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Viscosity approximation methods with weakly contractive mappings for nonexpansive mappings

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Abstract: Let K be a closed convex subset of a real reflexive Banach space E, $T:K\to K$ be a nonexpansive mapping, and $f:K\to K$ be a fixed weakly contractive (may not be contractive) mapping. Then for any $t\in(0, 1)$, let $x_t\in K$ be the unique fixed point of the weak contraction $x\mapsto tf(x)+(1-t)Tx$. If T has a fixed point and E admits a weakly sequentially continuous duality mapping from E to E^*, then it is shown that $\{x_t\}$ converges to a fixed point of T as $t\to 0$. The results presented here improve and generalize the corresponding results in (Xu, 2004).

Key words: Viscosity approximation methods, Weakly contractive mapping, Fixed point, Weakly sequentially continuous duality mapping
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INTRODUCTION

Let E be a real Banach space with dual space E^*, $\langle\cdot,\cdot\rangle$ be the dual pair between E and E^*, 2^{E^*} denote the family of all the nonempty subsets of E^* and K be a nonempty closed convex subset of E. When $\{x_n\}$ is a sequence in E, then $x_n\to x$ (respectively $x_n \xrightarrow{\text{weak}} x$, $x_n \xrightarrow{\text{weak}^*} x$) will denote strong (respectively weak, weak*) convergence of the sequence $\{x_n\}$ to x. The normalized duality mapping $J: E\to 2^{E^*}$ is defined by $J(x)=\{f\in E^*, \langle x,f\rangle=\|x\|\cdot\|f\|, \|x\|=\|f\|\}$, $\forall x\in E$. In the sequence, we shall denote the single-valued duality mapping by j, and denote $F(T)=\{x\in E, Tx=x\}$.

A self-mapping $T:K\to K$ is called contractive if there exists a constant $k\in(0,1)$ such that

$$\|Tx-Ty\|\le k\|x-y\|,\quad \forall x,y\in K, \tag{1}$$

while T is called nonexpansive if Eq.(1) holds for $k=1$. The self-mapping $f:K\to K$ is called weakly contractive of the class $C_{\psi(s)}$ if there exists a continuous and non-decreasing function $\psi(s)$ defined on $\mathbb{R}^*$ such that ψ is positive on $\mathbb{R}^*\setminus\{0\}$, $\psi(0)=0$, $\lim_{s\to\infty}\psi(s)=+\infty$ and for any $x,y\in K$, $\|f(x)-f(y)\|\le\|x-y\|-\psi(\|x-y\|)$.

Remark 1 Clearly a contractive mapping with constant k must be a weakly contractive mapping, where $\psi(s)=(1-k)s$, but the converse is not true.

Example 1 (Alber and Guerre-Delabriere, 1997) The mapping $Ax=\sin x$ from [0, 1] to [0, 1] is weakly contractive and $\psi(s)=s^3/8$. But A is not a contractive mapping.

Indeed, suppose that A is a contractive mapping with constant $k\in(0, 1)$, i.e.,

$$|\sin x-\sin y|\le k|x-y|,\quad \forall x,y\in[0,1]. \tag{2}$$

Since $\lim_{x\to 0}[(\sin x)/x]=1$, taking $\varepsilon=1-k$, there exists $\delta>0$ as $0<x<\delta$, we have $|(\sin x)/x-1|<1-k$. Therefore $k<|(\sin x-\sin 0)/(x-0)|$, i.e., $k|x-0|<|\sin x-\sin 0|$, which contradicts the assumption of Eq.(2). Thus A is not contractive.

