Journal of Zhejiang University SCIENCE A
ISSN 1673-565X (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
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# Paths of algebraic hyperbolic curves*

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Received Sept. 15, 2007; revision accepted Oct. 30, 2007

**Abstract:** Cubic algebraic hyperbolic (AH) Bézier curves and AH spline curves are defined with a positive parameter $\alpha$ in the space spanned by $\{1, t, \sinh t, \cosh t\}$. Modifying the value of $\alpha$ yields a family of AH Bézier or spline curves with the family parameter $\alpha$. For a fixed point on the original curve, it will move on a defined curve called "path of AH curve" (AH Bézier and AH spline curves) when $\alpha$ changes. We describe the geometric effects of the paths and give a method to specify a curve passing through a given point.

**Key words:** Algebraic hyperbolic (AH) Bézier curve, AH spline curve, Path, Shape modification
**doi:**10.1631/jzus.A071490 **Document code:** A **CLC number:** TP391.72

## INTRODUCTION

Bézier curves and surfaces are well known geometric modeling tools in Computer Aided Geometric Design (CAGD). Due to their several limitations in practical applications (Mainar *et al*., 2001), several new forms of curve and surface schemes have been proposed for geometric modeling in CAGD (Koch and Lyche, 1989; 1991; Zhang, 1996; 1997; 1999; Lü *et al*., 2002; Li and Wang, 2005). Pottmann and Wagner (1994) presented a kind of exponential splines in tension in the space spanned by $\{1, t, \sinh t, \cosh t\}$, while Koch and Lyche (1989; 1991) obtained its normalized B-basis by applying the blossoming principle and a de Casteljau type algorithm in extended Chebyshev spaces (Pottmann and Wagner, 1994). Lü *et al*.(2002) gave the explicit expressions for uniform splines which share most of the properties as those of the B-splines in polynomial space. Li and Wang (2005) generalized the curves and surfaces of exponential forms to algebraic hyperbolic (AH) Bézier and spline forms of any degree, which can represent exactly some remarkable curves and surfaces such as the hyperbola, the catenary, the hyperbolic spiral and the hyperbolic paraboloid.

In fact, cubic curves are most widely used in CAGD, so the geometric properties of the cubic curves are of great importance (Hoffmann and Juhász, 2003; 2006; Juhász and Hoffmann, 2004; Hoffmann *et al*., 2006). In the space $\{1, t, \sinh t, \cosh t\}$, the cubic AH Bézier curves and AH spline curves are defined with a positive parameter $\alpha$. For an AH Bézier curve, we get a family of AH Bézier curves with the family parameter $\alpha$ when $\alpha$ changes. Meanwhile, the curve points will move on some specially defined curves called paths. The geometric effects of the paths of the AH Bézier curve are described and a constrained method to modify the shape of the curve is given in this paper. Many conclusions about paths of the AH Bézier curve can be generalized to those of the AH spline curve. Furthermore, we give a simpler method to modify the shape of the AH spline curve.

This paper is organized as follows. In Section 2 we describe the paths of the AH Bézier curve and present the properties of the paths. The same method will be applied to the cubic AH spline curve in Sec-

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* Project supported by the National Natural Science Foundation of China (No. 60773179), the National Basic Research Program (973) of China (No. G2004CB318000), and the School Scientific Research Foundation of Hangzhou Dianzi University (No. KYS091507070), China

tion 3. In Section 4, a constrained method to modify the shape of the curve is given. Finally, Section 5 concludes this paper.

