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Robust exponential stability analysis of a larger class of discrete-time recurrent neural networks*

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Abstract: The robust exponential stability of a larger class of discrete-time recurrent neural networks (RNNs) is explored in this paper. A novel neural network model, named standard neural network model (SNNM), is introduced to provide a general framework for stability analysis of RNNs. Most of the existing RNNs can be transformed into SNNMs to be analyzed in a unified way. Applying Lyapunov stability theory method and S-Procedure technique, two useful criteria of robust exponential stability for the discrete-time SNNMs are derived. The conditions presented are formulated as linear matrix inequalities (LMIs) to be easily solved using existing efficient convex optimization techniques. An example is presented to demonstrate the transformation procedure and the effectiveness of the results.

Key words: Standard neural network model (SNNM), Robust exponential stability, Recurrent neural networks (RNNs), Discrete-time, Time-delay system, Linear matrix inequality (LMI)

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INTRODUCTION

Recurrent neural networks (RNNs) have been a subject of intense research activities over the decades and have found extensive applications in pattern recognition, image processing, association, system identification and control, etc. As dynamic systems, RNNs frequently need to be analyzed for stability. In practical applications, time delays, either constant or time-varying, are often encountered in various engineering, biological and economical systems. And the existence of time delays frequently causes oscillation, divergence or instability in neural networks. In recent years, the stability of RNNs with delay has been investigated by many researchers and many results on this topic have been reported in (Liao *et al*., 2002; Cao and Wang, 2003; Zeng *et al*., 2005; Liang and Cao, 2006) and references therein. Besides time-delayed features of such neural networks, there might also be some uncertainties such as perturbations and component variations, which might lead to very complex dynamical behavior. In the design of neural networks, it is important to ensure that systems be stable in the presence of these uncertainties. The robust stability of RNNs with various structures were considered (Ji *et al*., 2004; Singh, 2006; 2007), but they did not provide any information about the transient responses and decay rates (i.e. exponential convergence rates) of the system's states. The exponential stability property is particularly important when the exponential convergence rate is used to determine the speed of neural computations. The exponential stability property guarantees that, whatever transformation occurs, the networks' ability to rapidly store the activity pattern is left invariant by self-organization. Thus, it is not only theoretically interesting but also practically important to determine

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the exponential stability for recurrent neural networks in general. The global robust exponential stability for cellular neural networks with time delay was investigated in (Gau *et al.*, 2007). Ding and Huang (2006) studied the global exponential stability of delayed BAM neural network with uncertainties. In (Ou, 2007), a class of RNNs was concerned for global robust exponential stability, which can include Hopfield type neural networks and cellular neural networks.

It is noteworthy that all of the above results obtained take the form of linear matrix inequality (LMI). Recently, LMI-based techniques have been successfully used to tackle various stability problems for neural networks. The main advantage of the LMI-based approaches is that the LMI stability conditions can be solved numerically using the effective interior-point algorithm (Boyd *et al.*, 1994; Gahinet *et al.*, 1995).

The existing results regarding robust exponential stability for delayed neural networks pertain to a special neural network model (Gau *et al.*, 2007; Ding and Huang, 2006), or a class of neural network models (Ou, 2007). Most of the special neural network models, such as recurrent multilayer perceptrons (RMLP), CGNN, BAM, etc. are not included in the model described in (Ou, 2007). Furthermore, almost all the researches on this topic just focus on the continuous-time case. In most practical applications, however, discrete iteration process rather than continuous version is used. Generally speaking, the stability analysis of continuous-time case is not necessarily applicable to the discrete-time case. Therefore, the detailed analysis for discrete-time case is also necessary and important. This paper is supposed to solve the aforementioned problems.

Standard neural network model (SNNM) is a novel recurrent neural network model (Liu, 2006a; 2007). Like the nominal model in linear robust control theory, SNNM can be applied to either non-delayed systems or delayed systems. Most existing delayed (or non-delayed) recurrent neural networks can be transformed into SNNMs to be analyzed in a unified way. So SNNM provides a general framework to facilitate the stability analysis of RNNs. SNNM has been successfully used in stability analysis and controller synthesis for RNNs (Liu and Yan, 2003; Liu, 2006a; 2006b; 2007; Yan *et al.*, 2004; Zhang and Liu, 2005). This paper is concerned with the problem of global robust exponential stability for discrete-time SNNM with and without delays. Based on the Lyapunov-Krasovskii stability theory and S-Procedure, two sufficient conditions of global robust exponential stability for discrete-time SNNM are derived. The stability criteria are characterized in the form of a set of LMIs which allow for the application of convex optimization algorithms to be possible.

Notation: Throughout this paper, $\Re^n$ denotes the n dimensional Euclidean space, and $\Re^{n\times m}$ is the set of all $n\times m$ real matrices, $\boldsymbol{I}$ denotes identity matrix of appropriate order, $\lambda_{\rm M}(\boldsymbol{A})$ and $\lambda_{\rm m}(\boldsymbol{A})$ denote the maximal and minimal eigenvalue of a square matrix $\boldsymbol{A}$, respectively. $\|\boldsymbol{x}\|$ denotes the Euclidean norm of the vector $\boldsymbol{x}$, and $\|\boldsymbol{A}\|$ denotes the induced norm of the matrix $\boldsymbol{A}$, that is $\|\boldsymbol{A}\|=\sqrt{\lambda_{\rm M}(\boldsymbol{A}^{\rm T}\boldsymbol{A})}$. The notations $\boldsymbol{X}>\boldsymbol{Y}$ and $\boldsymbol{X}\geq\boldsymbol{Y}$, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are matrices of the same dimensions, mean that the matrix $\boldsymbol{X}-\boldsymbol{Y}$ is positive definite and positive semi-definite, respectively. If $\boldsymbol{X}\in\Re^p$ and $\boldsymbol{Y}\in\Re^q$, $\boldsymbol{C}(\boldsymbol{X};\boldsymbol{Y})$ denotes the space of all continuous functions mapping $\Re^p\rightarrow\Re^q$.

STANDARD NEURAL NETWORK MODEL

In linear robust control theory, systems with uncertainty can be transformed into a standard form, known as linear fractional transformation (LFT) (Chandrasekharan, 1996). Similar to the LFT, and referring to (Rios-Patron, 2000), SNNM can be used to describe a class of intelligent systems. The SNNM represents a neural network model as the interconnection of a linear dynamic system and static nonlinear operators consisting of bounded activation functions. In this paper, only discrete-time SNNM is concerned, similar architecture and results for continuous-time case can also be achieved. A discrete-time SNNM is shown in Fig.1. The block $\boldsymbol{\Phi}$ is a block diagonal operator composed of nonlinear activation functions $\phi_i(\xi_i(\cdot))$, which are typically continuous, differentiable, monotonically increasing, slope-restricted, and bounded. The matrix $\boldsymbol{N}$ represents a linear mapping between the inputs and outputs of the time delay $z^{-1}\boldsymbol{I}$ and the operator $\boldsymbol{\Phi}$. The vectors $\boldsymbol{\xi}(\cdot)$ and $\boldsymbol{\phi}(\boldsymbol{\xi}(\cdot))$ are the input and output of the

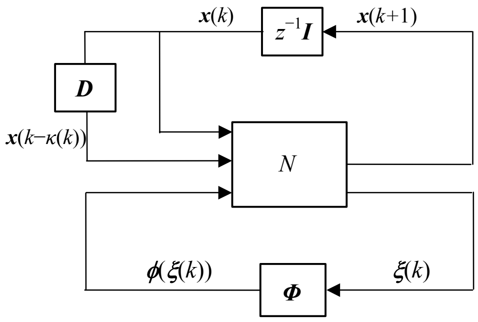

Fig.1 Discrete-time standard neural network model (SNNM) with time delay

nonlinear operator $\boldsymbol{\Phi}$, respectively. The block $\boldsymbol{D}$ represents the delayed element. $\kappa(\cdot)$ is the time-varying delay satisfying $0<\kappa(\cdot)\leq h$, where h is an integer representing the maximal delay.

If $\boldsymbol{N}$ in Fig.1 is partitioned as

$$\boldsymbol{N}=\begin{bmatrix}\boldsymbol{A} & \boldsymbol{A}_d & \boldsymbol{B}\\ \boldsymbol{C} & \boldsymbol{C}_d & \boldsymbol{D}\end{bmatrix}, \tag{1}$$

the discrete-time SNNM can be depicted as a linear difference inclusion (LDI):

$$\begin{cases}\boldsymbol{x}(k+1)=\boldsymbol{A}\boldsymbol{x}(k)+\boldsymbol{A}_d\boldsymbol{x}(k-\kappa(k))+\boldsymbol{B}\boldsymbol{\phi}(\xi(k)),\\ \xi(k)=\boldsymbol{C}\boldsymbol{x}(k)+\boldsymbol{C}_d\boldsymbol{x}(k-\kappa(k))+\boldsymbol{D}\boldsymbol{\phi}(\xi(k)),\end{cases} \tag{2}$$

with the initial condition function

$$\boldsymbol{x}(k)=\boldsymbol{\varpi}(k),\ \forall k\in[-h, 0], \tag{3}$$

where $\boldsymbol{x}\in\Re^n$ is the state vector, $\boldsymbol{A}$, $\boldsymbol{A}_d\in\Re^{n\times n}$, $\boldsymbol{B}\in\Re^{n\times L}$, $\boldsymbol{C}$, $\boldsymbol{C}_d\in\Re^{L\times n}$ and $\boldsymbol{D}\in\Re^{L\times L}$ are the corresponding state-space matrices, $\xi\in\Re^L$ is the input of nonlinear operator $\boldsymbol{\Phi}$, $\boldsymbol{\phi}\in\boldsymbol{C}(\Re^L;\Re^L)$ is the output of nonlinear operator $\boldsymbol{\Phi}$ satisfying $\boldsymbol{\phi}(0)=0$, $L\in\Re$ is the number of nonlinear activation functions (that is, the total number of neurons in the hidden layers and output layer of the neural network).