Recall that the norm of Banach space E is said to be "Gateaux differentiable", if

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \qquad (3)$$

exists for each x, y on the unit sphere $S(E)$ of E. Moreover, if for each y in $S(E)$ the limit defined by (3) is uniformly obtained for x in $S(E)$, we say that the norm of E is "uniformly Gateaux differentiable". A Banach space E is said to be strictly convex if $\|x\|=\|y\|=1$, $x\neq y$ implies $\|x+y\|<2$. A mapping $T:K\to K$ is called pseudocontractive (respectively, strongly pseudocontractive), if for any x, $y\in K$, there exists $j(x-y)\in J(x-y)$ such that $\langle Tx-Ty, j(x-y)\rangle\leq\|x-y\|^2$ (respectively, $\langle Tx-Ty, j(x-y)\rangle\leq\beta\|x-y\|^2$ for some $0<\beta<1$).

Xu (2004) studied the following viscosity iteration Eq.(4) in a uniformly smooth Banach space E for a fixed contractive mapping $f:K\to K$ and a nonexpansive mapping $T:K\to K$,

$$x_t = tf(x_t)+(1-t)Tx_t \qquad (4)$$

and proved that as t approaches 0 the sequence $\{x_t\}$ converges strongly to a fixed point of T, which is the unique solution to the following variational inequality:

$$\langle f(p)-p, j(u-p)\rangle\leq 0,\ \forall u\in F(T). \qquad (5)$$

Chen *et al.*(2006) continued this direction of research. They studied the viscosity iteration Eq.(4) in a real reflexive Banach space E for a fixed Lipschitzian strongly pseudocontractive mapping $f:K\to K$ and a continuous pseudocontractive mapping $T:K\to K$. If $F(T)\neq\varnothing$ and E admits a weakly sequentially continuous duality mapping, they proved that $\{x_t\}$ defined by Eq.(4) converges strongly to a fixed point of T, which is a unique solution to the inequality (5).

Recently, in a real reflexive and strictly convex Banach space E with a "uniformly Gateaux differentiable" norm, Song and Chen (2007) studied the viscosity iterative process Eq.(4) for continuous pseudocontractive self-mappings, and showed that the $\{x_t\}$ in Eq.(4) strongly converges $x^*\in F(T)$ as $t\to 0$ and x^* is a unique solution to the inequality (5).

In this paper, we will further study the strong convergence for the viscosity iterative sequence $\{x_t\}$ in Eq.(4). Here $f:K\to K$ is a fixed weakly contractive (may not be contractive) mapping and $T:K\to K$ is a nonexpansive mapping with $F(T)\neq\varnothing$. We prove that $\{x_t\}$ converges strongly to some $p\in F(T)$, where p is a unique solution to the variational inequality (5). Our results improve and extend the corresponding ones in (Xu, 2004).

If a Banach space E admits a sequentially continuous duality mapping J from weak topology to weak* topology, from Lemma 1 of (Gossez and Lami Dozo, 1972), it follows that the duality mapping J is single-valued. In this case, the duality mapping J is also said to be weakly sequentially continuous, i.e., for each $\{x_n\}\subset E$ with $x_n\xrightarrow{\text{weak}}x$, then $J(x_n)\xrightarrow{\text{weak}^*}J(x)$.

A Banach space E satisfies Opial condition if for each $\{x_n\}\subset E$ with $x_n\xrightarrow{\text{weak}}x$, then

$$\limsup_{n\to\infty}\|x_n-x\|<\limsup_{n\to\infty}\|x_n-y\|,\ \forall y\in E,\ y\neq x.$$

From Theorem 1 of (Gossez and Lami Dozo, 1972), if E admits a weakly sequentially continuous duality mapping, then E satisfies Opial condition.

Let C be a nonempty subset of a Banach space E, a mapping T on C is called demiclosed if for any $\{x_n\}\subset C$, as $n\to\infty$, $x_n\xrightarrow{\text{weak}}x$ and $Tx_n\to y$ imply $x\in C$ and $Tx=y$.

In what follows, we shall make use of the following lemmas:

Lemma 1 (Jung, 2005) Let C be a nonempty closed convex subset of a reflexive Banach space E satisfying Opial condition, and $T:C\to E$ be a nonexpansive mapping. Then the mapping I-T is demiclosed on C.

Lemma 2 (Rhoades, 2001) Let (X, d) be a complete metric space, $T:X\to X$ be a weakly contractive mapping. Then T has a unique fixed point p in X.