## PATHS OF AH BÉZIER CURVE

Given control points $\boldsymbol{p}_i$ ($i$=0, 1, 2, 3), a cubic AH Bézier curve can be given as (Li and Wang, 2005)

$$p(\alpha,t)=\sum_{i=0}^{3}B_i(\alpha,t)\boldsymbol{p}_i,\ t\in[0,\ \alpha],\ \alpha>0,$$

where the AH Bézier basis functions $B_i(\alpha, t)$ ($i$=0, 1, 2, 3) are defined as

$$\begin{cases} U=\dfrac{(1+\cosh\alpha)t+\sinh(\alpha-t)-\sinh\alpha-\sinh t}{\alpha+\alpha\cosh\alpha-2\sinh\alpha}, \\ B_0(\alpha,t)=\dfrac{(\alpha-t)-\sinh(\alpha-t)}{\alpha-\sinh\alpha}, \\ B_1(\alpha,t)=1-B_0(\alpha,t)-U, \\ B_2(\alpha,t)=U-B_3(\alpha,t), \\ B_3(\alpha,t)=(t-\sinh t)/(\alpha-\sinh\alpha). \end{cases} \tag{1}$$

Modifying the value of $\alpha$ within the interval (0, +∞) yields a family of AH Bézier curves with the family parameter $\alpha$ (Fig.1). From Fig.1, we know that the curve will approach the control polygon more closely as $\alpha$ grows to +∞, so we call $\alpha$ the shape parameter.

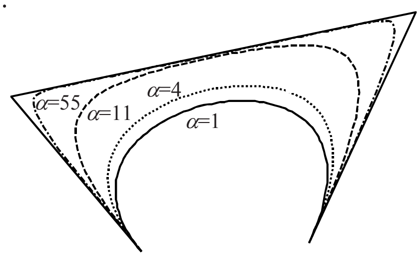


**Fig.1 A family of AH Bézier curves with family parameter *α***

To study the performance of the family curves, we substitute $\alpha r$ for $t$ in Eq.(1), then the parameter $t\in[0, \alpha]$ defined on the value of $\alpha$ is normalized into the parameter $r\in[0, 1]$. Thus the family of AH Bézier curves with the family parameter $\alpha$ can be defined as

$$s(\alpha,r)=\sum_{i=0}^{3}B_i(\alpha,\alpha r)\boldsymbol{p}_i,\ r\in[0,\ 1],\ \alpha>0.$$

For every fixed value of the parameter $r$, $s(\alpha, r)$ represents a curve called path of AH Bézier curve (Fig.2) with the running parameter $\alpha$>0. Let $\alpha\to0$, the limits of the AH Bézier basis functions $B_i(\alpha, \alpha r)$ are just the cubic Bézier basis functions:

$$B_{0,0}(r)=(1-r)^3,\quad B_{1,0}(r)=3r(1-r)^2,$$
$$B_{2,0}(r)=3r^2(1-r),\ B_{3,0}(r)=r^3.$$

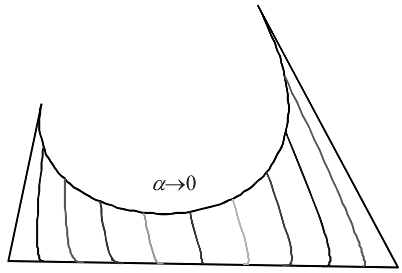


**Fig 2 Paths of an AH Bézier curve**

So it is easy to know that the AH Bézier curve $\sum_{i=0}^{3}B_{i,0}(r)\boldsymbol{p}_i$ is just a cubic Bézier curve with the same control polygon. Furthermore, all the points on the Bézier curve $\sum_{i=0}^{3}B_{i,0}(r)\boldsymbol{p}_i$ are just the starting endpoints of the paths. To obtain the other endpoints of the paths, we let $\alpha\to\infty$ and $r\neq0, 1$, then the limits of the AH Bézier basis functions $B_i(\alpha, \alpha r)$ can be computed as

$$\begin{cases} B_{0,\infty}(r)=0,\quad B_{1,\infty}(r)=(1-r), \\ B_{2,\infty}(r)=r,\quad B_{3,\infty}(r)=0. \end{cases} \tag{2}$$

By Eq.(2) and the endpoint interpolation property of AH Bézier curves (Li and Wang, 2005), we have the following theorems:

**Theorem 1** For a cubic AH Bézier curve

$$p(\alpha,\alpha r)=\sum_{i=0}^{3}B_i(\alpha,\alpha r)\boldsymbol{p}_i,\ r\in[0,1],\ \alpha>0,$$

let $\alpha\to\infty$, then the limit of the curve is the control polygon.

