for $Pr$=7.0 |
|---|---|---|---|---|---|
| $f$=0.4, $\Lambda$=0.2 | 0.2 | 5.8771 | 0.9657 | 9.6605 | 1.4321 |
| | 0.4 | 5.8905 | 1.0163 | 9.6586 | 1.1387 |
| | 0.6 | 5.9047 | 1.0446 | 9.6568 | 0.7801 |
| $f$=0.5, $\Lambda$=0.2 | 0.2 | 5.8900 | 0.9897 | 9.6939 | 1.5610 |
| | 0.4 | 5.9157 | 1.0347 | 9.6916 | 1.3239 |
| | 0.6 | 5.9417 | 1.0602 | 9.6893 | 1.1290 |
| $f$=0.6, $\Lambda$=0.2 | 0.2 | 5.9028 | 1.0098 | 9.7271 | 1.6526 |
| | 0.4 | 5.9408 | 1.0505 | 9.7243 | 1.4255 |
| | 0.6 | 5.9783 | 1.0738 | 9.7216 | 1.2651 |

**Table 4 Variations of $T_x$, $T_z$ and *Nu* for $Pr$=0.7 and $Pr$=7.0 with various values of $\Lambda$**

| | $\Lambda$ | $T_x$ | $T_z$ | *Nu* for $Pr$=0.7 | *Nu* for $Pr$=7.0 |
|---|---|---|---|---|---|
| $f$=0.2, $\lambda$=0.5 | 0.10 | 10.1276 | 1.5290 | 19.1161 | 3.0722 |
| | 0.11 | 9.2472 | 1.4235 | 17.3844 | 2.9277 |
| | 0.12 | 8.5301 | 1.3367 | 15.9412 | 2.8048 |
| $f$=0.4, $\lambda$=0.5 | 0.10 | 10.2186 | 1.5840 | 19.1841 | 2.2546 |
| | 0.11 | 9.3361 | 1.4765 | 17.4522 | 2.0452 |
| | 0.12 | 8.6166 | 1.3880 | 16.0089 | 1.8673 |
| $f$=0.6, $\lambda$=0.5 | 0.10 | 10.3081 | 1.6241 | 19.2517 | 2.5167 |
| | 0.11 | 9.4236 | 1.5148 | 17.5195 | 2.3082 |
| | 0.12 | 8.7016 | 1.4250 | 16.0759 | 2.1337 |

($Pr$=7.0) than in the case of air ($Pr$=0.7) for both dusty fluid and clean fluid.

When the concentration parameter of the dust particles is neglected it was found that our results are in perfect agreement with the results obtained by Singh (1999).

## CONCLUSION

(1) The effect of mass concentration parameter on $u$ and $u_{\mathrm{p}}$ is similar. But this effect is opposite to $T_x$, $T_z$ and $Nu$.

(2) The effect of injection parameter on $u$, $u_{\mathrm{p}}$ and $Nu$ is the same. But this effect is opposite to that on $T_x$ and $T_z$.

(3) The effect of relaxation time parameter on $T_x$, $T_z$ and $Nu$ is the same.

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## APPENDIX A

$$v_{\mathrm{p11}}(y,z)=\left\{c_6\mathrm{e}^{-y/\Lambda}+\frac{c_3\mathrm{e}^{r_1y}}{\Lambda r_1+1}+\frac{c_4\mathrm{e}^{r_2y}}{\Lambda r_2+1}+\frac{c_5\mathrm{e}^{r_3y}}{\Lambda r_3+1}-\frac{d_1c_1}{a}\mathrm{e}^{\pi y}-\frac{d_2c_2}{b}\mathrm{e}^{-\pi y}\right\}\cos\pi z, \quad \text{(A1)}$$

$$w_{\mathrm{p11}}(y,z)=\frac{-1}{\pi}\left\{\frac{-c_6}{\Lambda}\mathrm{e}^{-(1/\Lambda)}+\frac{c_3r_1\mathrm{e}^{r_1y}}{\Lambda r_1+1}+\frac{c_4r_2\mathrm{e}^{r_2y}}{\Lambda r_2+1}+\frac{c_5r_3\mathrm{e}^{r_3y}}{\Lambda r_3+1}-\frac{d_1c_1\pi}{a}\mathrm{e}^{\pi y}-\frac{d_2c_2\pi}{b}\mathrm{e}^{-\pi y}\right\}\sin\pi z, \quad \text{(A2)}$$

