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On modal energy in civil structural control*

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Abstract: A new control strategy based on modal energy criterion is proposed to demonstrate the effectiveness of the control system in reducing structural earthquake responses. The modal control algorithm combining LQR (linear quadratic regulator) control algorithm is adopted in the discrete time-history analysis. The various modal energy forms are derived by definition of the generalized absolute displacement vector. A preliminary numerical study of the effectiveness of this control strategy is carried out on a 20-storey framed steel structural model. The controlled performance of the model is studied from the perspectives of both response and modal energy. Results show that the modal energy-based control strategy is very effective in reducing structural responses as well as in consuming a large amount of modal energy, while augmentation of additional generalized control force corresponding to the modes that contain little modal energy is unnecessary, as it does little help to improve the controlled structural performance.

Key words: Modal energy, Aseismic control, LQR (linear quadratic regulator) control algorithm, Modal space, Generalized single-degree-of-freedom (SDOF) system
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INTRODUCTION

To prevent structural damage from earthquake remains challenging for civil engineering practices. A general consensus is that the technique of structural control, in which certain mechanical devices are implemented on structural systems, can be an effective way to improve both structural serviceability and safety (Housner *et al*., 1997). A structural control system in civil engineering applications generally consists of a mechanical actuating device, a sensing system, and a kernel control unit that is normally composed of a computer that deals with numerical signal processing and computation (Soong, 1990). The crucial part of the kernel control unit is the control algorithm that decides how the actuators should react to external disturbance on the controlled structure. The control algorithm decides the effectiveness of the whole control system.

In the modal control which is one of many control algorithms, the motion of a structure is modified by controlling the dominating vibration modes. There are two reasons for this: (1) For practical civil engineering structures, asymptotic stability of structural vibration is normally ensured; (2) Although civil engineering structures are mostly large structural systems which may incorporate a large number of vibration modes, their modal energy forms are mostly concentrated on the first few modes (Ou, 2003). Therefore, suppression of these dominant modal energy forms may significantly ameliorate structural earthquake responses as well as reduce most of the earthquake input energy in the structural system. The application of modal space reduction techniques and the control of the critical modes of vibration of practical civil engineering structural models have been discussed in many published works (e.g.,Yang and Lin, 1982; Adhikari *et al*., 1998; Lu and Chung, 2001; Barbone *et al*., 2003).

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From an energy perspective, the structural vibration procedure under earthquake excitation can be viewed as an energy transferring process. Although the motion of the ground at a specific site is independent of the structure built at this site, the responses of the structure and the amount of energy inputted to the structure highly depend on the structural characteristics such as natural frequency, damping and stiffness distributions. The input energy is then transformed in different forms, such as kinetic energy, damping energy and strain energy. Therefore, it is valuable to delve into the energy aspects of the structural earthquake responses.

Studies on energy mechanism and its application to structural analysis in earthquake engineering have been one of the most attended fields since Uang and Bertero (1990) derived the energy equations and compared the two energy forms, i.e., absolute energy and relative energy for single-degree-of-freedom (SDOF) structural models. Later, Chou and Uang (2003) proposed a procedure to evaluate the structural total energy demand and distribute it along the height of a frame based on energy spectral analysis. The main advantage of this method is that the energy demand of a multi-degree-of-freedom (MDOF) structure can be estimated without performing an inelastic time-history analysis. Surahman (2007) proposed an energy-based earthquake-resistant structural design method that uses input energy spectra, modal or time-history analyses, and energy distribution among structural members. In recent years, more efforts have been made to study the controlled structural performance based on an energy perspective. Tomlinson *et al*.(2006) conceptually proposed a new approach to suppress structural resonant vibration via energy transfer concepts. Wong and Yang (2001a; 2001b) and Wong and Zhao (2005) derived a computational method of energy evaluation to study structural earthquake responses and energy distribution of actively controlled structures, and the control efficiency of the method was demonstrated and evaluated by a numerical study.

Although adequate investigations on controlled structural performance and energy evaluations have been carried out as stated above, few studies have focused on the modal energy perspective in the generalized modal space coordinate systems, and no derivation of modal energy forms has been found.