MAIN RESULTS

Theorem 1 Let K be a nonempty closed convex subset of a real Banach space E. Suppose that $T:K\to K$ is a nonexpansive mapping with $F(T)\neq\varnothing$, and $f:K\to K$ is a fixed weakly contractive mapping of the class $C_{\psi(s)}$. Then

(1) for each $t\in(0,1)$, there exists a unique point $x_t\in K$ satisfying Eq.(4);

(2) for any fixed $q\in F(T)$, $\psi(\|x_t-q\|)\|x_t-q\|\le\langle f(q)-q,j(x_t-q)\rangle$;

(3) $\{x_t\}$ is bounded;

(4) for any fixed $q\in F(T)$, $\langle x_t-f(x_t),j(x_t-q)\rangle\le 0$.

Proof For each $t\in(0, 1)$, set $F=tf+(1-t)T$ and for any $x, y\in K$, since f is weakly contractive and T is nonexpansive, we have

$$\begin{aligned}&\|F(x)-F(y)\|\\&=\|t[f(x)-f(y)]+(1-t)(Tx-Ty)\|\\&\le t\|f(x)-f(y)\|+(1-t)\|Tx-Ty\|\\&\le t\|x-y\|-t\psi(\|x-y\|)+(1-t)\|x-y\|\\&=\|x-y\|-t\psi(\|x-y\|).\end{aligned}$$

This implies that F is a weakly contractive mapping of the class $C_{t\psi(s)}$ and maps K in itself because $f(K)\subset K$, $T(K)\subset K$ and K is convex. Thus from Lemma 2, F has a unique fixed point $x_t\in K$. So Theorem 1(1) holds.

For any fixed $q\in F(T)$, we have

$$\begin{aligned}&\|x_t-q\|^2\\&=\langle t(f(x_t)-q)+(1-t)(Tx_t-Tq),j(x_t-q)\rangle\\&\le\langle t(f(x_t)-f(q))+t(f(q)-q),j(x_t-q)\rangle\\&\quad+(1-t)\|Tx_t-Tq\|\cdot\|x_t-q\|\\&\le t\|x_t-q\|^2-t\psi(\|x_t-q\|)\|x_t-q\|\\&\quad+(1-t)\|x_t-q\|^2+t\langle f(q)-q,j(x_t-q)\rangle\\&\le\|x_t-q\|^2-t\psi(\|x_t-q\|)\|x_t-q\|\\&\quad+t\langle f(q)-q,j(x_t-q)\rangle.\end{aligned}$$

Thus $\psi(\|x_t-q\|)\|x_t-q\|\le\langle f(q)-q,j(x_t-q)\rangle$. This establishes Theorem 1(2).

From Theorem 1(2), we have $\psi(\|x_t-q\|)\|x_t-q\|\le\|f(q)-q\|\cdot\|x_t-q\|$. If $\|x_t-q\|=0$, the result is clearly obtained. If $\|x_t-q\|>0$, then $\psi(\|x_t-q\|)\le\|f(q)-q\|$. This implies that $\|x_t-q\|\le\psi^{-1}(\|f(q)-q\|)$. So Theorem 1(3) is proved.

For any fixed $q\in F(T)$,

$$\begin{aligned}&\langle x_t-f(x_t),j(x_t-q)\rangle\\&=(1-t)\langle Tx_t-f(x_t),j(x_t-q)\rangle\\&\le(1-t)\langle Tx_t-Tq,j(x_t-q)\rangle+\\&\quad(1-t)\langle q-x_t+x_t-f(x_t),j(x_t-q)\rangle\\&\le(1-t)\langle x_t-f(x_t),j(x_t-q)\rangle.\end{aligned}$$

Then $\langle x_t-f(x_t),j(x_t-q)\rangle\le 0$. This completes the proof.