$$u_{\mathrm{p11}}(y,z)=\left\{\frac{L\mathrm{e}^{r_1y}}{\Lambda(r_1+1)}+\frac{M\mathrm{e}^{r_2y}}{\Lambda r_2+1}+\frac{N\mathrm{e}^{r_3y}}{\Lambda r_3+1}+R\mathrm{e}^{-y/\Lambda}\right.$$
$$+\frac{K_1\mathrm{e}^{(r_1+m_1)y}}{(r_1+m_1)\Lambda+1}+\frac{K_2\mathrm{e}^{(r_2+m_1)y}}{(r_2+m_1)\Lambda+1}+\frac{K_3\mathrm{e}^{(r_3+m_1)y}}{(m_1+r_3)\Lambda+1}$$
$$+\frac{K_4\mathrm{e}^{(m_1+\pi)y}}{(m_1+\pi)\Lambda+1}+\frac{K_5\mathrm{e}^{(m_1-\pi)y}}{(m_1-\pi)\Lambda+1}+\frac{K_6\mathrm{e}^{(m_2+r_1)y}}{(m_2+r_1)\Lambda+1}$$
$$+\frac{K_7\mathrm{e}^{(m_2+r_2)y}}{(m_2+r_2)\Lambda+1}+\frac{K_8\mathrm{e}^{(m_2+r_3)y}}{\Lambda(m_2+r_3)+1}+\frac{K_9\mathrm{e}^{(m_2+\pi)y}}{(m_2+\pi)\Lambda+1}$$
$$+\frac{K_{10}\mathrm{e}^{(m_2-\pi)y}}{(m_2-\pi)\Lambda+1}+\frac{K_{11}\mathrm{e}^{(m_1-1/\Lambda)y}}{(m_1-1/\Lambda)\Lambda+1}+\frac{K_{12}\mathrm{e}^{(m_2-1/\Lambda)y}}{(m_2-1/\Lambda)\Lambda+1}$$
$$-K_{13}\mathrm{e}^{-2y/\Lambda}+\frac{K_{14}\mathrm{e}^{(r_1-1/\Lambda)y}}{(r_1-1/\Lambda)\Lambda+1}+\frac{K_{15}\mathrm{e}^{(r_2-1/\Lambda)y}}{(r_2-1/\Lambda)\Lambda+1}$$

$$+\frac{K_{16}\mathrm{e}^{(r_3-1/\Lambda)y}}{(r_3-1/\Lambda)\Lambda+1}+\frac{K_{17}\mathrm{e}^{(\pi-1/\Lambda)y}}{\pi\Lambda}-\frac{K_{18}\mathrm{e}^{-(\pi+1/\Lambda)y}}{\pi\Lambda}$$

$$-\Lambda\left\{\frac{Am_1c_6\mathrm{e}^{(m_1-1/\Lambda)y}}{\Lambda(\Lambda m_1+1)m_1}+\frac{Am_1c_3\mathrm{e}^{(m_1+r_1)y}}{(\Lambda r_1+1)(\Lambda m_1+r_1)[(m_1+r_1)\Lambda+1]}\right.$$

$$+\frac{Am_1c_4\mathrm{e}^{(m_1+r_2)y}}{(\Lambda r_2+1)(\Lambda m_1+1)[(m_1+r_2)\Lambda+1]}$$

$$+\frac{Am_1c_5\mathrm{e}^{(m_1+r_3)y}}{(\Lambda m_1+1)(\Lambda r_3+r_1)[(m_1+r_3)\Lambda+1]}$$

$$-\frac{Am_1d_1c_1\mathrm{e}^{(m_1+\pi)y}}{a(\Lambda m_1+1)[(m_1+\pi)\Lambda+1]}-\frac{Am_1d_2c_2\mathrm{e}^{(m_1-\pi)y}}{b(\Lambda m_1+1)[(\Lambda(m_1-\pi)+1]}$$

$$+\frac{Bm_2c_6\mathrm{e}^{(m_2-1/\Lambda)y}}{(\Lambda m_2+1)[(m_2-1/\Lambda)\Lambda+1]}$$