In this paper, we propose a new control strategy based on modal energy criterion to demonstrate the effectiveness of the control system in reducing structural earthquake responses. The various modal energy forms are derived by definition of the generalized absolute displacement vector. Through numerical analysis of modal energy composition using a 20-storey framed structure, the MDOF structural control problems are transferred into a problem that deals with control of SDOF structural models in the generalized modal space coordinate systems. The controlled structural performance is evaluated by comparing three cases: no control, 2-mode control and 5-mode control. Based on the numerical results obtained, conclusions are drawn to generalize the relevant significance of this modal energy-based control strategy.

MODAL SPACE ANALYSIS

The dynamic equilibrium equation of a controlled structure with n degree-of-freedoms (DOFs) and p control actuators can be written as

$$\boldsymbol{M}\ddot{\boldsymbol{X}}+\boldsymbol{C}\dot{\boldsymbol{X}}+\boldsymbol{K}\boldsymbol{X}=-\boldsymbol{M}\ddot{\boldsymbol{X}}_{\rm g}+\boldsymbol{D}\boldsymbol{U}_{\rm c}(t), \tag{1}$$

where $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ are the $n\times n$ mass, damping and stiffness matrices, respectively; $\dot{\boldsymbol{X}}$ and $\ddot{\boldsymbol{X}}$ are the $n\times 1$ velocity and acceleration vectors, respectively; $\boldsymbol{U}_{\rm c}(t)$ is the $p\times 1$ control force vector; $\boldsymbol{D}$ is the $n\times p$ distribution matrix that relates the effect of each control force with each DOF; and $\ddot{\boldsymbol{X}}_{\rm g}$ is the earthquake ground acceleration vector corresponding to each DOF.

Non-proportional damping

Consider the non-proportional damping case, and define

$$\boldsymbol{Z}(t)=\begin{bmatrix}\boldsymbol{X}(t)\\ \dot{\boldsymbol{X}}(t)\end{bmatrix}, \tag{2}$$

where $\boldsymbol{Z}(t)$ is the $2n\times 2n$ state vector of the MDOF system. Then Eq.(1) becomes

$$\dot{\boldsymbol{Z}}(t)=\boldsymbol{A}\boldsymbol{Z}(t)+\boldsymbol{B}\boldsymbol{U}(t)+\boldsymbol{F}_{\ddot{X}_{\rm g}}, \tag{3}$$

where

$$\boldsymbol{A}=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{I}\\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C}\end{bmatrix},\quad \boldsymbol{B}=\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{M}^{-1}\boldsymbol{D}\end{bmatrix},$$

$$\boldsymbol{F}_{\ddot{X}_g}=\begin{bmatrix}\boldsymbol{0}\\ -\ddot{\boldsymbol{X}}_g\end{bmatrix}.$$

Following the usual method of modal analysis, the response of the structure in the state space notations $\boldsymbol{Z}$ can be represented in the modal space coordinate systems as

$$\boldsymbol{Z}(t)=\boldsymbol{\Psi}\boldsymbol{p}(t), \tag{4}$$

where $\boldsymbol{\Psi}$ is the $2n\times2n$ eigenvector matrix of $\boldsymbol{A}$, and $\boldsymbol{p}(t)$ is the $2n\times1$ modal co-ordinates. It should be noted that both $\boldsymbol{\Psi}$ and $\boldsymbol{p}(t)$ appear as complex conjugate pairs. Substituting Eq.(4) into the state Eq.(3), pre-multiplying the modal matrix $\boldsymbol{\Psi}^{\mathrm{T}}$ and making use of the following biorthonormal properties of the modal matrix

$$\boldsymbol{\Psi}\boldsymbol{\Psi}^{\mathrm{T}}=\boldsymbol{I},\ \boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{\Psi}=\boldsymbol{\Lambda}, \tag{5}$$

Eq.(3) becomes

$$\dot{\boldsymbol{p}}(t)=\boldsymbol{\Lambda}\boldsymbol{p}(t)+\boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{U}(t)+\boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{F}_{\ddot{X}_g}, \tag{6}$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix consisting of the complex eigenvalue ω_k of the matrix $\boldsymbol{A}$, and $\boldsymbol{I}$ is an identity matrix.