In this paper, we assume that the activation functions in the SNNM satisfy the sector conditions $\phi_i(\xi_i(k))/\xi_i(k)\in[q_i, u_i]$, i.e., $[\phi_i(\xi_i(k))-q_i\xi_i(k)]\cdot[\phi_i(\xi_i(k))-u_i\xi_i(k)]\leq 0$. $u_i>q_i\geq 0$, $i=1, \ldots, L$, and the delays in SNNM are constant, i.e., $\kappa(\cdot)=h>0$. Since $\boldsymbol{x}=\boldsymbol{0}$, $\xi=\boldsymbol{0}$ is a solution of Eq.(2), there exists at least one equilibrium point located at the origin, i.e., $\boldsymbol{x}_{\rm eq}=\boldsymbol{0}$, $\xi_{\rm eq}=\boldsymbol{0}$.

MAIN RESULTS

In this section, the discrete-time SNNM with structured uncertainties and constant time delay in state is concerned:

$$\begin{cases}\boldsymbol{x}(k+1)=(\boldsymbol{A}+\Delta\boldsymbol{A})\boldsymbol{x}(k)+(\boldsymbol{A}_d+\Delta\boldsymbol{A}_d)\boldsymbol{x}(k-h)\\ \qquad\qquad +(\boldsymbol{B}+\Delta\boldsymbol{B})\boldsymbol{\phi}(\xi(k)),\\ \xi(k)=(\boldsymbol{C}+\Delta\boldsymbol{C})\boldsymbol{x}(k)+(\boldsymbol{C}_d+\Delta\boldsymbol{C}_d)\boldsymbol{x}(k-h)\\ \qquad\qquad +(\boldsymbol{D}+\Delta\boldsymbol{D})\boldsymbol{\phi}(\xi(k)),\end{cases} \tag{4}$$

where $\Delta\boldsymbol{A}$, $\Delta\boldsymbol{B}$, $\Delta\boldsymbol{A}_d$, $\Delta\boldsymbol{C}$, $\Delta\boldsymbol{D}$, $\Delta\boldsymbol{C}_d$ denote the parametric uncertainties in $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{A}_d$, $\boldsymbol{C}$, $\boldsymbol{D}$ and $\boldsymbol{C}_d$, respectively, and are assumed to satisfy

$$\begin{cases}[\Delta\boldsymbol{A}\ \ \Delta\boldsymbol{B}\ \ \Delta\boldsymbol{A}_d]=\boldsymbol{H}_x\boldsymbol{F}(k)[\boldsymbol{E}_{x1}\ \ \boldsymbol{E}_{x2}\ \ \boldsymbol{E}_{x3}],\\ [\Delta\boldsymbol{C}\ \ \Delta\boldsymbol{D}\ \ \Delta\boldsymbol{C}_d]=\boldsymbol{H}_q\boldsymbol{F}(k)[\boldsymbol{E}_{q1}\ \ \boldsymbol{E}_{q2}\ \ \boldsymbol{E}_{q3}],\end{cases} \tag{5}$$

where $\boldsymbol{H}_x$, $\boldsymbol{H}_q$, $\boldsymbol{E}_{xi}$ and $\boldsymbol{E}_{qi}$ (i=1,2,3) are some known constant matrices of appropriate dimensions that represent the structure of uncertainties, and $\boldsymbol{F}(k)$, representing the parameter uncertainty, is an unknown real valued time-varying matrix with appropriate dimension satisfying

$$\boldsymbol{F}^{\rm T}(k)\boldsymbol{F}(k)\leq\boldsymbol{I}. \tag{6}$$

Remark 1 The uncertainty structure satisfying both Eqs.(5) and (6) has been widely adopted in robust control and filtering for uncertain systems and part of the reason can be found in (Khargonekar *et al.*, 1990).

Definition 1 Uncertainties, which satisfy Eqs.(5) and (6), are defined as admissible uncertainties.

Definition 2 If there exist $\gamma>0$ and $f(\gamma)>0$ such that

$$\|\boldsymbol{x}(k)\|\leq f(\gamma){\rm e}^{-\gamma k}, \tag{7}$$

then SNNM Eq.(4) is said to be exponentially stable at the equilibrium point, where γ is called the exponential convergence rate.

Definition 3 The SNNM Eq.(4) is said to be globally robustly exponentially stable if it is globally exponentially stable with respect to all admissible uncertainties.

Before stating the main results, the following lemmas are needed:

Lemma 1 (Khargonekar *et al*., 1990) For any given matrices $\boldsymbol{H}$ and $\boldsymbol{E}$, $\varepsilon>0$, and a time-varying matrix $\boldsymbol{F}(k)$ satisfying $\boldsymbol{F}^{\mathrm{T}}(k)\boldsymbol{F}(k)\leq\boldsymbol{I}$, we have

$$\boldsymbol{H}^{\mathrm{T}}\boldsymbol{F}(k)\boldsymbol{E}+\boldsymbol{E}^{\mathrm{T}}\boldsymbol{F}(k)\boldsymbol{H}\leq\varepsilon^{-1}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}. \quad (8)$$

Lemma 2 (Schur complement) (Boyd *et al*., 1994) Given constant symmetric matrices $\boldsymbol{S}=\begin{bmatrix}\boldsymbol{S}_1 & \boldsymbol{S}_2\\ \boldsymbol{S}_2^{\mathrm{T}} & \boldsymbol{S}_3\end{bmatrix}$, where $\boldsymbol{S}_1=\boldsymbol{S}_1^{\mathrm{T}}$, $\boldsymbol{S}_3=\boldsymbol{S}_3^{\mathrm{T}}$. Then the following inequalities are equivalent:

(1) $\boldsymbol{S}<\boldsymbol{0}$;

(2) $\boldsymbol{S}_1<\boldsymbol{0}$, $\boldsymbol{S}_3-\boldsymbol{S}_2^{\mathrm{T}}\boldsymbol{S}_1^{-1}\boldsymbol{S}_2<\boldsymbol{0}$;

(3) $\mathrm{S}_3<\boldsymbol{0}$, $\boldsymbol{S}_1-\boldsymbol{S}_2\boldsymbol{S}_3^{-1}\boldsymbol{S}_2^{\mathrm{T}}<\boldsymbol{0}$.

Lemma 3 (Xie and de Souza, 1992) Given any matrices $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ with appropriate dimensions and $\boldsymbol{Y}>\boldsymbol{0}$. Then, we have

$$\boldsymbol{X}^{\mathrm{T}}\boldsymbol{Z}+\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{X}\leq\boldsymbol{X}^{\mathrm{T}}\boldsymbol{Y}\boldsymbol{X}^{\mathrm{T}}+\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{Y}^{-1}\boldsymbol{Z}. \quad (9)$$

Lemma 4 (S-Procedure) (Boyd *et al*., 1994) Let $\boldsymbol{T}_0$, $\boldsymbol{T}_1$, …, $\boldsymbol{T}_p$ be symmetric matrices. If there exist $\tau_i\geq0$ (i=1, 2, …, p) such that

$$\boldsymbol{T}_0-\sum_{i=1}^{p}\tau_i\boldsymbol{T}_i<\boldsymbol{0}, \quad (10)$$

then $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{T}_0\boldsymbol{x}<\boldsymbol{0}$ $\forall\boldsymbol{x}\neq\boldsymbol{0}$ such that $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{T}_i\boldsymbol{x}\leq\boldsymbol{0}$ (i=1, 2, …, p).