$$+\frac{Bm_2c_3\mathrm{e}^{(m_2+r_1)y}}{(\Lambda r_1+1)(\Lambda m_1+1)[(m_2+r_1)\Lambda+1]}$$

$$+\frac{Bm_2c_4\mathrm{e}^{(m_2+r_2)y}}{(\Lambda r_2+1)[(\Lambda m_2+1)[\Lambda(m_2+r_2)+1]}$$

$$+\frac{Bm_2c_5\mathrm{e}^{(m_2+r_3)y}}{(\Lambda m_2+1)(\Lambda r_3+1)[\Lambda(m_2+r_3)+1]}$$

$$+\frac{-Bm_2d_1c_1\mathrm{e}^{(m_2+\pi)y}}{a(\Lambda m_2+1)[(m_2+\pi)\Lambda+1]}-\frac{Bm_2d_2c_2\mathrm{e}^{(m_2-\pi)y}}{b(\Lambda m_2+1)[(m_2-\pi)\Lambda+1]}$$

$$+\frac{Dc_6\mathrm{e}^{-2y/\Lambda}}{\Lambda}-\frac{Dc_3\mathrm{e}^{(r_1-1/\Lambda)y}}{\Lambda^2(\Lambda r_2+1)r_1}-\frac{Dc_4\mathrm{e}^{(r_2-1/\Lambda)y}}{\Lambda^2(\Lambda r_1+1)r_2}$$

$$-\frac{Dc_5\mathrm{e}^{(r_3-1/\Lambda)y}}{\Lambda^2(\Lambda r_3+1)r_3}+\frac{d_1c_1D\mathrm{e}^{(\pi-1/\Lambda)y}}{a\Lambda^2\pi}$$

$$\left.\left.-\frac{c_2d_2D}{b\Lambda^2\pi}\mathrm{e}^{-(\pi+1/\Lambda)y}\right\}\right\}\cos\pi z, \qquad \text{(A3)}$$

$$\theta_{\mathrm{p}1}(y,z)=\frac{b_1R_1\mathrm{e}^{s_1y}}{s_1+b_1}+\frac{b_1R_2\mathrm{e}^{s_2y}}{s_2+b_1}+\frac{b_1R_3\mathrm{e}^{s_3y}}{s_3+b_1}+R_4\mathrm{e}^{-b_1y}$$

$$+b_1\left[\frac{t_1\mathrm{e}^{(m_1+r_1)y}}{(m_1+r_1+b_1)}+\frac{t_2\mathrm{e}^{(m_1+r_2)y}}{(m_2+r_2+b_1)}+\frac{t_3\mathrm{e}^{(m_1+r_3)y}}{(m_1+r_3+b_1)}\right.$$

$$+\frac{t_4\mathrm{e}^{(m_1+\pi)y}}{(m_1+\pi+b_1)}+\frac{t_5\mathrm{e}^{(m_1-\pi)y}}{(m_1-\pi+b_1)}+\frac{t_6\mathrm{e}^{(m_2+r_1)y}}{(m_2+r_1+b_1)}$$

$$+\frac{t_7\mathrm{e}^{(m_2+r_2)y}}{(m_1+r_2+b_1)}+\frac{t_8\mathrm{e}^{(m_3+r_3)y}}{(m_1+r_3+b_1)}+\frac{t_9\mathrm{e}^{(m_2+\pi)y}}{(m_2+\pi+b_1)}$$

$$+\frac{t_{10}\mathrm{e}^{(m_2-\pi)y}}{(m_2-\pi+b_1)}+\frac{t_{11}\mathrm{e}^{-(b_1+1/\Lambda)y}}{1/\Lambda}-\frac{t_{12}\mathrm{e}^{(r_1-b_1)y}}{r_1}-\frac{t_{13}\mathrm{e}^{(r_2-b_1)y}}{r_2}$$

$$-\frac{t_{14}\mathrm{e}^{(r_3-b_1)y}}{r_3}+\frac{t_{15}\mathrm{e}^{(\pi-b_1)y}}{\pi}-\frac{t_{16}\mathrm{e}^{-(\pi+b_1)y}}{\pi}+\frac{t_{17}\mathrm{e}^{(m_1-1/\Lambda)y}}{m_1-1/\Lambda+b_1}$$

