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Probabilistic analysis of linear elastic cracked structures

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Abstract: This paper presents a probabilistic methodology for linear fracture mechanics analysis of cracked structures. The main focus is on probabilistic aspect related to the nature of crack in material. The methodology involves finite element analysis; statistical models for uncertainty in material properties, crack size, fracture toughness and loads; and standard reliability methods for evaluating probabilistic characteristics of linear elastic fracture parameter. The uncertainty in the crack size can have a significant effect on the probability of failure, particularly when the crack size has a large coefficient of variation. Numerical example is presented to show that probabilistic methodology based on Monte Carlo simulation provides accurate estimates of failure probability for use in linear elastic fracture mechanics.

Key words: Probabilistic fracture mechanics, Linear elastic fracture mechanics, Failure probability, First-order reliability methods

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INTRODUCTION

Probabilistic fracture mechanics is becoming increasingly popular for realistic evaluation of fracture response and reliability of cracked structures. Using probabilistic fracture mechanics, statistical uncertainties can be incorporated in engineering design and evaluation (Rahman, 2001). The theory of fracture mechanics provides a mechanistic relationship between the maximum permissible load acting on a structural component to the size and location of a crack in that component. Currently, there are many methods and applications for probabilistic fracture mechanics in oil and gas, nuclear, automotive, naval, aerospace, and other industries, nearly all of which have been developed based on linear elastic fracture mechanics models (Rahman and Kim, 2001). Probability theory determines how the uncertainties in crack size, loads, and material properties, when modelled accurately, affect the integrity of cracked structures. Probabilistic fracture mechanics provides a more rational means to describe the actual behaviour and reliability of structures than traditional deterministic methods (Provan, 1987).

A number of methods have been developed or implemented for estimating statistics of various fracture response and reliability. Most of these methods are based on linear elastic fracture mechanics and finite element method (FEM) that employs the stress intensity factor as the primary crack driving force (Besterfield *et al*., 1990). Although finite element based methods are well developed, research in probabilistic analysis has not been widespread and is only currently gaining attention. Grigoriu *et al*.(1990) applied first- and second-order reliability methods (FORM/SORM) to predict the probability of fracture initiation and a confidence interval of the direction of crack extension. The methods can account for random loads, material properties, and crack geometry. However, the randomness in crack geometry was modelled by response surface approximations of stress intensity factor as explicit functions of crack geometry. Furthermore, the usefulness of response surface based methods is limited, since they cannot be applied for general fracture mechanics analysis (Chen *et al*., 2001). Rahman and Kim (2001) developed PROFAC code based on Monte Carlo with importance sampling to calculate the probability of failure based on initiation of crack growth.

This paper presents a computational methodology for probabilistic characterization of fracture initiation in cracked structures. The methodology is based on linear FEM for deterministic stress analysis, statistical models for loads and material properties and Monte Carlo method for probabilistic analysis. Example is presented to illustrate the proposed methodology for 2D cracked structures. The results from these examples show that the methodology is capable of predicting deterministic and probabilistic characteristics for use in linear elastic fracture mechanics.

FINITE ELEMENT CALCULATION

In order to perform linear elastic analysis, the finite element analysis needs to be well developed (Tada *et al*., 2000). In this study triangular mesh generation using the advancing front method was used (Zienkiewicz and Zhu, 1987). The mesh finally optimised by smoothing and associated boundary conditions are found by interpolation from the initial geometry conditions, then finally producing the output files. The remeshing algorithms place a rosette of quarter point elements around the crack tip, and then rebuild the mesh around the crack tip. A computer code has been developed using FORTRAN programming language for finite element analysis calculation processes, which is based on displacement control for linear elastic crack propagation modelling. The stress intensity factors during crack propagation steps were calculated by using the displacement extrapolation method, which shown to be highly accurate.

The mesh refinement is guided by a characteristic size of each element, and is predicted according to a given error rate and the degree of the element interpolation function. The error estimation for the simulation is based on stress smoothing. It was a point wise error in stress indicator (ESI) to evaluate the accuracy of the finite element solution.

In general, the smaller mesh sizes in a finite element mesh give more accurate finite element approximate solution. However, reduction in the mesh size leads to greater computational effort. The error estimator used in this paper was based on stress error norm by Zienkiewicz *et al*.(2005). The adaptive remeshing technique was used when the element shape became highly distorted due to large displacement. After a few deformation increments, the whole domain was remeshed based on a stress error norm. The strategy used to refine the mesh during analysis process is adopted from (Alshoaibi *et al*., 2007).

PROBABILISTIC ANALYSIS AND RELIABILITY

Consider a linear elastic cracked structure under uncertain mechanical and geometric characteristics subject to random loads. Denote by $\boldsymbol{X}$, an N-dimensional random vector with components X_1, $X_2,\ldots,X_N$ characterizing uncertainties in the load, crack geometry and material properties. For example, if the crack size a, elastic modulus E, far field applied stress magnitude σ^{∞}, and mode I fracture toughness at crack initiation K_{Ic}, are modelled as input random variables, then $\boldsymbol{X}=\{a, E, \sigma^{\infty}, K_{\mathrm{Ic}}\}$. Let stress intensity factor K be a relevant crack driving force that can be calculated using standard finite element analysis. Suppose that the structure fails when $K>K_{\mathrm{Ic}}$ (Guinea *et al*., 2000). This requirement cannot be satisfied with certainty, since K is dependent on the input vector $\boldsymbol{X}$ which is random, and K_{Ic} itself to be a random variable.