Theorem 1 The SNNM described by Eqs.(4)~(6) is globally robustly exponentially stable if there exist symmetric positive definite matrices $\boldsymbol{R}$ and $\boldsymbol{\Gamma}$, diagonal positive semi-definite matrices $\boldsymbol{\Lambda}$ and $\boldsymbol{T}$, and positive scalars ε_1, ε_2, such that the following LMI holds:

$$\boldsymbol{M}=\begin{bmatrix}\boldsymbol{M}_{11} & \boldsymbol{M}_{12}\\ * & \boldsymbol{M}_{13}\end{bmatrix}<\boldsymbol{0}. \quad (11)$$

The sub-matrices of $\boldsymbol{M}$ are

$$\boldsymbol{M}_{11}=\begin{bmatrix}-\boldsymbol{R} & \boldsymbol{RA} & \boldsymbol{RB} & \boldsymbol{RA}_d\\ * & (\boldsymbol{M}_{11})_{22} & (\boldsymbol{M}_{11})_{23} & (\boldsymbol{M}_{11})_{24}\\ * & * & (\boldsymbol{M}_{11})_{33} & (\boldsymbol{M}_{11})_{34}\\ * & * & * & (\boldsymbol{M}_{11})_{44}\end{bmatrix},$$

$$\boldsymbol{M}_{12}=\begin{bmatrix}\boldsymbol{RH}_x & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{\Sigma H}_q\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix},\quad \boldsymbol{M}_{22}=\mathrm{diag}(-\varepsilon_1\boldsymbol{I},\ -\varepsilon_2\boldsymbol{I}),$$

where

$$(\boldsymbol{M}_{11})_{22}=-\mathrm{e}^{-2\gamma}\boldsymbol{R}+\boldsymbol{\Gamma}+\varepsilon_1\boldsymbol{E}_{x1}^{\mathrm{T}}\boldsymbol{E}_{x1}+\varepsilon_2\boldsymbol{E}_{q1}^{\mathrm{T}}\boldsymbol{E}_{q1},$$
$$(\boldsymbol{M}_{11})_{23}=\boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Sigma}+\varepsilon_1\boldsymbol{E}_{x1}^{\mathrm{T}}\boldsymbol{E}_{x2}+\varepsilon_2\boldsymbol{E}_{q1}^{\mathrm{T}}\boldsymbol{E}_{q2},$$
$$(\boldsymbol{M}_{11})_{24}=\varepsilon_1\boldsymbol{E}_{x1}^{\mathrm{T}}\boldsymbol{E}_{x3}+\varepsilon_2\boldsymbol{E}_{q1}^{\mathrm{T}}\boldsymbol{E}_{q3},$$
$$(\boldsymbol{M}_{11})_{33}=\boldsymbol{\Sigma D}+\boldsymbol{D}^{\mathrm{T}}\boldsymbol{\Sigma}-2\boldsymbol{T}+\varepsilon_1\boldsymbol{E}_{x2}^{\mathrm{T}}\boldsymbol{E}_{x2}+\varepsilon_2\boldsymbol{E}_{q2}^{\mathrm{T}}\boldsymbol{E}_{q2},$$
$$(\boldsymbol{M}_{11})_{34}=\boldsymbol{\Sigma C}_d+\varepsilon_1\boldsymbol{E}_{x2}^{\mathrm{T}}\boldsymbol{E}_{x3}+\varepsilon_2\boldsymbol{E}_{q2}^{\mathrm{T}}\boldsymbol{E}_{q3},$$
$$(\boldsymbol{M}_{11})_{44}=-\mathrm{e}^{-2\gamma h}\boldsymbol{\Gamma}+\varepsilon_1\boldsymbol{E}_{x3}^{\mathrm{T}}\boldsymbol{E}_{x3}+\varepsilon_2\boldsymbol{E}_{q3}^{\mathrm{T}}\boldsymbol{E}_{q3},$$
$$\boldsymbol{\Sigma}=\boldsymbol{\Lambda}+\boldsymbol{T}(\boldsymbol{Q}+\boldsymbol{U}),\ \boldsymbol{Q}=\mathrm{diag}(q_1,\ q_2,\ ...,\ q_L),$$
$$\boldsymbol{U}=\mathrm{diag}(u_1,\ u_2,\ ...,\ u_L).$$

Moreover,

$$\boldsymbol{x}(k)\leq\sqrt{\frac{\lambda_{\mathrm{M}}(\boldsymbol{P})\left\|\boldsymbol{x}(0)\right\|^2+\lambda_{\mathrm{M}}(\boldsymbol{\Gamma})\left\|\boldsymbol{\Omega}\right\|^2\dfrac{\mathrm{e}^{-2\gamma}-\mathrm{e}^{-2\gamma(h+1)}}{1-\mathrm{e}^{-2\gamma}}+\Pi}{\lambda_{\mathrm{M}}(\boldsymbol{P})}}\mathrm{e}^{-\gamma k} \quad (12)$$

where $\Pi=2\lambda_{\mathrm{M}}(\boldsymbol{U\Lambda})[\mathrm{e}^{-2\gamma}\left\|\boldsymbol{\xi}(-1)\right\|^2+\left\|\boldsymbol{\xi}(0)\right\|^2]$.

Proof For simplicity, denote $\boldsymbol{x}(k)$ as $\boldsymbol{x}_k$, $\boldsymbol{x}(k-h)$ as $\boldsymbol{x}_{k,h}$, $\boldsymbol{\zeta}(k)$ as $\boldsymbol{\zeta}_k$, $\zeta_i(k)$ as $\zeta_{k,i}$, $\boldsymbol{\phi}(\boldsymbol{\xi}(k))$ as $\boldsymbol{\phi}_k$, $\phi_i(\xi_i(k))$ as $\phi_{k,i}$, and $\tilde{\boldsymbol{A}}=\boldsymbol{A}+\Delta\boldsymbol{A}$, $\tilde{\boldsymbol{A}}_d=\boldsymbol{A}_d+\Delta\boldsymbol{A}_d$, $\tilde{\boldsymbol{B}}=\boldsymbol{B}+\Delta\boldsymbol{B}$, $\tilde{\boldsymbol{C}}=\boldsymbol{C}+\Delta\boldsymbol{C}$, $\tilde{\boldsymbol{C}}_d=\boldsymbol{C}_d+\Delta\boldsymbol{C}_d$, $\tilde{\boldsymbol{D}}=\boldsymbol{D}+\Delta\boldsymbol{D}$. Then SNNM Eq.(4) can be rewritten as:

$$\begin{cases}\boldsymbol{x}_{k+1}=\tilde{\boldsymbol{A}}\boldsymbol{x}_k+\tilde{\boldsymbol{A}}_d\boldsymbol{x}_{k,h}+\tilde{\boldsymbol{B}}\boldsymbol{\phi}_k,\\ \boldsymbol{\xi}_k=\tilde{\boldsymbol{C}}\boldsymbol{x}_k+\tilde{\boldsymbol{C}}_d\boldsymbol{x}_{k,h}+\tilde{\boldsymbol{D}}\boldsymbol{\phi}_k.\end{cases} \quad (13)$$

Choose the following positive definite Lyapunov functional:

$$V(\boldsymbol{x}_k,\boldsymbol{\xi}_k)=\mathrm{e}^{2\gamma k}\boldsymbol{x}_k^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}_k+\sum_{i=k-h}^{k-1}\mathrm{e}^{2\gamma i}\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\Gamma}\boldsymbol{x}_i+2\sum_{i=1}^{L}\lambda_i\sum_{j=0}^{k-1}\mathrm{e}^{2\gamma j}\phi_i(\xi_i(j))\xi_i(j), \quad (14)$$

where $\boldsymbol{P}=\boldsymbol{P}^{\mathrm{T}}>\boldsymbol{0}$, $\boldsymbol{\Gamma}\geq\boldsymbol{0}$, and $\lambda_i>0$, i=1, 2, …, L. The time derivative of $V(\boldsymbol{x}_k,\boldsymbol{\xi}_k)$ along the trajectories of Eq.(13) takes the form:

$$\begin{aligned}
\Delta V(\boldsymbol{x}_k,\xi_k) &= \mathrm{e}^{2\gamma(k+1)}\boldsymbol{x}_{k+1}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}_{k+1} - \mathrm{e}^{2\gamma k}\boldsymbol{x}_k^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}_k \\
&+2\mathrm{e}^{2\gamma k}\sum_{i=1}^{L}\lambda_i\phi_{k,i}\xi_{k,i} + \mathrm{e}^{2\gamma k}\boldsymbol{x}_k^{\mathrm{T}}\boldsymbol{\Gamma}\boldsymbol{x}_k - \mathrm{e}^{2\gamma(k-h)}\boldsymbol{x}_{k,h}^{\mathrm{T}}\boldsymbol{\Gamma}\boldsymbol{x}_{k,h} \\
&= \mathrm{e}^{2\gamma(k+1)}(\tilde{\boldsymbol{A}}\boldsymbol{x}_k + \tilde{\boldsymbol{A}}_d\boldsymbol{x}_{k,h} + \tilde{\boldsymbol{B}}\boldsymbol{\phi}_k)^{\mathrm{T}}\boldsymbol{P}(\tilde{\boldsymbol{A}}\boldsymbol{x}_k + \tilde{\boldsymbol{A}}_d\boldsymbol{x}_{k,h} + \tilde{\boldsymbol{B}}\boldsymbol{\phi}_k) \\
&-\mathrm{e}^{2\gamma k}\boldsymbol{x}_k^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}_k + 2\mathrm{e}^{2\gamma k}\sum_{i=1}^{L}\lambda_i\phi_{k,i}(\tilde{\boldsymbol{C}}_i\boldsymbol{x}_k + \tilde{\boldsymbol{C}}_{d,i}\boldsymbol{x}_{k,h} + \tilde{\boldsymbol{D}}_i\boldsymbol{\phi}_k) \\
&+\mathrm{e}^{2\gamma k}\boldsymbol{x}_k^{\mathrm{T}}\boldsymbol{\Gamma}\boldsymbol{x}_k - \mathrm{e}^{2\gamma(k-h)}\boldsymbol{x}_{k,h}^{\mathrm{T}}\boldsymbol{\Gamma}\boldsymbol{x}_{k,h} \\
&= \mathrm{e}^{2\gamma k}[\boldsymbol{x}_k^{\mathrm{T}}(\tilde{\boldsymbol{A}}^{\mathrm{T}}\mathrm{e}^{2\gamma}\boldsymbol{P}\tilde{\boldsymbol{A}} - \boldsymbol{P} + \boldsymbol{\Gamma})\boldsymbol{x}_k + \boldsymbol{x}_k^{\mathrm{T}}(\tilde{\boldsymbol{A}}^{\mathrm{T}}\mathrm{e}^{2\gamma}\boldsymbol{P}\tilde{\boldsymbol{B}} + \tilde{\boldsymbol{C}}^{\mathrm{T}}\boldsymbol{\Lambda})\boldsymbol{\phi}_k \\
&+\boldsymbol{x}_k^{\mathrm{T}}\tilde{\boldsymbol{A}}^{\mathrm{T}}\mathrm{e}^{2\gamma}\boldsymbol{P}\tilde{\boldsymbol{A}}_d\boldsymbol{x}_{k,h} + \boldsymbol{\phi}_k^{\mathrm{T}}(\tilde{\boldsymbol{B}}^{\mathrm{T}}\mathrm{e}^{2\gamma}\boldsymbol{P}\tilde{\boldsymbol{A}} + \boldsymbol{\Lambda}\tilde{\boldsymbol{C}})\boldsymbol{x}_k + \boldsymbol{\phi}_k^{\mathrm{T}}(\tilde{\boldsymbol{B}}^{\mathrm{T}}\mathrm{e}^{2\gamma}\boldsymbol{P}\tilde{\boldsymbol{B}} \\
&+\boldsymbol{\Lambda}\tilde{\boldsymbol{D}} + \tilde{\boldsymbol{D}}^{\mathrm{T}}\boldsymbol{\Lambda})\boldsymbol{\phi}_k + \boldsymbol{\phi}_k^{\mathrm{T}}(\tilde{\boldsymbol{B}}^{\mathrm{T}}\mathrm{e}^{2\gamma}\boldsymbol{P}\tilde{\boldsymbol{A}}_d + \boldsymbol{\Lambda}\tilde{\boldsymbol{C}}_d)\boldsymbol{x}_{k,h} + \boldsymbol{x}_{k,h}^{\mathrm{T}}\tilde{\boldsymbol{A}}_d^{\mathrm{T}}\mathrm{e}^{2\gamma}\boldsymbol{P}\tilde{\boldsymbol{A}}\boldsymbol{x}_k \\
&+\boldsymbol{x}_{k,h}^{\mathrm{T}}(\tilde{\boldsymbol{A}}_d^{\mathrm{T}}\mathrm{e}^{2\gamma}\boldsymbol{P}\tilde{\boldsymbol{B}} + \tilde{\boldsymbol{C}}_d^{\mathrm{T}}\boldsymbol{\Lambda})\boldsymbol{\phi}_k + \boldsymbol{x}_{k,h}^{\mathrm{T}}(\tilde{\boldsymbol{A}}_d^{\mathrm{T}}\mathrm{e}^{2\gamma}\boldsymbol{P}\tilde{\boldsymbol{A}}_d - \mathrm{e}^{-2\gamma h}\boldsymbol{\Gamma})\boldsymbol{x}_{k,h}] \\
&= \mathrm{e}^{2\gamma k}\begin{bmatrix}\boldsymbol{x}_k\\ \boldsymbol{\phi}_k\\ \boldsymbol{x}_{k,h}\end{bmatrix}^{\mathrm{T}}\boldsymbol{M}\begin{bmatrix}\boldsymbol{x}_k\\ \boldsymbol{\phi}_k\\ \boldsymbol{x}_{k,h}\end{bmatrix},
\end{aligned}$$

where

$$\boldsymbol{M} = \underbrace{\begin{bmatrix}\tilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}} - \mathrm{e}^{-2\gamma}\boldsymbol{R} + \boldsymbol{\Gamma} & \tilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{B}} + \tilde{\boldsymbol{C}}^{\mathrm{T}}\boldsymbol{\Lambda} & \tilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}}_d \\ \tilde{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}} + \boldsymbol{\Lambda}\tilde{\boldsymbol{C}} & \tilde{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{B}} + \boldsymbol{\Lambda}\tilde{\boldsymbol{D}} + \tilde{\boldsymbol{D}}^{\mathrm{T}}\boldsymbol{\Lambda} & \tilde{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}}_d + \boldsymbol{\Lambda}\tilde{\boldsymbol{C}}_d \\ \tilde{\boldsymbol{A}}_d^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}} & \tilde{\boldsymbol{A}}_d^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{B}} + \tilde{\boldsymbol{C}}_d^{\mathrm{T}}\boldsymbol{\Lambda} & \tilde{\boldsymbol{A}}_d^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}}_d - \mathrm{e}^{2\gamma h}\boldsymbol{\Gamma}\end{bmatrix}}_{T_0}$$

$\tilde{\boldsymbol{C}}_i$ and $\tilde{\boldsymbol{D}}_i$ are the ith row of the matrices $\tilde{\boldsymbol{C}}$ and $\tilde{\boldsymbol{D}}$, respectively. $\tilde{\boldsymbol{C}}_{d,i}$ is the ith row of the matrix $\tilde{\boldsymbol{C}}_d$, $\boldsymbol{R}=\mathrm{e}^{2\gamma}\boldsymbol{P}$, $\boldsymbol{\Lambda}=\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_L)$, and $\boldsymbol{\Lambda}\geq\boldsymbol{0}$.

The sector conditions, $(\phi_{k,i} - q_i\xi_{k,i})(\phi_{k,i} - u_i\xi_{k,i}) \leq 0$, can be rewritten as follows:

$$\begin{aligned}&(\phi_{k,i} - q_i\tilde{\boldsymbol{C}}_i\boldsymbol{x}_k - q_i\tilde{\boldsymbol{C}}_{d,i}\boldsymbol{x}_{k,h} - q_i\tilde{\boldsymbol{D}}_i\boldsymbol{\phi}_k) \\ &\quad\cdot(\phi_{k,i} - u_i\tilde{\boldsymbol{C}}_i\boldsymbol{x}_k - u_i\tilde{\boldsymbol{C}}_{d,i}\boldsymbol{x}_{k,h} - u_i\tilde{\boldsymbol{D}}_i\boldsymbol{\phi}_k) \leq \boldsymbol{0},\end{aligned} \tag{15}$$

which is equivalent to

$$\begin{aligned}&2\phi_{k,i}^2 - 2\phi_{k,i}(q_i+u_i)\tilde{\boldsymbol{C}}_i\boldsymbol{x}_k - 2\phi_{k,i}(q_i+u_i)\tilde{\boldsymbol{D}}_i\boldsymbol{\phi}_k - \\ &2\phi_{k,i}(q_i+u_i)\tilde{\boldsymbol{C}}_{d,i}\boldsymbol{x}_{k,h} + 2\boldsymbol{x}_k^{\mathrm{T}}\tilde{\boldsymbol{C}}_i^{\mathrm{T}}q_iu_i\tilde{\boldsymbol{C}}_i\boldsymbol{x}_k + \\ &2\boldsymbol{\phi}_k^{\mathrm{T}}\tilde{\boldsymbol{D}}_i^{\mathrm{T}}q_iu_i\tilde{\boldsymbol{D}}_i\boldsymbol{\phi}_k + 2\boldsymbol{x}_{k,h}^{\mathrm{T}}\tilde{\boldsymbol{C}}_{d,i}^{\mathrm{T}}q_iu_i\tilde{\boldsymbol{C}}_{d,i}\boldsymbol{x}_{k,h} + \\ &2\boldsymbol{\phi}_k^{\mathrm{T}}\tilde{\boldsymbol{D}}_i^{\mathrm{T}}q_iu_i\tilde{\boldsymbol{C}}_{d,i}\boldsymbol{x}_{k,h} + 2\boldsymbol{x}_{k,h}^{\mathrm{T}}\tilde{\boldsymbol{C}}_{d,i}^{\mathrm{T}}q_iu_i\tilde{\boldsymbol{D}}_i\boldsymbol{\phi}_k + \\ &2\boldsymbol{x}_k^{\mathrm{T}}\tilde{\boldsymbol{C}}_i^{\mathrm{T}}q_iu_i\tilde{\boldsymbol{D}}_i\boldsymbol{\phi}_k + 2\boldsymbol{x}_k^{\mathrm{T}}\tilde{\boldsymbol{C}}_i^{\mathrm{T}}q_iu_i\tilde{\boldsymbol{C}}_{d,i}\boldsymbol{x}_{k,h} + \\ &2\boldsymbol{\phi}_k^{\mathrm{T}}\tilde{\boldsymbol{D}}_i^{\mathrm{T}}q_iu_i\tilde{\boldsymbol{C}}_i\boldsymbol{x}_k + 2\boldsymbol{x}_{k,h}^{\mathrm{T}}\tilde{\boldsymbol{C}}_{d,i}^{\mathrm{T}}q_iu_i\tilde{\boldsymbol{C}}_i\boldsymbol{x}_k \leq \boldsymbol{0}.\end{aligned} \tag{16}$$