$$\left.+\frac{t_{18}\mathrm{e}^{(m_2-1/\Lambda)y}}{m_2-(1/\Lambda)+b_1}\right]-\Lambda DC_6b_1\mathrm{e}^{-(b_1+1/\Lambda)y}+\frac{C_3Db_1\mathrm{e}^{(r_1-b_1)y}}{r_1(\Lambda r_1+1)}$$

$$+\frac{C_4Db_1\mathrm{e}^{(r_2-b_1)y}}{(\Lambda r_2+1)\Lambda}+\frac{Db_1c_5\mathrm{e}^{(r_3-b_1)y}}{(\Lambda r_3+1)r_3}-\frac{d_1c_1Db_1}{\pi a}\mathrm{e}^{(\pi-b_1)y}$$

$$+\frac{d_2c_2Db_1}{\pi b}\mathrm{e}^{(b_1-\pi)y}-\frac{b_1Am_1c_6\mathrm{e}^{(m_1-1/\Lambda)y}}{(m_1+b_1)(m_1-1/\Lambda+b_1)}$$

$$-\frac{b_1Am_1c_3\mathrm{e}^{(r_1-m_1)y}}{(m_1+b_1)(\Lambda r_1+1)(r_1+m_1+b_1)}$$

$$-\frac{b_1Am_1c_4\mathrm{e}^{(m_1-r_2)y}}{(m_1+b_1)(\Lambda r_2+1)(m_2+r_2+b_1)}$$

$$-\frac{b_1Am_1c_5\mathrm{e}^{(m_1+r_2)y}}{(m_1+b_1)(\Lambda r_3+1)(m_1+r_3+b_1)}$$

$$+\frac{d_1b_1Am_1c_1\mathrm{e}^{(m_1+\pi)y}}{a(m_1+b_1)(m_1+\pi+b_1)}+\frac{d_2b_1Am_1c_2\mathrm{e}^{(m_1-\pi)y}}{b(m_1+b_1)(m_1-\pi+b_1)}$$

$$-\frac{b_1m_2Bc_6\mathrm{e}^{(m_2-1/\Lambda)y}}{(m_2+b_1)(m_2-1/\Lambda+b_1)}$$

$$-\frac{b_1m_2Bc_3\mathrm{e}^{(r_1-m_2)y}}{(m_2+b_1)(\Lambda r_1+1)(r_1+m_2+b_1)}$$

$$-\frac{c_4b_1m_2B\mathrm{e}^{(r_2-m_2)y}}{(m_2+b_1)(\Lambda r_2+1)(r_2+m_2+b_1)}$$

$$-\frac{b_1m_2Bc_5\mathrm{e}^{(m_2+r_3)y}}{(m_2+b_1)(\Lambda r_3+1)(m_2+r_3+b_1)}$$

$$\left.+\frac{d_1c_1b_1m_2B\mathrm{e}^{(m_2+\pi)y}}{a(m_2+b_1)(m_2+\pi+b_1)}+\frac{d_2c_2b_1m_2B\mathrm{e}^{(m_2+\pi)y}}{b(m_2+b_1)(m_2+\pi+b_1)}\right]\cos\pi z, \qquad \text{(A4)}$$

where $A$, $B$, $C$, $D$ satisfy the equations

$$A+B+C=0,\ A\mathrm{e}^{m_1}+B\mathrm{e}^{m_2}+C=1,$$

$$\frac{A}{\Lambda m_1+1}+\frac{B}{\Lambda m_2+1}+C+D=0,$$

$$\frac{A\mathrm{e}^{m_1}}{\Lambda m_1+1}+\frac{B\mathrm{e}^{m_2}}{\Lambda m_2+1}+C+D\mathrm{e}^{-1/\Lambda}=1,$$

$$b_1=\frac{2}{3Pr\gamma\Lambda},\ d=\frac{2f}{3\Lambda}.$$

$A_1$, $B_1$, $C_1$, $D_1$ satisfy the equations

$$A_1+B_1+C_1=0,\ A_1\mathrm{e}^{m_1}+B_1\mathrm{e}^{m_2}+C_1=1,$$

$$\frac{b_1A_1}{m_1+b_1}+\frac{b_1B_1}{m_2+b_1}+C_1+D_1=0,$$

$$\frac{b_1 A_1 \mathrm{e}^{m_3}}{m_1+b_1}+\frac{b_1 B_1 \mathrm{e}^{m_4}}{m_2+b_1}+C_1+D\mathrm{e}^{-b_1}=1,$$