The performance of the cracked structure with the above uncertainties consideration can be evaluated by the Monte Carlo failure analysis. In this methodology, the random variables are generated by some prescribed probability distribution functions, such as Lognormal and Gaussian. Then a statistical analysis is carried out for each of the Monte Carlo samples to obtain some parameters such as mean and coefficient of variation (COV) (Soong, 2004). The calculation of stress intensity factor is affected by the randomness of crack size, elastic modulus and far field applied stress magnitude. Then the analysis for each of Monte Carlo samples is evaluated to see if a failure has occurred. Failure occurred when K calculated from finite element analysis exceeds the value of K_{Ic}.

Consider the limit state represented by $\boldsymbol{X}=g(a, E, \sigma^{\infty}, K_{\mathrm{Ic}})$ corresponding to a failure mode for a structure. With all the random variables assumed to be statistically independent, the Monte Carlo simulation approach consists of drawing samples of the variables according to their probability distribution functions and then feeding them into the mathematical model.

The samples thus obtained gave the probabilistic characteristics of the response random variable $\boldsymbol{X}$. It is known that if the value of K is less than K_{Ic}, it indicates failure. Let N_{F} be the number of simulation cycles when K is less than K_{Ic} and let N be the total number of simulation cycles. Therefore, an estimate of the probability of failure P_{F} can be expressed as

$$P_{\mathrm{F}}=N_{\mathrm{F}}/N. \quad (1)$$

Then the probability of failure obtained from Monte Carlo failure analysis is compared with FORM methodology.

The FORM is based on linear approximation of the limit state surface $\boldsymbol{g}(\boldsymbol{x})=0$ tangent to the closest point of the surface to the origin of the space. The FORM algorithm involves several steps. First, the space $\boldsymbol{x}$ of uncertain parameters $\boldsymbol{X}$ is transformed into a new N-dimensional space $\boldsymbol{u}$, consisting of independent standard Gaussian variables $\boldsymbol{U}$. The original limit state $\boldsymbol{g}(\boldsymbol{x})=0$ is then mapped into the new limit state $\boldsymbol{g_U}(\boldsymbol{u})=0$ in the $\boldsymbol{u}$ space. Second, the point on the limit state $\boldsymbol{g_U}(\boldsymbol{u})=0$ having the shortest distance to the origin of the $\boldsymbol{u}$ space is determined. This point is referred to as the most probable point or the beta point, and has a distance $\boldsymbol{\beta_{HL}}$ (known as reliability index) to the origin of the $\boldsymbol{u}$ space. Third, the limit state $\boldsymbol{g_U}(\boldsymbol{u})=0$ is approximated by a hyperplane $\boldsymbol{g_L}(\boldsymbol{u})=0$, tangent to it at the beta point. The probability of failure P_{F} is thus approximated by $P_{\mathrm{F,1}}=Pr[\boldsymbol{g_L}(\boldsymbol{U})<0]$ in FORM and is given by (Madsen *et al.*, 1986)

$$P_{\mathrm{F,1}}=\Phi(-\boldsymbol{\beta_{HL}}), \quad (2)$$

where

$$\Phi(\boldsymbol{u})=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\boldsymbol{u}}\exp(-\xi^2/2)\mathrm{d}\xi, \quad (3)$$

which is the cumulative probability distribution function. A Hasofer and Lind algorithm (Madsen *et al.*, 1986) was used to calculate analytically the probability of failure.

RESULTS AND DISCUSSION

Consider a 2D double edged notched tension (DENT) specimen subjected to quasi-static far field tension stress σ^{∞}. The geometry of the DENT specimen, shown in Fig.1a, has width $2W$, length $2L$ and crack length a. The load, crack size and material properties were treated as statistically independent random variables. Table 1 presents the mean, COV and probability distribution for each of these parameters. The Poisson's ratio of $\upsilon=0.3$ was assumed to be deterministic. The mean of far field tensile stress was arbitrarily varied from 140 MPa until 350 MPa and the COV of normalised crack length was arbitrarily varied from 0 until 0.4. The probabilistic distribution is adopted from (Chen *et al.*, 2001).