Rewrite Eq.(16) in the matrix form:

$$\begin{bmatrix}\boldsymbol{x}_k\\ \phi_{k,1}\\ \vdots\\ \phi_{k,i}\\ \vdots\\ \phi_{k,L}\\ \boldsymbol{x}_{k,h}\end{bmatrix}^{\mathrm{T}}\boldsymbol{T}_i^1\begin{bmatrix}\boldsymbol{x}_k\\ \phi_{k,1}\\ \vdots\\ \phi_{k,i}\\ \vdots\\ \phi_{k,L}\\ \boldsymbol{x}_{k,h}\end{bmatrix} + \begin{bmatrix}\boldsymbol{x}_k\\ \boldsymbol{\phi}_k\\ \boldsymbol{x}_{k,h}\end{bmatrix}^{\mathrm{T}}\boldsymbol{T}_i^2\begin{bmatrix}\boldsymbol{x}_k\\ \boldsymbol{\phi}_k\\ \boldsymbol{x}_{k,h}\end{bmatrix} \leq \boldsymbol{0}, \tag{17}$$

where

$\boldsymbol{T}_i^1$ is as shown at the bottom of this page,

$$\boldsymbol{T}_i^2 = \begin{bmatrix}2\tilde{\boldsymbol{C}}_i^{\mathrm{T}}q_iu_i\tilde{\boldsymbol{C}}_i & 2\tilde{\boldsymbol{C}}_i^{\mathrm{T}}q_iu_i\tilde{\boldsymbol{D}}_i & 2\tilde{\boldsymbol{C}}_i^{\mathrm{T}}q_iu_i\tilde{\boldsymbol{C}}_{d,i} \\ 2\tilde{\boldsymbol{D}}_i^{\mathrm{T}}q_iu_i\tilde{\boldsymbol{C}}_i & 2\tilde{\boldsymbol{D}}_i^{\mathrm{T}}q_iu_i\tilde{\boldsymbol{D}}_i & 2\tilde{\boldsymbol{D}}_i^{\mathrm{T}}q_iu_i\tilde{\boldsymbol{C}}_{d,i} \\ 2\tilde{\boldsymbol{C}}_{d,i}^{\mathrm{T}}q_iu_i\tilde{\boldsymbol{C}}_i & 2\tilde{\boldsymbol{C}}_{d,i}^{\mathrm{T}}q_iu_i\tilde{\boldsymbol{D}}_i & 2\tilde{\boldsymbol{C}}_{d,i}^{\mathrm{T}}q_iu_i\tilde{\boldsymbol{C}}_{d,i}\end{bmatrix},$$

where d_{ij} is the element of matrix $\tilde{\boldsymbol{D}}$ at the ith row and jth column.

By the S-Procedure, if there exist $\tau_i\geq0$ (i=1, 2, …, L), such that the following inequality (18) holds, where $\boldsymbol{T}=\mathrm{diag}(\tau_1, \tau_2, \ldots, \tau_L)$ and $\boldsymbol{T}\geq\boldsymbol{0}$, then $\boldsymbol{T}_0$ is negative definite (i.e., $\Delta V(\boldsymbol{x}_k,\xi_k)<0$). In other words, Eq.(18) is a sufficient condition for the global asymptotical stability of the origin of system (4). Eq.(18) can be rewritten as Eq.(19).

$$\boldsymbol{T}_i^1 = \begin{bmatrix} 0 & 0 & \cdots & -\tilde{\boldsymbol{C}}_i^{\mathrm{T}}(q_i+u_i) & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -d_{i,1}(q_i+u_i) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 \\ -(q_i+u_i)\tilde{\boldsymbol{C}} & -(q_i+u_i)d_{i,1} & \cdots & 2-2(q_i+u_i)d_{i,i} & \cdots & -(q_i+u_i)d_{i,L} & -(q_i+u_i)\tilde{\boldsymbol{C}}_{d,i} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -d_{i,L}(q_i+u_i) & 0 & 0 & 0 \\ 0 & 0 & \cdots & -\tilde{\boldsymbol{C}}_{d,i}^{\mathrm{T}}(q_i+u_i) & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{T}_0-\sum_{i=1}^{L}\tau_i(\boldsymbol{T}_i^1+\boldsymbol{T}_i^2)=\underbrace{\begin{bmatrix}\tilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}}-\mathrm{e}^{-2\gamma}\boldsymbol{R}+\boldsymbol{\Gamma} & \tilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{B}}+\tilde{\boldsymbol{C}}^{\mathrm{T}}\boldsymbol{\Lambda} & \tilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}}_d\\ \tilde{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}}+\boldsymbol{\Lambda}\tilde{\boldsymbol{C}} & \tilde{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{B}}+\boldsymbol{\Lambda}\tilde{\boldsymbol{D}}+\tilde{\boldsymbol{D}}^{\mathrm{T}}\boldsymbol{\Lambda} & \tilde{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}}_d+\boldsymbol{\Lambda}\tilde{\boldsymbol{C}}_d\\ \tilde{\boldsymbol{A}}_d^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}} & \tilde{\boldsymbol{A}}_d^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{B}}+\tilde{\boldsymbol{C}}_d^{\mathrm{T}}\boldsymbol{\Lambda} & \tilde{\boldsymbol{A}}_d^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}}_d-\mathrm{e}^{2\gamma h}\boldsymbol{\Gamma}\end{bmatrix}}_{T_0}-$$
$$\begin{bmatrix}2\tilde{\boldsymbol{C}}^{\mathrm{T}}\boldsymbol{TQU}\tilde{\boldsymbol{C}} & 2\tilde{\boldsymbol{C}}^{\mathrm{T}}\boldsymbol{TQU}\tilde{\boldsymbol{D}}-\tilde{\boldsymbol{C}}^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{U})\boldsymbol{T} & 2\tilde{\boldsymbol{C}}^{\mathrm{T}}\boldsymbol{TQU}\tilde{\boldsymbol{C}}_d\\ \begin{pmatrix}-2\tilde{\boldsymbol{D}}^{\mathrm{T}}\boldsymbol{TQU}\tilde{\boldsymbol{C}}\\ -\boldsymbol{T}(\boldsymbol{Q}+\boldsymbol{U})\tilde{\boldsymbol{C}}\end{pmatrix} & \begin{pmatrix}2\tilde{\boldsymbol{D}}^{\mathrm{T}}\boldsymbol{TQU}\tilde{\boldsymbol{D}}+2\boldsymbol{T}\\ -\tilde{\boldsymbol{D}}^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{U})\boldsymbol{T}-\boldsymbol{T}(\boldsymbol{Q}+\boldsymbol{U})\tilde{\boldsymbol{D}}\end{pmatrix} & \begin{pmatrix}2\tilde{\boldsymbol{D}}^{\mathrm{T}}\boldsymbol{TQU}\tilde{\boldsymbol{C}}_d\\ -\boldsymbol{T}(\boldsymbol{Q}+\boldsymbol{U})\tilde{\boldsymbol{C}}_d\end{pmatrix}\\ 2\tilde{\boldsymbol{C}}_d^{\mathrm{T}}\boldsymbol{TQU}\tilde{\boldsymbol{C}} & 2\tilde{\boldsymbol{C}}_d^{\mathrm{T}}\boldsymbol{TQU}\tilde{\boldsymbol{D}}-\tilde{\boldsymbol{C}}_d^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{U})\boldsymbol{T} & 2\tilde{\boldsymbol{C}}_d^{\mathrm{T}}\boldsymbol{TQU}\tilde{\boldsymbol{C}}_d\end{bmatrix}<\boldsymbol{0},\tag{18}$$

$$\begin{bmatrix}\tilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}}-\mathrm{e}^{-2\gamma}\boldsymbol{R}+\boldsymbol{\Gamma} & \tilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{B}}+\tilde{\boldsymbol{C}}^{\mathrm{T}}\boldsymbol{\Sigma} & \tilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}}_d\\ \tilde{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}}+\boldsymbol{\Lambda}\tilde{\boldsymbol{C}} & \begin{pmatrix}\tilde{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{B}}+\boldsymbol{\Sigma}\tilde{\boldsymbol{D}}\\ +\tilde{\boldsymbol{D}}^{\mathrm{T}}\boldsymbol{\Sigma}-2\boldsymbol{T}\end{pmatrix} & \tilde{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}}_d+\boldsymbol{\Sigma}\tilde{\boldsymbol{C}}_d\\ \tilde{\boldsymbol{A}}_d^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}} & \tilde{\boldsymbol{A}}_d^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{B}}+\tilde{\boldsymbol{C}}_d^{\mathrm{T}}\boldsymbol{\Sigma} & \tilde{\boldsymbol{A}}_d^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}}_d-\mathrm{e}^{-2\gamma h}\boldsymbol{\Gamma}\end{bmatrix}-\begin{bmatrix}\tilde{\boldsymbol{C}}^{\mathrm{T}}\boldsymbol{TQU}\\ \tilde{\boldsymbol{D}}^{\mathrm{T}}\boldsymbol{TQU}\\ \tilde{\boldsymbol{C}}_d^{\mathrm{T}}\boldsymbol{TQU}\end{bmatrix}\begin{bmatrix}\frac{1}{2}\boldsymbol{TQU}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{TQU}\tilde{\boldsymbol{C}} & \boldsymbol{TQU}\tilde{\boldsymbol{D}} & \boldsymbol{TQU}\tilde{\boldsymbol{C}}_d\end{bmatrix}<\boldsymbol{0}.\tag{19}$$

In view of Lemma 3, the second term of the left side of Eq.(18) is positive semi-definite. Therefore, if Eq.(20) holds, then Eq.(18) holds. Using the Schur complement lemma, Eq.(20) is equivalent to Eq.(21).