$$c_1=A_3/A_0,\ c_2=A_4/A_0,$$

$$A_3=(r_1-r_2)\mathrm{e}^{r_1+r_2}+(r_2-\pi)\mathrm{e}^{r_1-\pi}-(r_1-\pi)\mathrm{e}^{r_2-\pi},$$

$$A_4=(r_1-r_2)\mathrm{e}^{r_1+r_2}+(r_2+\pi)\mathrm{e}^{r_1+\pi}-(r_1+\pi)\mathrm{e}^{r_2+\pi},$$

$$A_0=2(r_2-r_1)\{1+\mathrm{e}^{r_1+r_2}\}-\{(r_2-r_1)+2\pi\}\{\mathrm{e}^{r_1+\pi}+\mathrm{e}^{r_2-\pi}\}$$
$$-\{\mathrm{e}^{r_1-\pi}+\mathrm{e}^{r_2+\pi}\}\{(r_2-r_1)-2\pi\},$$

$$r_1=\frac{(1+f)\lambda+\Lambda\pi^2+\sqrt{\{(1+f)\lambda+\Lambda\pi^2\}^2+4\pi^2(1-\lambda\Lambda)}}{2(1-\lambda\Lambda)},$$

$$r_2=\frac{(1+f)\lambda+\Lambda\pi^2-\sqrt{\{(1+f)\lambda+\Lambda\pi^2\}^2+4\pi^2(1-\lambda\Lambda)}}{2(1-\lambda\Lambda)},$$

$$r_3=\lambda-\frac{1}{\Lambda}-r_2-r_3,\ a=1+\Lambda\pi,\ b=1-\Lambda\pi,$$

$$d_1=\frac{a}{a+f},\ d_2=\frac{b}{b+f}.$$

$c_3$, $c_4$ and $c_5$ satisfy the following equations:

$$(1-\mathrm{e}^{r_1})c_3+(1-\mathrm{e}^{r_2})c_4+(1-\mathrm{e}^{r_3})c_5=X_1-X_2,$$

$$r_1c_3+r_2c_4+r_3c_5=X_3,\quad r_1\mathrm{e}^{r_1}c_3+r_2\mathrm{e}^{r_2}c_4+r_3\mathrm{e}^{r_3}c_5=X_4,$$

$$X_1=1+d_1c_1+d_2c_2,\ X_2=d_1c_1\mathrm{e}^{\pi}+d_2c_2\mathrm{e}^{-\pi},$$

$$X_3=\pi(d_1c_1-d_2c_2),\ X_4=\pi(d_1c_1\mathrm{e}^{\pi}-d_2c_2\mathrm{e}^{-\pi}).$$

$c_3$, $c_4$, $c_5$ and $c_6$ satisfy the following equations:

$$c_6+\frac{c_3}{\Lambda r_1+1}+\frac{c_4}{\Lambda r_2+1}+\frac{c_5}{\Lambda r_3+1}=X_5,$$

$$c_6\mathrm{e}^{-1/\Lambda}+\frac{c_3\mathrm{e}^{r_1}}{\Lambda r_1+1}+\frac{c_4\mathrm{e}^{r_2}}{\Lambda r_2+1}+\frac{c_5\mathrm{e}^{r_3}}{\Lambda r_3+1}=X_6,$$

$$\frac{-c_6}{\Lambda}+\frac{c_3r_1}{\Lambda r_1+1}+\frac{c_4r_2}{\Lambda r_2+1}+\frac{c_5r_3}{\Lambda r_3+1}=X_7,$$

$$\frac{-c_6}{\Lambda}\mathrm{e}^{-1/\Lambda}+\frac{c_3r_1\mathrm{e}^{r_1}}{\Lambda r_1+1}+\frac{c_4r_2\mathrm{e}^{r_2}}{\Lambda r_2+1}+\frac{c_5r_3\mathrm{e}^{r_3}}{\Lambda r_3+1}=X_8,$$

$$X_5=\frac{d_1c_1}{a}+\frac{d_2c_2}{b},\ X_6=\frac{d_1c_1\mathrm{e}^{\pi}}{a}+\frac{d_2c_2\mathrm{e}^{-\pi}}{b},$$

$$X_7=\frac{d_1c_1\pi}{a}+\frac{d_2c_2\pi}{b},\ X_8=\frac{d_1c_1\pi}{a}\mathrm{e}^{\pi}-\frac{d_2c_2\pi\mathrm{e}^{-\pi}}{b}.$$