By this way, the original $2n$-dimensional problem becomes n 2-dimensional control problems.

Proportional damping

Similarly, for the case of proportional damping (Rayleigh damping), let

$$\boldsymbol{X}(t)=\boldsymbol{\Phi}\boldsymbol{q}(t), \tag{7}$$

where $\boldsymbol{\Phi}$ is the $n\times n$ matrix of natural vibration mode shapes and $\boldsymbol{q}(t)$ indicates the modal co-ordinates. Then the MDOF model control can be transformed into such a form as

$$\boldsymbol{M}^*\ddot{\boldsymbol{q}}(t)+\boldsymbol{C}^*\dot{\boldsymbol{q}}(t)+\boldsymbol{K}^*\boldsymbol{q}(t)=\boldsymbol{F}^*_{\ddot{X}_g}+\boldsymbol{U}^*(t), \tag{8}$$

where $\boldsymbol{M}^*$, $\boldsymbol{C}^*$ and $\boldsymbol{K}^*$ are the $n\times n$ generalized mass, damping and stiffness matrices, respectively; and

$$\boldsymbol{F}^*_{\ddot{X}_g}=-\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{M}\ddot{\boldsymbol{X}}_g, \tag{9}$$

$$\boldsymbol{U}^*(t)=\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{U}_c(t), \tag{10}$$

which are the generalized earthquake excitation and control force vectors, respectively.

The original n-DOF structural model becomes n independent SDOF models in the generalized modal space coordinate system. For the ith generalized SDOF system, the corresponding m_i^*, c_i^* and k_i^* represent the generalized mass, damping and stiffness, respectively. Let

$$\boldsymbol{z}_i^*(t)=\begin{bmatrix}q_i\\ \dot{q}_i\end{bmatrix}, \tag{11}$$

where $\boldsymbol{z}_i^*(t)$ is the 2×1 generalized state vector of the ith generalized modal space SDOF system. Then the corresponding dynamic equilibrium equation becomes

$$\dot{\boldsymbol{z}}_i^*(t)=\boldsymbol{A}_i\boldsymbol{z}_i^*(t)+\boldsymbol{B}_iu_i^*(t)+\boldsymbol{H}_i\boldsymbol{F}^*_{i,\ddot{X}_g}, \tag{12}$$

where

$$\boldsymbol{A}_i=\begin{bmatrix}0 & 1\\ -\dfrac{k_i^*}{m_i^*} & -\dfrac{c_i^*}{m_i^*}\end{bmatrix},\quad \boldsymbol{B}_i=\begin{bmatrix}0\\ -1\end{bmatrix}, \tag{13}$$

and $u_i^*(t)$ is the ith generalized control force. Eq.(12) is the governing first order differential equation for control of the ith generalized SDOF system. The solution to this equation is

$$\boldsymbol{z}_i^*(t)=\mathrm{e}^{\boldsymbol{A}_i(t-t_0)}\boldsymbol{z}_i^*(t_0)+\mathrm{e}^{\boldsymbol{A}_it}\int_{t_0}^{t}\mathrm{e}^{-\boldsymbol{A}_is}\left[\boldsymbol{B}_iu_i^*(s)+\boldsymbol{H}_iF^*_{i,\ddot{X}_g}(s)\right]\mathrm{d}s. \tag{14}$$

LQR CONTROL OF GENERALIZED SDOF SYSTEM

The generalized control force in Eq.(14) is usually calculated based on the optimization procedure similar to that in a physical coordinate system. A common approach for the determination of a suitable control law is based on the solution of the linear quadratic regulator (LQR) problem. In this approach, for any of the generalized SDOF systems, the quadratic performance function J_i is defined in the form of

$$J_i=\frac{1}{2}\int_{t_0}^{\infty}\{\boldsymbol{z}_i^{*\mathrm{T}}(t)\boldsymbol{Q}_i\boldsymbol{z}_i^*(t)+r_i[u_i^*(t)]^2\}\mathrm{d}t, \tag{15}$$

where $\boldsymbol{Q}_i$ is a 2×2 positive semi-definite weighting matrix associated with the components of the state vector $\boldsymbol{z}_i^*$, and r_i is a positive-definite weighting parameter associated with the ith generalized control force u_i^*. The selection of the magnitude of weighting matrix $\boldsymbol{Q}_i$ and r_i reflects the trade-off between the desired performance and the applied control actions, which provides the designer with significant flexibility. The control law is obtained by minimizing J_i, subject to Eq.(12).