**Theorem 2** Let $\alpha\to\infty$ and $r\neq0, 1$, then points of the AH Bézier path $s(\alpha, r)$ will converge to a point on the line $\boldsymbol{p}_1\boldsymbol{p}_2$ with the barycentric coordinates $(r, 1-r)$, i.e.,

$$\lim_{\alpha\to\infty}s(\alpha,r)=(1-r)\boldsymbol{p}_1+r\boldsymbol{p}_2,\ r\neq0,\ 1.$$

Noting that the basis functions of the original AH Bézier curve are symmetric with respect to $t$ for the parameter $t=\alpha/2$ (Li and Wang, 2005), the paths of AH Bézier curves have also a symmetric property with respect to $r$ for the parameter $r=1/2$. The path with $r=1/2$ is the combination of four control points with the coefficients:

$$B_0 = B_3 = [\alpha/2 - \sinh(\alpha/2)]/(\alpha - \sinh\alpha),$$
$$B_1 = B_2 = [2\sinh(\alpha/2) - (\sinh\alpha)/2]/(\alpha - \sinh\alpha),$$

which obviously yield that this path is a straight line segment. Furthermore, the line segment is exactly on the line connecting the two midpoints of $\boldsymbol{p}_0\boldsymbol{p}_3$ and $\boldsymbol{p}_1\boldsymbol{p}_2$. Unfortunately, as shown by Fig.2 one can clearly see that the other paths are not straight lines, though it seems that most parts of the paths can be regarded as straight lines.

## PATHS OF AH SPLINE CURVE

### Paths of AH spline curve

Consider a cubic AH spline curve (Li and Wang, 2005):

$$q(\alpha,t) = \sum_{i=0}^{3} Z_i(\alpha,t)\boldsymbol{q}_i,\ t\in[0,\alpha],\ \alpha>0,$$

where $\boldsymbol{q}_i$ are control points and AH spline basis functions $Z_i(\alpha, t)$ ($i$=0, 1, 2, 3) are defined on a uniform knot sequence $\{\dots, -1, 0, 1, \dots\}$:

$$\begin{cases} Z_0(\alpha,t) = \dfrac{(\alpha-t)-\sinh(\alpha-t)}{2\alpha(1-\cosh\alpha)}, \\ Z_1(\alpha,t) = Z_3(\alpha,t) + \dfrac{\sinh(\alpha-t)-(\alpha-t)\cosh\alpha}{\alpha(1-\cosh\alpha)}, \\ Z_2(\alpha,t) = Z_0(\alpha,t) + \dfrac{\sinh t - t\cosh\alpha}{\alpha(1-\cosh\alpha)}, \\ Z_3(\alpha,t) = \dfrac{t-\sinh t}{2\alpha(1-\cosh\alpha)}. \end{cases} \tag{3}$$

Similar to the cubic AH Bézier curves, we can obtain a family of AH spline curves when $\alpha$ changes its way (Fig.3). Furthermore, the larger $\alpha$ is, the closer the curve will approach the control polygon. In the same way, we substitute $\alpha r$ for $t$ in Eq.(3), and the parameter $t\in[0, \alpha]$ will be normalized into the parameter $r\in[0, 1]$. Let $\alpha\to 0$ and $\alpha\to\infty$ respectively, then the furthest curve and the nearest curve away from the control polygon will be described by the following basis functions:

$$\begin{cases} Z_{0,0}(r) = (1-r)^3/6, \\ Z_{1,0}(r) = (4+3r^3-6r^2)/6, \\ Z_{2,0}(r) = (1+3r+3r^2-3r^3)/6, \\ Z_{3,0}(r) = r^3/6, \end{cases} \tag{4}$$

$$\begin{cases} Z_{0,\infty}(r) = 0,\ Z_{1,\infty}(r) = 1-r, \\ Z_{2,\infty}(r) = r,\ Z_{3,\infty}(r) = 0. \end{cases} \tag{5}$$