$L$, $M$, $N$, $R$ satisfy the following equations

$$L+M+N=X_9,\ L\mathrm{e}^{r_1}+M\mathrm{e}^{r_2}+N\mathrm{e}^{r_3}=X_0,$$

$$\frac{L\mathrm{e}^{r1}}{\Lambda r_1+1}+\frac{M\mathrm{e}^{r_2}}{\Lambda r_2+1}+\frac{N\mathrm{e}^{r_3}}{\Lambda r_3+1}+R\,\mathrm{e}^{-1/\Lambda}=X_{11},$$

$$\frac{L}{\Lambda r_1+1}+\frac{M}{\Lambda r_2+1}+\frac{N}{\Lambda r_3+1}+R=X_{12},$$

where

$$X_9=-(K_1+K_2+K_3+K_4+K_5+K_6+K_7+K_8+K_9+K_{10}$$
$$+K_{11}+K_{12}+K_{13}+K_{14}+K_{15}+K_{16}+K_{17}+K_{18}),$$

$$X_{10}=-[K_1\mathrm{e}^{r_1+m_1}+K_2\mathrm{e}^{r_2+m_1}+K_3\mathrm{e}^{r_3+m_1}+K_4\mathrm{e}^{\pi+m_1}$$
$$+K_5\mathrm{e}^{m_1-\pi}+K_6\mathrm{e}^{r_1+m_2}+K_7\mathrm{e}^{r_2+m_2}+K_8\mathrm{e}^{r_3+m_2}$$
$$+K_9\mathrm{e}^{m_2+\pi}+K_{10}\mathrm{e}^{m_2-\pi}+K_{11}\mathrm{e}^{m_1-1/\Lambda}+K_{12}\mathrm{e}^{m_2-1/\Lambda}$$
$$+K_{13}\mathrm{e}^{-2/\Lambda}+K_{14}\mathrm{e}^{r_1-1/\Lambda}+K_{15}\mathrm{e}^{r_2-1/\Lambda}+K_{16}\mathrm{e}^{r_3-1/\Lambda}$$
$$+K_{17}\mathrm{e}^{\pi-1/\Lambda}-K_{18}\mathrm{e}^{-(\pi+1/\Lambda)}.$$

$R_1$, $R_2$, $R_3$, $R_4$ satisfy the following equations.

$$R_1+R_2+R_3=X_{13},\ R_1\mathrm{e}^{s_1}+R_2\mathrm{e}^{s_2}+R_3\mathrm{e}^{s_3}=X_{14},$$

$$R_4+\frac{b_1R_1}{s_1+b_1}+\frac{b_1R_2}{s_2+b_1}+\frac{b_1R_3}{s_3+b_1}=X_{15},$$

$$R_4\mathrm{e}^{-b_1}+\frac{b_1R_1\mathrm{e}^{s_1}}{s_1+b_1}+\frac{b_1R_2\mathrm{e}^{s_2}}{s_2+b_1}+\frac{b_1R_3\mathrm{e}^{s_3}}{s_3+b_1}=X_{16},$$

$$s_1=\frac{(\lambda Pr-b_1)+\sqrt{(\lambda Pr-b_1)^2+4(\pi+2f\lambda/(3\Lambda)+\lambda Prb_1)}}{2},$$

$$s_2=\frac{(\lambda Pr-b_1)-\sqrt{(\lambda Pr-b_1)^2+4(\pi+2f\lambda/(3\Lambda)+\lambda Prb_1)}}{2},$$

$$s_3=(\lambda Pr-b_1)-s_1-s_2.$$

The expressions for $t_1$, $t_2$, …, $t_{18}$ are known constant which are not presented for the sake of brevity.