Since $\tilde{\boldsymbol{A}}=\boldsymbol{A}+\Delta\boldsymbol{A}$, $\tilde{\boldsymbol{A}}_d=\boldsymbol{A}_d+\Delta\boldsymbol{A}_d$, $\tilde{\boldsymbol{B}}=\boldsymbol{B}+\Delta\boldsymbol{B}$, $\tilde{\boldsymbol{C}}=\boldsymbol{C}+\Delta\boldsymbol{C}$, $\tilde{\boldsymbol{C}}_d=\boldsymbol{C}_d+\Delta\boldsymbol{C}_d$, $\tilde{\boldsymbol{D}}=\boldsymbol{D}+\Delta\boldsymbol{D}$, then we will obtain Eq.(22). By Lemma 1, Eq.(22) is equivalent to Eq.(23) (as shown at the top of the next page).

Using the well-known Schur complement lemma, the above condition can be rearranged as Eq.(11). It is known that if Eq.(11) holds, then $\Delta V(\boldsymbol{x}(k))<\boldsymbol{0}$, i.e., $V(\boldsymbol{x}(k))\le V(\boldsymbol{x}(0))$. Furthermore,

$$\begin{aligned}&V(\boldsymbol{x}(0),\boldsymbol{\xi}(0))\\&=\boldsymbol{x}^{\mathrm{T}}(0)\boldsymbol{Px}(0)+\sum_{i=-h}^{-1}\mathrm{e}^{2\gamma i}\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\Gamma}\boldsymbol{x}_i+2\sum_{i=1}^{L}\lambda_i\sum_{j=0}^{-1}\mathrm{e}^{2\gamma j}\phi_i(\xi_i(j))\xi_i(j)\\&\le\lambda_{\mathrm{M}}(\boldsymbol{P})\|\boldsymbol{x}(0)\|^2+\lambda_{\mathrm{M}}(\boldsymbol{\Gamma})\|\boldsymbol{\Omega}\|^2\frac{\mathrm{e}^{-2\gamma}-\mathrm{e}^{-2\gamma(h+1)}}{1-\mathrm{e}^{-2\gamma}}\\&\quad+2\lambda_{\mathrm{M}}(\boldsymbol{U\Lambda})[\mathrm{e}^{-2\gamma}\|\boldsymbol{\xi}(-1)\|^2+\|\boldsymbol{\xi}(0)\|^2].\end{aligned}\tag{24}$$

$$\begin{bmatrix}\tilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}}-\mathrm{e}^{-2\gamma}\boldsymbol{R}+\boldsymbol{\Gamma} & \tilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{B}}+\tilde{\boldsymbol{C}}^{\mathrm{T}}\boldsymbol{\Sigma} & \tilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}}_d\\ \tilde{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}}+\boldsymbol{\Lambda}\tilde{\boldsymbol{C}} & \begin{pmatrix}\tilde{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{B}}+\boldsymbol{\Sigma}\tilde{\boldsymbol{D}}\\ +\tilde{\boldsymbol{D}}^{\mathrm{T}}\boldsymbol{\Sigma}-2\boldsymbol{T}\end{pmatrix} & \tilde{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}}_d+\boldsymbol{\Sigma}\tilde{\boldsymbol{C}}_d\\ \tilde{\boldsymbol{A}}_d^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}} & \tilde{\boldsymbol{A}}_d^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{B}}+\tilde{\boldsymbol{C}}_d^{\mathrm{T}}\boldsymbol{\Sigma} & \boldsymbol{A}_d^{\mathrm{T}}\boldsymbol{R}\tilde{\boldsymbol{A}}_d-\mathrm{e}^{-2\gamma h}\boldsymbol{\Gamma}\end{bmatrix}<\boldsymbol{0}\tag{20}$$

$$\boldsymbol{M}_1=\begin{bmatrix}-\boldsymbol{R} & \boldsymbol{R}\tilde{\boldsymbol{A}} & \boldsymbol{R}\tilde{\boldsymbol{B}} & \boldsymbol{R}\tilde{\boldsymbol{A}}_d\\ * & -\mathrm{e}^{-2\gamma}\boldsymbol{R}+\boldsymbol{\Gamma} & \tilde{\boldsymbol{C}}^{\mathrm{T}}\boldsymbol{\Sigma} & \boldsymbol{0}\\ * & * & \boldsymbol{\Sigma}\tilde{\boldsymbol{D}}+\tilde{\boldsymbol{D}}^{\mathrm{T}}\boldsymbol{\Sigma}-2\boldsymbol{T} & \boldsymbol{\Sigma}\tilde{\boldsymbol{C}}_d\\ * & * & * & -\mathrm{e}^{-2\gamma h}\boldsymbol{\Gamma}\end{bmatrix}<\boldsymbol{0}.\tag{21}$$

$$\begin{aligned}\boldsymbol{M}_1=&\begin{bmatrix}-\boldsymbol{R} & \boldsymbol{RA} & \boldsymbol{RB} & \boldsymbol{RA}_d\\ * & -\mathrm{e}^{-2\gamma}\boldsymbol{R}+\boldsymbol{\Gamma} & \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Sigma} & \boldsymbol{0}\\ * & * & \boldsymbol{\Sigma D}+\boldsymbol{D}^{T}\boldsymbol{\Sigma}-2\boldsymbol{T} & \boldsymbol{\Sigma C}_d\\ * & * & * & -\mathrm{e}^{-2\gamma h}\boldsymbol{\Gamma}\end{bmatrix}+\begin{bmatrix}\boldsymbol{PH}_x\\ \boldsymbol{0}\\ \boldsymbol{0}\\ \boldsymbol{0}\end{bmatrix}\boldsymbol{F}(k)\begin{bmatrix}\boldsymbol{0} & \boldsymbol{E}_{x1} & \boldsymbol{E}_{x2} & \boldsymbol{E}_{x3}\end{bmatrix}+\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{E}_{x1}^{\mathrm{T}}\\ \boldsymbol{E}_{x2}^{\mathrm{T}}\\ \boldsymbol{E}_{x3}^{\mathrm{T}}\end{bmatrix}\boldsymbol{F}^{\mathrm{T}}(k)\begin{bmatrix}\boldsymbol{H}_1^{\mathrm{T}}\boldsymbol{P} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\\&+\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{0}\\ \boldsymbol{\Sigma H}_q\\ \boldsymbol{0}\end{bmatrix}\boldsymbol{F}(k)\begin{bmatrix}\boldsymbol{0} & \boldsymbol{E}_{q1} & \boldsymbol{E}_{q2} & \boldsymbol{E}_{q3}\end{bmatrix}+\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{E}_{q1}^{\mathrm{T}}\\ \boldsymbol{E}_{q2}^{\mathrm{T}}\\ \boldsymbol{E}_{q3}^{\mathrm{T}}\end{bmatrix}\boldsymbol{F}^{\mathrm{T}}(k)\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{H}_q^{\mathrm{T}}\boldsymbol{\Sigma} & \boldsymbol{0}\end{bmatrix}<\boldsymbol{0},\end{aligned}\tag{22}$$

$$\begin{bmatrix} -\boldsymbol{R}+\varepsilon_1^{-1}\boldsymbol{R}\boldsymbol{H}_x\boldsymbol{H}_x^{\mathrm{T}}\boldsymbol{R} & \boldsymbol{R}\boldsymbol{A} & \boldsymbol{R}\boldsymbol{B} & \boldsymbol{R}\boldsymbol{A}_d \\ * & \begin{pmatrix} -\mathrm{e}^{-2\gamma}\boldsymbol{R}+\boldsymbol{\Gamma}+ \\ \varepsilon_1\boldsymbol{E}_{x1}^{\mathrm{T}}\boldsymbol{E}_{x1}+\varepsilon_2\boldsymbol{E}_{q1}^{\mathrm{T}}\boldsymbol{E}_{q1} \end{pmatrix} & \begin{pmatrix} \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Sigma}+\varepsilon_1\boldsymbol{E}_{x1}^{\mathrm{T}}\boldsymbol{E}_{x2} \\ +\varepsilon_2\boldsymbol{E}_{q1}^{\mathrm{T}}\boldsymbol{E}_{q2} \end{pmatrix} & \varepsilon_1\boldsymbol{E}_{x1}^{\mathrm{T}}\boldsymbol{E}_{x3}+\varepsilon_2\boldsymbol{E}_{q1}^{\mathrm{T}}\boldsymbol{E}_{q3} \\ * & * & \begin{pmatrix} \boldsymbol{\Sigma}\boldsymbol{D}+\boldsymbol{D}^{\mathrm{T}}\boldsymbol{\Sigma}-2\boldsymbol{T}+\varepsilon_1\boldsymbol{E}_{x2}^{\mathrm{T}}\boldsymbol{E}_{x2} \\ +\varepsilon_2\boldsymbol{E}_{q2}^{\mathrm{T}}\boldsymbol{E}_{q2}\varepsilon_2^{-1}\boldsymbol{\Sigma}\boldsymbol{H}_q\boldsymbol{H}_q^{\mathrm{T}}\boldsymbol{\Sigma} \end{pmatrix} & \begin{pmatrix} \boldsymbol{\Sigma}\boldsymbol{C}_d+\varepsilon_1\boldsymbol{E}_{x2}^{\mathrm{T}}\boldsymbol{E}_{x3} \\ +\varepsilon_2\boldsymbol{E}_{q2}^{\mathrm{T}}\boldsymbol{E}_{q3} \end{pmatrix} \\ * & * & * & \begin{pmatrix} -\mathrm{e}^{-2\gamma h}\boldsymbol{\Gamma}+\varepsilon_1\boldsymbol{E}_{x3}^{\mathrm{T}}\boldsymbol{E}_{x3} \\ +\varepsilon_2\boldsymbol{E}_{q3}^{\mathrm{T}}\boldsymbol{E}_{q3} \end{pmatrix} \end{bmatrix}<\boldsymbol{0}. \quad (23)$$

In Eq.(24), $\|\boldsymbol{\Omega}\|=\sup_{-h\le\theta\le 0}\|\boldsymbol{x}(\theta)\|$. Meanwhile,

$$V(\boldsymbol{x}_k,\xi_k)\ge \mathrm{e}^{2\gamma k}\boldsymbol{x}_k^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}_k\ge \mathrm{e}^{2\gamma k}\lambda_{\mathrm{M}}(\boldsymbol{P})\|\boldsymbol{x}_k\|^2. \quad (25)$$

From Eqs.(24) and (25), Eq.(12) can be obtained. The proof of Theorem 1 is completed.