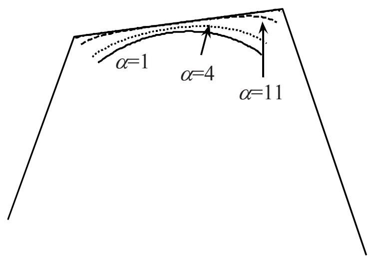


**Fig.3 A family of AH spline curves with family parameter α**

Eqs.(4) and (5) obviously provide evidence for the following theorem:

**Theorem 3** For a cubic AH spline curve

$$q(\alpha,\alpha r) = \sum_{i=0}^{3} Z_i(\alpha,\alpha r)\boldsymbol{q}_i,\ r\in[0, 1],\ \alpha>0,$$

let $\alpha\to 0$ and $\alpha\to\infty$, then the limits of the curve are the cubic B-spline curve and the line $\boldsymbol{q}_1\boldsymbol{q}_2$, respectively.

For a fixed value of $r$ and running parameter $\alpha$, the expression of the family curves represents a curve we call path (Fig.4). By Theorem 3, we know that the paths are all finite curves with two endpoints on a cubic AH spline curve and on the control polygon, respectively.

**Theorem 4** Let $\alpha\to\infty$ and $r\neq 0, 1$, then the points on the path $s(\alpha, r) = \sum_{i=0}^{3} Z_i(\alpha,\alpha r)\boldsymbol{q}_i$ of an AH spline curve will converge to a point on the line $\boldsymbol{q}_1\boldsymbol{q}_2$ with the barycentric coordinates $(r, 1-r)$, i.e.,

$$\lim_{\alpha\to\infty} s(\alpha,r) = (1-r)\boldsymbol{p}_1 + r\boldsymbol{p}_2,\quad r\neq 0, 1.$$

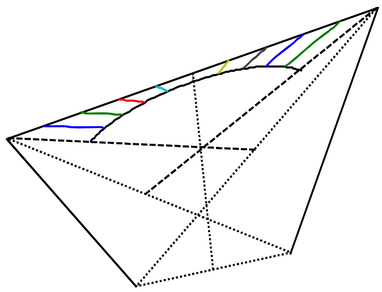

**Fig.4 Paths of the AH spline curve**

It is easy to know that the starting points of the AH spline family curves are not at the same position as those of the AH Bézier family curves. Simply letting $r$=0 (i.e., $t$=$r\alpha$=0) in Eq.(3), we have

$$Z_0 = Z_2 = \frac{\alpha - \sinh\alpha}{2\alpha(1-\cosh\alpha)},$$
$$Z_1 = \frac{\sinh\alpha - \alpha\cosh\alpha}{\alpha(1-\cosh\alpha)},\quad Z_3 = 0,$$

which yield that the path with $r$=0 is a line segment on the line connecting the point $\boldsymbol{q}_1$ and the midpoint of $\boldsymbol{q}_0\boldsymbol{q}_2$ (Fig.4). In the same way, we let $r$=1 ($t$=$r\alpha$=$\alpha$) in Eq.(3), and the basis functions will turn out to be

$$Z_0 = 0,\quad Z_2 = \frac{\sinh\alpha - \alpha\cosh\alpha}{\alpha(1-\cosh\alpha)},$$
$$Z_1 = Z_3 = \frac{\alpha - \sinh\alpha}{2\alpha(1-\cosh\alpha)}.$$

So the path when $r$=1 is a line segment on the line connecting the point $\boldsymbol{q}_2$ and the midpoint of $\boldsymbol{q}_1\boldsymbol{q}_3$ (Fig.4). Finally we get the basis functions with $r$=1/2:

$$Z_0 = Z_3 = \frac{\alpha/2 - \sinh(\alpha/2)}{2\alpha(1-\cosh\alpha)},$$
$$Z_1 = Z_2 = \frac{\alpha/2 + \sinh(\alpha/2) - \alpha\cosh(\alpha/2)}{2\alpha(1-\cosh\alpha)}.$$

Clearly, the path when $r$=1/2 is also a straight line segment with the two endpoints being the midpoints of $\boldsymbol{q}_0\boldsymbol{q}_3$ and $\boldsymbol{q}_1\boldsymbol{q}_2$, respectively (Fig.4).