Theorem 2 Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from E to E^*, and K be a nonempty closed convex subset of E. Assume that $T:K\to K$ is a nonexpansive mapping with $F(T)\neq\varnothing$, and $f:K\to K$ is a fixed weakly contractive mapping of the class $C_{\psi(s)}$. Then $\{x_t\}$ defined by Eq.(4) converges strongly to a fixed point p of T, where p is the unique solution in $F(T)$ to the variational inequality (5).

Proof We first show that the uniqueness of a solution to the variational inequality (5). In fact, suppose p, $q\in F(T)$ satisfy inequality (5), i.e.,

$$\langle f(p)-p,j(q-p)\rangle\le 0,\tag{6}$$

$$\langle f(q)-q,j(p-q)\rangle\le 0.\tag{7}$$

Adding Eqs.(6) and (7) up, we obtain $\langle (I-f)(p)-(I-f)(q),j(p-q)\rangle\le 0$. From this inequality, we have

$$\begin{aligned}&\|p-q\|^2\le\|f(p)-f(q)\|\cdot\|q-p\|\\&\le\|p-q\|^2-\psi(\|p-q\|)\cdot\|p-q\|.\end{aligned}$$

This implies that $p=q$ and the uniqueness is proved.

Next we show that $\|x_t-Tx_t\|\to 0$ as $t\to 0$. Indeed, it follows from Theorem 1(3) that $\{x_t\}$, $\{f(x_t)\}$ and $\{Tx_t\}$ are bounded. As a result, $\lim_{t\to 0}\|x_t-Tx_t\|=\lim_{t\to 0}t\|f(x_t)-Tx_t\|=0$.

We claim that $\{x_t\}$ is sequentially compact.

Since E is reflexive and $\{x_t\}$ is bounded, there exists a weakly convergent subsequence $\{x_{t_n}\}$. Put $x_n:=x_{t_n}$. We suppose $x_n\xrightarrow{\text{weak}}p$ as $n\to\infty$. And from $\lim_{t\to 0}\|x_t-Tx_t\|=0$ and Lemma 1, we obtain $p=Tp$. By Theorem 1(2) and the weakly sequential continuity of J, as $n\to\infty$, we have $\psi(\|x_n-p\|)\|x_n-p\|\to 0$. This implies that $x_n\to p$ $(n\to\infty)$.

Finally we prove that p is the unique solution to the variational inequality (5). Indeed, for any $u\in F(T)$, since the set $\{x_n-u\}$ is bounded, J is weakly sequentially continuous and $x_n\to p$ $(n\to\infty)$; as $n\to\infty$, we obtain

$$\| f(x_n) - x_n - [f(p) - p] \| \to 0,$$

$$\left| \left\langle f(x_n) - x_n, j(u - x_n) \right\rangle - \left\langle f(p) - p, j(u - p) \right\rangle \right|$$
$$\le \left| \left\langle f(x_n) - x_n - [f(p) - p], j(u - x_n) \right\rangle \right| +$$
$$\left| \left\langle f(p) - p, j(u - x_n) - j(u - p) \right\rangle \right|$$
$$\le \| f(x_n) - x_n - [f(p) - p] \| \cdot \| u - x_n \| +$$
$$\left| \left\langle f(p) - p, j(u - x_n) - j(u - p) \right\rangle \right| \to 0.$$

Thus by Theorem 1(4), we get

$$\left\langle f(p) - p, j(u - p) \right\rangle = \lim_{n \to \infty} \left\langle f(x_n) - x_n, j(u - x_n) \right\rangle \le 0.$$

So p is a unique solution to the variational inequality (5). The proof is completed.

Since a contractive mapping is a weakly contractive mapping, we easily get the following result:

Corollary 1 Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from E to E^*, and K be a nonempty closed convex subset of E. Assume that T:$K \to K$ is a nonexpansive mapping with $F(T) \neq \varnothing$, and f:$K \to K$ is a fixed contractive mapping. Then $\{x_t\}$ defined by Eq.(4) converges strongly to a fixed point p of T, where p is the unique solution in $F(T)$ to the variational inequality (5).

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