A non-delayed SNNM with structured uncertainties can be represented as:

$$\begin{cases} \boldsymbol{x}(k+1)=(\boldsymbol{A}+\Delta\boldsymbol{A})\boldsymbol{x}(k)+(\boldsymbol{B}+\Delta\boldsymbol{B})\boldsymbol{\phi}(\boldsymbol{\xi}(k)), \\ \boldsymbol{\xi}(k)=(\boldsymbol{C}+\Delta\boldsymbol{C})\boldsymbol{x}(k)+(\boldsymbol{D}+\Delta\boldsymbol{D})\boldsymbol{\phi}(\boldsymbol{\xi}(k)), \end{cases} \quad (26)$$

where

$$\begin{cases} [\Delta\boldsymbol{A}\ \Delta\boldsymbol{B}]=\boldsymbol{H}_x\boldsymbol{F}(k)[\boldsymbol{E}_{x1}\ \ \boldsymbol{E}_{x2}], \\ [\Delta\boldsymbol{C}\ \Delta\boldsymbol{D}]=\boldsymbol{H}_q\boldsymbol{F}(k)[\boldsymbol{E}_{q1}\ \ \boldsymbol{E}_{q2}], \end{cases} \quad (27)$$

and

$$\boldsymbol{F}^{\mathrm{T}}(k)\boldsymbol{F}(k)\le\boldsymbol{I}. \quad (28)$$

Like Theorem 1, the following corollary can be obtained:

Corollary 1 The SNNM described in Eqs.(26)~(28) is globally robustly exponentially stable if there exist a symmetric matrix $\boldsymbol{R}$, diagonal semi-positive definite matrices $\boldsymbol{\Lambda}$ and $\boldsymbol{T}$, and positive scalars ε_1, ε_2, such that the following LMI condition holds:

$$\boldsymbol{N}=\begin{bmatrix} \boldsymbol{N}_{1,1} & \boldsymbol{N}_{1,2} \\ * & \boldsymbol{N}_{2,2} \end{bmatrix}<\boldsymbol{0}, \quad (29)$$

where

$$\boldsymbol{N}_{12}=\begin{bmatrix} \boldsymbol{R}\boldsymbol{H}_x & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Sigma}\boldsymbol{H}_q \end{bmatrix},\quad \boldsymbol{N}_{22}=\mathrm{diag}(-\varepsilon_1\boldsymbol{I},-\varepsilon_2\boldsymbol{I}).$$

$$\boldsymbol{N}_{11}= \begin{bmatrix} -\boldsymbol{R} & \boldsymbol{R}\boldsymbol{A} & \boldsymbol{R}\boldsymbol{B} \\ * & \begin{pmatrix} -\mathrm{e}^{-2\gamma}\boldsymbol{R}+\varepsilon_1\boldsymbol{E}_{x1}^{\mathrm{T}}\boldsymbol{E}_{x1} \\ +\varepsilon_2\boldsymbol{E}_{q1}^{\mathrm{T}}\boldsymbol{E}_{q1} \end{pmatrix} & \begin{pmatrix} \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Sigma}+\varepsilon_1\boldsymbol{E}_{x1}^{\mathrm{T}}\boldsymbol{E}_{x2} \\ +\varepsilon_2\boldsymbol{E}_{q1}^{\mathrm{T}}\boldsymbol{E}_{q2} \end{pmatrix} \\ * & * & \begin{pmatrix} \boldsymbol{\Sigma}\boldsymbol{D}+\boldsymbol{D}^{\mathrm{T}}\boldsymbol{\Sigma}-2\boldsymbol{T}+ \\ \varepsilon_1\boldsymbol{E}_{x2}^{\mathrm{T}}\boldsymbol{E}_{x2}+\varepsilon_2\boldsymbol{E}_{q2}^{\mathrm{T}}\boldsymbol{E}_{q2} \end{pmatrix} \end{bmatrix}$$

Moreover,

$$\boldsymbol{x}(k)\le\sqrt{\frac{\lambda_{\mathrm{M}}(\boldsymbol{P})\|\boldsymbol{x}(0)\|^2+\Pi}{\lambda_{\mathrm{M}}(\boldsymbol{P})}}\mathrm{e}^{-\gamma k}, \quad (30)$$

where $\Pi=2\lambda_{\mathrm{M}}(\boldsymbol{U\Lambda})[\mathrm{e}^{-2\gamma}\|\boldsymbol{\xi}(-1)\|^2+\|\boldsymbol{\xi}(0)\|^2]$.

Proof The proof of Corollary 1 follows the steps of proof of Theorem 1, and choosing the Lyapunov functional in the form

$$V(\boldsymbol{x}_k,\boldsymbol{\xi}_k)=\mathrm{e}^{2\gamma k}\boldsymbol{x}_k^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}_k+2\sum_{i=1}^{L}\lambda_i\sum_{j=0}^{k-1}\mathrm{e}^{2\gamma j}\phi_i(\xi_i(j))\xi_i(j). \quad (31)$$

ROBUST EXPONENTIAL STABILITY OF DISCRETE-TIME BAM NETWORKS

In this section, the robust exponential stability for discrete-time BAM neural network described by the following model is concerned:

$$\begin{cases} \boldsymbol{v}(k+1)=(\boldsymbol{M}+\Delta\boldsymbol{M})\boldsymbol{v}(k)+ \\ \qquad (\boldsymbol{W}_1+\Delta\boldsymbol{W}_1)g(\boldsymbol{u}(k-\tau))+\boldsymbol{J}, \\ \boldsymbol{u}(k+1)=(\boldsymbol{E}+\Delta\boldsymbol{E})\boldsymbol{u}(k)+ \\ \qquad (\boldsymbol{W}_2+\Delta\boldsymbol{W}_2)f(\boldsymbol{v}(k-\sigma))+\boldsymbol{U}, \end{cases} \quad (32)$$

where $\boldsymbol{v}(k)=[v_1(k), v_2(k), \ldots, v_n(k)]^{\mathrm{T}}$, $\boldsymbol{u}(k)=[u_1(k), u_2(k), \ldots, u_n(k)]^{\mathrm{T}}$ are the activations of neurons. $\boldsymbol{M}=\mathrm{diag}(m_1, m_2, \ldots, m_n)$, $\boldsymbol{E}=\mathrm{diag}(e_1, e_2, \ldots, e_n)$ denote the neuron charging time constants and decay rates, respectively. $\boldsymbol{W}_1=(w_{ij}^{(1)})_{n\times n}$ and $\boldsymbol{W}_2=(w_{ij}^{(2)})_{n\times n}$ are the synaptic connection strengths. $\Delta\boldsymbol{M}=\mathrm{diag}(\Delta m_1, \Delta m_2, \ldots, \Delta m_n)$, $\Delta\boldsymbol{E}=\mathrm{diag}(\Delta e_1, \Delta e_2, \ldots, \Delta e_n)$, $\Delta\boldsymbol{W}_1=(\Delta w_{ij}^{(1)})_{n\times n}$, $\Delta\boldsymbol{W}_2=(\Delta w_{ij}^{(2)})_{n\times n}$ indicate the parametric uncertainties of $\boldsymbol{M}$, $\boldsymbol{E}$, $\boldsymbol{W}_1$, $\boldsymbol{W}_2$, respectively. $\boldsymbol{J}$ and $\boldsymbol{U}$ are the exogenous inputs. Time delays τ and σ correspond to the finite speeds of the axonal transmission of signals. $\boldsymbol{g}(\boldsymbol{u}(k-\tau))=[g_1(u_1(k-\tau_1)), g_2(u_2(k-\tau_2)), \ldots, g_n(u_n(k-\tau_n))]^{\mathrm{T}}$, $\boldsymbol{f}(\boldsymbol{v}(k-\sigma))=[f_1(v_1(k-\sigma_1)), f_2(v_2(k-\sigma_2)), \ldots, f_n(v_n(k-\sigma_n))]^{\mathrm{T}}$ respectively represent the activation functions of the neurons and the propagational functions, satisfying the sector conditions.