**Approximate lines of the paths**

From above we know that the two endpoints of the paths are on the two family curves when $\alpha\to 0$ and $\alpha\to\infty$, respectively. Connecting every couple of points on these two curves $s(0, r)=\sum_{i=0}^{3} Z_{i,0}(r)\boldsymbol{q}_i$ and $s(\infty, r)=\sum_{i=\infty}^{3} Z_{i,\infty}(r)\boldsymbol{q}_i$, we obtain a family of straight lines with parameter $r$ (Fig.5). Every point on these lines can be described by the barycentric coordinates $(1-\lambda, \lambda)$ of these two endpoints $s(0, r)$ and $s(\infty, r)$. Thus the lines can be defined as

$$E(\lambda, r) = (1-\lambda)\sum_{i=0}^{3} Z_{i,0}(r)\boldsymbol{q}_i + \lambda\sum_{i=0}^{3} Z_{i,\infty}(r)\boldsymbol{q}_i.$$

Let $E_i(\lambda, r)=(1-\lambda)Z_{i,0}(r)+\lambda Z_{i,\infty}(r)$, then the lines can be also denoted as

$$E(\lambda, r) = \sum_{i=0}^{3} E_i(\lambda, r)\boldsymbol{q}_i.$$

In fact, the paths of the AH spline curve, as we can see, can be approximated by these lines closely.

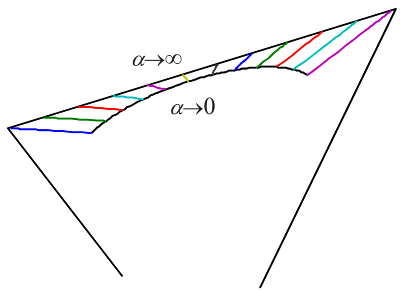


**Fig.5 Approximate lines of paths of the AH spline curve**

To prove that the paths can be closely approximated by these lines, we have to prove that the distance from every point $s(\alpha_0, r_0)$ of the paths to its corresponding approximate line $E(\lambda, r_0)$ is short enough. The distance can be enlarged to an approximate value by the distance from the point $s(\alpha_0, r_0)$ to a point on the $E(\lambda_0, r_0)$ with the barycentric coordinates of $E(\lambda_0, r_0)$ being $(1-\lambda_0, \lambda_0)$, here

$$\lambda_0 = \frac{\sum_{i=0}^{3}\{[Z_i(\alpha_0, \alpha_0 r_0) - Z_{i,0}(r_0)][Z_{i,0}(r_0) - Z_{i,\infty}(r_0)]\}}{-\sum_{i=0}^{3}[Z_{i,0}(r_0) - Z_{i,\infty}(r_0)]^2}.$$

The enlarged distance can be represented as

$$dis(\lambda_0, r_0) = |s(\alpha_0, r_0) - E(\lambda_0, r_0)|$$
$$= \left|\sum_{i=0}^{3}[Z_i(\alpha_0, \alpha_0 r_0) - E_i(\lambda_0, r_0)]\boldsymbol{q}_i\right|.$$

Similarly as approximate lines of paths of C-curves and without considering the control points (Hoffmann *et al.*, 2006), difference between the approximate lines and the paths can be described as the following standard deviation function:

$$stdev(\lambda_0, r_0) = \sum_{i=0}^{3} [Z_i(\alpha_0, \alpha_0 r_0) - E_i(\lambda_0, r_0)]^2,$$

where $\alpha_0>0$, $r_0 \in [0, 1]$. The graph of the difference is shown in Fig.6.