Fig.1b depicts a finite element mesh of DENT specimen. A total of 1184 elements and 2451 nodes were used in the mesh. Both plane stress and plane strain conditions were studied. Focused elements were used in the vicinity of crack tip. Using Monte Carlo analyses, a number of probabilistic analyses were performed to calculate the probability of failure P_{F} of the DENT specimen, as a function of mean far field tensile stress, $\mu[\sigma^{\infty}]$, where $\mu[\cdot]$ is the expectation (mean) operator. Fig.2 plots the P_{F} versus $\mu[\sigma^{\infty}]$ re-

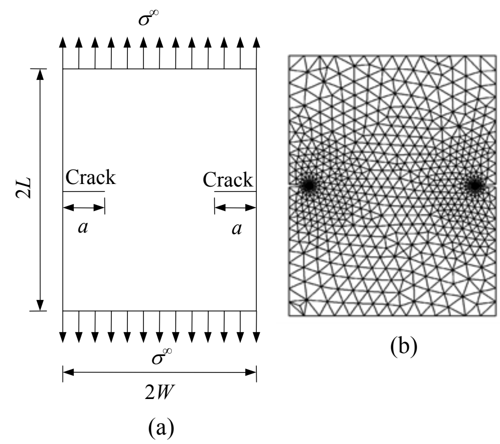

Fig.1 A DENT specimen under far-field uniform tension; (a) Geometry and loads; (b) Finite element mesh

Table 1 Statistical properties of random input for DENT specimen

Random variable	Mean	COV[a]	Probability distribution	Reference
Normalised crack length a/w	0.5	Variable[b]	Lognormal	c
Elastic modulus E (GPa)	75.2	0.06	Gaussian	c
Initiation fracture toughness K_{Ic} ($\mathrm{MPa\cdot m^{1/2}}$)	24.83	0.51	Lognormal	b
Far field tensile stress σ^{∞}	Variable[b]	0.15	Gaussian	c

[a]: COV=standard deviation/mean; [b]: arbitrarily varied; [c]: arbitrarily assumed

sults for $v_{a/W}$=0.2 and the plane stress condition, where $v_{a/W}$ is the COV of the normalised crack length a/W. As can be seen in Fig.2, the probability of failure by (Chen *et al.*, 2001) and FORM are in good agreement with the present study results.

Figs.3a and 3b indicate the plots of P_F versus $E[\sigma^{\infty}]$ using FORM and present study methodology for plane stress and plane strain conditions, for both deterministic ($v_{a/W}$=0) and random ($v_{a/W}$=0.1, 0.2, 0.4) crack sizes. The results indicate that the failure probability increases with $v_{a/W}$, and can be much larger than the probabilities calculated for a deterministic crack size, particularly when the uncertainty of a/W is large. The probability of failure in plane stress is slightly larger than that in plane strain, regardless of the load intensity, since K in plane stress is $(1-\nu^2)^{-1}$ times larger than K in plane strain. The predicted finite element results from this study matched well with the FORM results.

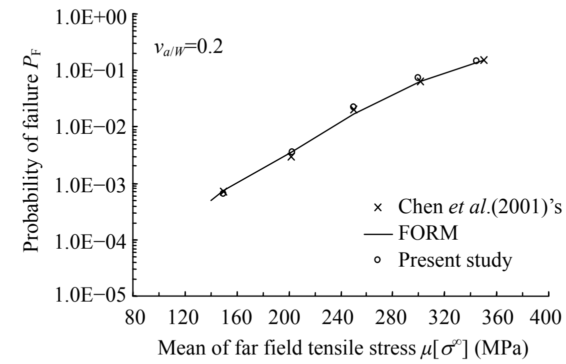

Fig.2 Failure probability of DENT specimen by Chen *et al.*(2001)'s, FORM and present study for plane stress condition

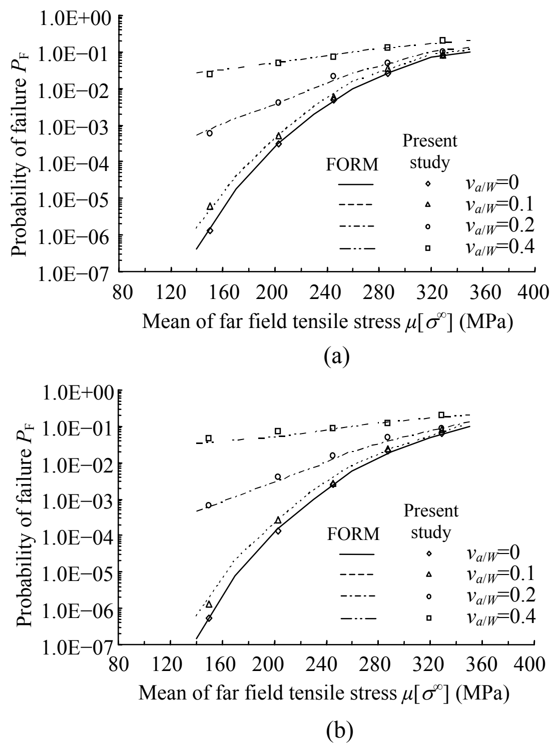

Fig.3 Failure probability of DENT specimen by FORM and present study for various uncertainties in crack size. (a) Plane stress; (b) Plain strain

CONCLUSION

The probabilistic method has been presented for fracture mechanics analysis of linear elastic cracked structures. The methodology involves development of finite element analysis codes, and statistical models for uncertainty and probabilistic analyses using Monte Carlo simulation. The numerical example has been presented to illustrate the proposed methodology for 2D cracked structures. The results from this example indicate that the methodology is capable of determining accurate probabilistic analyses in linear elastic fracture mechanics.

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