The parametric uncertainties in system (32) satisfy:

$$\begin{cases}[\Delta\boldsymbol{M} \ \ \Delta\boldsymbol{W}_1]=\boldsymbol{H}_v\boldsymbol{F}(k)[\boldsymbol{E}_{v1} \ \ \boldsymbol{E}_{v2}],\\ [\Delta\boldsymbol{E} \ \ \Delta\boldsymbol{W}_2]=\boldsymbol{H}_u\boldsymbol{F}(k)[\boldsymbol{E}_{u1} \ \ \boldsymbol{E}_{u2}],\end{cases} \tag{33}$$

and

$$\boldsymbol{F}^{\mathrm{T}}(k)\boldsymbol{F}(k)\le\boldsymbol{I}.$$

Assume that SNNM system (32) has an equilibrium point $(\boldsymbol{u}^*, \boldsymbol{v}^*)$. Let $\hat{\boldsymbol{u}}(k)=\boldsymbol{u}(k)-\boldsymbol{u}^*$, $\hat{\boldsymbol{v}}(k)=\boldsymbol{v}(k)-\boldsymbol{v}^*$, system (32) can be transformed into:

$$\begin{cases}\hat{\boldsymbol{v}}(k+1)=(\boldsymbol{M}+\Delta\boldsymbol{M})\hat{\boldsymbol{v}}(k)+(\boldsymbol{W}_1+\Delta\boldsymbol{W}_1)\hat{\boldsymbol{g}}(\hat{\boldsymbol{u}}(k-\tau)),\\ \hat{\boldsymbol{u}}(k+1)=(\boldsymbol{E}+\Delta\boldsymbol{E})\hat{\boldsymbol{u}}(k)+(\boldsymbol{W}_2+\Delta\boldsymbol{W}_2)\hat{\boldsymbol{f}}(\hat{\boldsymbol{v}}(k-\sigma)),\end{cases} \tag{34}$$

where

$$\hat{\boldsymbol{g}}(\hat{\boldsymbol{u}}(k-\tau))=\boldsymbol{g}(\boldsymbol{u}(t-\tau)+\boldsymbol{u}^*)-\boldsymbol{g}(\boldsymbol{u}^*),$$
$$\hat{\boldsymbol{f}}(\hat{\boldsymbol{v}}(k-\sigma))=\boldsymbol{f}(\boldsymbol{v}(k-\sigma)+\boldsymbol{v}^*)-\boldsymbol{f}(\boldsymbol{v}^*).$$

Let

$$\boldsymbol{x}(k)=[v_1(k), \ldots, v_n(k), u_1(k), \ldots, u_n(k)]^{\mathrm{T}},$$
$$\boldsymbol{\phi}(\cdot)=[\hat{g}_1(\cdot), \ldots, \hat{g}_n(\cdot), \hat{f}_1(\cdot), \ldots, \hat{f}_n(\cdot)]^{\mathrm{T}},$$
$$\boldsymbol{A}=\begin{bmatrix}\boldsymbol{M} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{E}\end{bmatrix}, \ \boldsymbol{B}=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{W}_1\\ \boldsymbol{W}_2 & \boldsymbol{0}\end{bmatrix}, \ \boldsymbol{C}=\boldsymbol{0}_{2n\times 2n}, \boldsymbol{D}=\boldsymbol{0}_{2n\times 2n},$$
$$\boldsymbol{A}_d=\boldsymbol{0}_{2n\times 2n}, \boldsymbol{C}_d=\boldsymbol{I}_{2n\times 2n}, \Delta\boldsymbol{A}_d=\boldsymbol{0}_{2n\times 2n}, \Delta\boldsymbol{C}_d=\boldsymbol{I}_{2n\times 2n},$$
$$\Delta\boldsymbol{A}=\begin{bmatrix}\boldsymbol{H}_v & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{H}_u\end{bmatrix}\begin{bmatrix}\boldsymbol{F}(k) & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{F}(k)\end{bmatrix}\begin{bmatrix}\boldsymbol{E}_{v1} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{E}_{u1}\end{bmatrix},$$
$$\Delta\boldsymbol{B}=\begin{bmatrix}\boldsymbol{H}_v & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{H}_u\end{bmatrix}\begin{bmatrix}\boldsymbol{F}(k) & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{F}(k)\end{bmatrix}\begin{bmatrix}\boldsymbol{0} & \boldsymbol{E}_{v2}\\ \boldsymbol{E}_{u2} & \boldsymbol{0}\end{bmatrix},$$
$$\Delta\boldsymbol{C}=\boldsymbol{0}_{2n\times 2n}, \Delta\boldsymbol{D}=\boldsymbol{0}_{2n\times 2n}, \quad L=2n.$$

The system (32) is transformed into the form of SNNM system (4). Therefore, Theorem 1 can be used to analyze the robust exponential stability for system (32).

As a numerical example, considering the following discrete-time BAM neural network:

$$\begin{cases}\boldsymbol{v}(k+1)=\left(\begin{bmatrix}0.2 & 0\\ 0 & 0.2\end{bmatrix}+\Delta\boldsymbol{M}\right)\boldsymbol{v}(k)+\\ \quad\left(\begin{bmatrix}0 & 0.125\\ 0.125 & 0\end{bmatrix}+\Delta\boldsymbol{W}_1\right)\tanh(\boldsymbol{u}(k-2)),\\ \boldsymbol{u}(k+1)=\left(\begin{bmatrix}0.1 & 0\\ 0 & 0.1\end{bmatrix}+\Delta\boldsymbol{E}\right)\boldsymbol{u}(k)+\\ \quad\left(\begin{bmatrix}0 & -0.05\\ -0.05 & 0\end{bmatrix}+\Delta\boldsymbol{W}_2\right)\tanh(\boldsymbol{v}(k-2)),\end{cases} \tag{35}$$

$$\boldsymbol{H}_v=\begin{bmatrix}0.10 & 0\\ 0 & 0.15\end{bmatrix}, \ \boldsymbol{H}_u=\begin{bmatrix}0.10 & 0\\ 0 & 0.15\end{bmatrix},$$
$$\boldsymbol{E}_{v1}=\begin{bmatrix}0.04 & 0\\ 0 & 0.08\end{bmatrix}, \ \boldsymbol{E}_{v2}=\begin{bmatrix}0 & 0.01\\ 0.02 & 0\end{bmatrix},$$
$$\boldsymbol{E}_{u1}=\begin{bmatrix}0.01 & 0\\ 0 & 0.03\end{bmatrix}, \ \boldsymbol{E}_{u2}=\begin{bmatrix}0 & -0.02\\ 0.01 & 0\end{bmatrix}.$$

With the above transformation, system (35) can be transformed into the SNNM system (4) with

$$\boldsymbol{A}=\mathrm{diag}(0.2, 0.2, 0.1, 0.1),$$
$$\boldsymbol{B}=\begin{bmatrix}0 & 0 & 0 & 0.125\\ 0 & 0 & 0.125 & 0\\ 0 & -0.05 & 0 & 0\\ -0.05 & 0 & 0 & 0\end{bmatrix},$$
$$\boldsymbol{E}_{x2}=\begin{bmatrix}0 & 0 & 0 & 0.01\\ 0 & 0 & 0.02 & 0\\ 0 & -0.02 & 0 & 0\\ 0.01 & 0 & 0 & 0\end{bmatrix},$$
$$\boldsymbol{E}_{x1}=\mathrm{diag}(0.04, 0.08, 0.01, 0.03),$$
$$\boldsymbol{H}_x=\mathrm{diag}(0.10, 0.15, 0.10, 0.15),$$
$$\boldsymbol{A}_d=\boldsymbol{C}=\boldsymbol{D}=\boldsymbol{H}_q=\boldsymbol{E}_{x3}=\boldsymbol{E}_{q1}=\boldsymbol{E}_{q2}=\boldsymbol{E}_{q3}=\boldsymbol{0}_{4\times 4},$$
$$\boldsymbol{C}_d=\boldsymbol{U}=\boldsymbol{I}_{4\times 4}, \boldsymbol{Q}=\boldsymbol{0}_{4\times 4}.$$

Using the MATLAB LMI toolbox, the optimal solution for the convex optimization problem (11) with γ=0.25 can be obtained:

$$\boldsymbol{P}=\text{diag}(0.6482, 0.6395, 0.6676, 0.6590),$$
$$\boldsymbol{\Gamma}=\text{diag}(0.2386, 0.2346, 0.2571, 0.2538),$$
$$\boldsymbol{\Lambda}=\text{diag}(0.0191, 0.0191, 0.0191, 0.0191),$$
$$\boldsymbol{T}=\text{diag}(0.1237, 0.1237, 0.1237, 0.1237),$$
$$\varepsilon_1=0.3619,\ \varepsilon_2=0.3669.$$

So, the BAM system (35) is globally robustly exponentially stable with convergence rate γ=0.25.

CONCLUSION

In this work, the problem of robust exponential stability for discrete-time standard neural network models (SNNMs) has been studied in detail. The criteria obtained in this paper are derived by means of Lyapunov functionals and S-Procedure, and possess the structure of LMI, so they can be easily solved using MATLAB LMI toolbox. SNNM provides a general framework to facilitate the stability analysis of RNNs. The last example illustrates how a BAM neural network is transformed into SNNM and how it is analyzed using the presented approach.

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