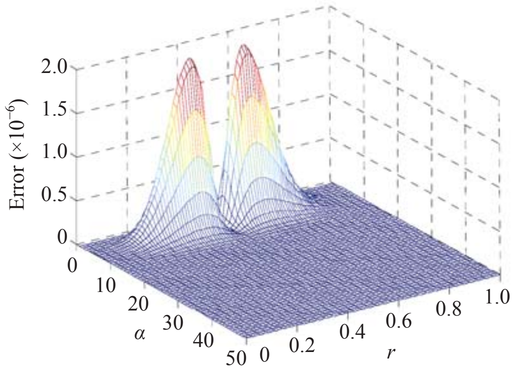


**Fig.6 Graph of the difference between paths and approximate lines**

As we can see from Fig.6, the maximum deviation is smaller than $2.0\times10^{-6}$ (at about $\alpha$=4). It is possible that we can obtain smaller deviation by another $\lambda$, but certainly, it is enough to get the result that the paths can be approximated by these lines within the deviation $2.0\times10^{-6}$. We would like to give a geometric viewpoint of this deviation as the difference analysis. Denote $\Delta_i = Z_i(\alpha, \alpha r) - E_i(\lambda, r)$ and see the graph of $\Delta_i$ (Fig.7), distance from the paths to the lines can be described as

$$dis(\lambda, r_0) = \left|\sum\nolimits_{i=0}^{3} \Delta \boldsymbol{q}_i\right| \le \left|\sum\nolimits_{i=0}^{3} |\Delta_i|\right| \cdot \max\{|\boldsymbol{q}_i|\}$$
$$\le 4\sqrt{\sum\nolimits_{i=0}^{3} \Delta_i^2} \cdot \max\{|\boldsymbol{q}_i|\} \le 0.0057 \max\{|\boldsymbol{q}_i|\}.$$

By simple calculation, the sum of functions $\Delta_i$ equals 0 for all $\alpha$, $r$ and $\lambda$. Thus the difference is overestimated if the functions $\Delta_i$ are replaced by their overall maximum. From the geometric invariance property of AH curves (Li and Wang, 2005), we let the first point $\boldsymbol{q}_0$ be the origin of the coordinate system, so the enlarged distance can be described as

$$dis(\lambda, r_0) \le 0.0057 \max\{|\boldsymbol{q}_i - \boldsymbol{q}_0|\}.$$

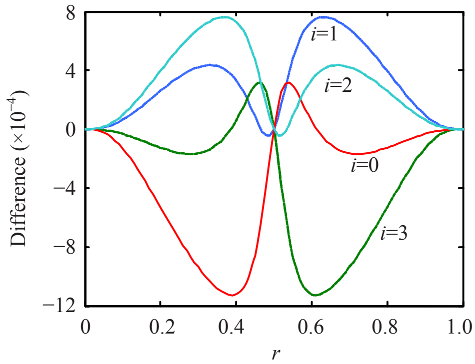


**Fig.7 Graph of four difference functions $\Delta_i$ at $\alpha$=4**

## PASSING THROUGH A POINT

With the running parameters $\alpha$ and $r$, there is no doubt that points on the paths of the AH Bézier curves will fill the shadow region in Fig.8. Then comes the constraint-based modification problem of passing a given point in the constrained region, which is used extensively in practical applications. That is, we give a point $\boldsymbol{Q}$ in the constrained region through which a family curve has to pass, and we should find the value of parameter $\alpha$ for the family curve, which can be implemented by some iteration methods, such as the least squares method (Fig.9). But for the AH spline curve, there is a simpler method to find the passing curve.

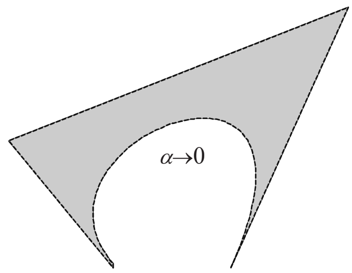


**Fig.8 Constrained region of an AH Bézier curve**

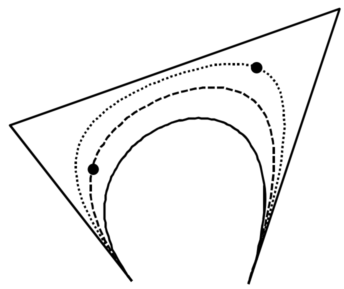

**Fig.9 AH Bézier curves passing through a point**

The same as the paths of an AH Bézier curve, the points on the paths of an AH spline curve will fill a shadow region encircled by a family curve with $\alpha\to0$ and three lines—$\boldsymbol{q}_1\boldsymbol{q}_2$ and the path lines when $r$=0, $r$=1. Given a point in the region, we can find a family curve passing through it. Here we give the algorithm in detail.

Given an AH spline curve $s(\alpha, t)$ with four control points $\boldsymbol{q}_i$ and a given point $\boldsymbol{Q}$, and a given error smaller than 0.0057max{|$\boldsymbol{q}_i$|}, we want to find a curve $s(\alpha_0, t_0)$ with two fixed parameters $\alpha_0$, $t_0$ passing through $\boldsymbol{Q}$, that is, $s(\alpha_0, t_0)=\boldsymbol{Q}$.

Firstly, we find the parameter $r_0$ at which the line $E(\lambda_0, r_0)$ passes through the given point $\boldsymbol{Q}$ with $\lambda_0$ being a certain fixed value. Denote [*first*, *end*]=[0, 1], *middle*=(*first*+*end*)/2=1/2, this step can be solved numerically as follows:

Considering the approximate line when $r$=*middle*. If $\boldsymbol{Q}$ is on the line within the given error, then $r_0$=*middle*. Else if $\boldsymbol{Q}$ is on the same side of the approximate line with the control point $\boldsymbol{q}_0$, then *first* remains unchanged and *end*=*middle*; else *first*=*middle* and *end* remains unchanged. Go to the first step of the algorithm until $\boldsymbol{Q}$ is on the line when $r$=*middle* within the given error, then $r_0$=*middle*.

From Fig.1 we can see that the path will be on the control leg $\boldsymbol{q}_1\boldsymbol{q}_2$ when $\alpha$ is large enough. Thus the value of $\alpha_0$ can be found by a similar method:

Denote [*first*, *end*]=[0, 100], *middle*=(*first*+*end*)/2=50. Consider the point $s(\alpha, r_0)$ when $\alpha$=*middle*. If the distance from point $s(\alpha, r_0)$ to $\boldsymbol{Q}$ is smaller than the given error, then $\alpha_0$=*middle*. Else if $\boldsymbol{Q}$ is on the same side of the point $s(\alpha, r_0)$ with the point $s(0, r_0)$, then *first* remains unchanged and *end*=*middle*; else *first*=*middle* and *end* remains unchanged. Go to the first step of the algorithm until the distance from the point $s(\alpha, r_0)$ to $\boldsymbol{Q}$ is smaller than the given error when $\alpha$=*middle*, then $\alpha_0$=*middle*. So $t_0=\alpha_0 r_0$ can be obtained (Fig.10).

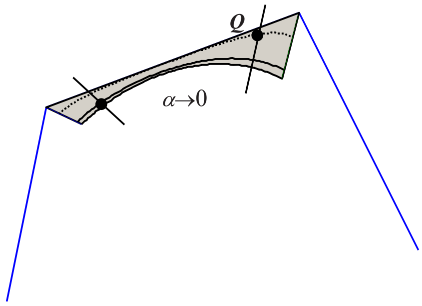


**Fig.10 AH spline curve passing through a point**

## CONCLUSION AND FUTURE WORK

AH Bézier curves and AH spline curves share most optimal properties as those of Bézier curves and B-spline curves, respectively. Modifying the value of the parameter $\alpha$ within the interval (0, +∞), the curve points will move on the defined curves called paths. In this paper, we studied the properties of paths of the AH Bézier curves and AH spline curves and gave a constraint-based modification of passing a given point in the constrained region. The generalization to tensor product surfaces is our future work.

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