To minimize the cost function J_i, let the Hamiltonian function be the form of

$$\begin{aligned}H_i=\int_{t_0}^{\infty}\Big[&(1/2)[\boldsymbol{z}_i^{*\mathrm{T}}\boldsymbol{Q}_i\boldsymbol{z}_i^*+r_i(u_i^*)^2]\\&+\lambda_i(\boldsymbol{A}_i\boldsymbol{z}_i^*+\boldsymbol{B}_iu_i^*+\boldsymbol{H}_iF_{i,\ddot{x}_\mathrm{g}}^*-\dot{\boldsymbol{z}}_i^*)\Big]\mathrm{d}t,\end{aligned} \tag{16}$$

where λ_i is the Lagrange multiplier for the ith generalized SDOF system. Minimizing the cost function J_i in Eq.(15) can be performed by differentiating the Hamiltonian function in Eq.(16) with respect to λ_i, $\boldsymbol{z}_i^*$ and u_i^*, i.e.,

$$\left(\frac{\partial H_i}{\partial \lambda_i}\right)^{\mathrm{T}}=0, \tag{17}$$

$$\left(\frac{\partial H_i}{\partial \boldsymbol{z}_i^*}\right)^{\mathrm{T}}=0, \tag{18}$$

$$\left(\frac{\partial H_i}{\partial u_i^*}\right)^{\mathrm{T}}=0. \tag{19}$$

The optimal solution is given by

$$u_i^*(t)=-\boldsymbol{K}_{\mathrm{c},i}(t)\boldsymbol{z}_i^*(t)+u_{\ddot{x}_\mathrm{g},i}^*(t). \tag{20}$$

The first term of the solution Eq.(20) represents the feedback term, which depends on the system states through the time-varying 1×2 matrix $\boldsymbol{K}_{\mathrm{c},i}(t)$, given by

$$\boldsymbol{K}_{\mathrm{c},i}(t)=\frac{1}{r_i}\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}_{\mathrm{c},i}(t), \tag{21}$$

where the 2×2 matrix $\boldsymbol{P}_{\mathrm{c},i}$ is the solution of the corresponding matrix differential Riccati equation that must be solved subject to the following terminal boundary condition,

$$\boldsymbol{P}_{\mathrm{c},i}(\infty)=0. \tag{22}$$

The second term in the control law Eq.(20) represents a component of the total generalized control force, which is dependent on 'future' values of the ground motion. However, in practical earthquake engineering applications, it is impossible to know the time-history of the disturbance acting on the structural system. Therefore, the control law in Eq.(20) is usually implemented using only the first (feedback) term.

For sufficiently large time t, the time-varying matrix $\boldsymbol{P}_{\mathrm{c},i}(t)$ is usually replaced by its steady-state value $\overline{\boldsymbol{P}}_{\mathrm{c},i}$, obtained as the solution of the corresponding algebraic Riccati matrix equation

$$-\overline{\boldsymbol{P}}_{\mathrm{c},i}\boldsymbol{A}_i-\boldsymbol{A}_i^{\mathrm{T}}\overline{\boldsymbol{P}}_{\mathrm{c},i}+\frac{1}{r_i}\overline{\boldsymbol{P}}_{\mathrm{c},i}\boldsymbol{B}_i\boldsymbol{B}_i^{\mathrm{T}}\overline{\boldsymbol{P}}_{\mathrm{c},i}-\boldsymbol{Q}_i=0. \tag{23}$$

The control law is then defined as a constant linear combination of the system states (constant linear feedback) of the form

$$u_i^*(t)=-\overline{\boldsymbol{K}}_{\mathrm{c},i}\boldsymbol{z}_i^*(t), \tag{24}$$

where the feed-forward term $u_{\ddot{x}_\mathrm{g},i}^*$ has been omitted and $\overline{\boldsymbol{K}}_{\mathrm{c},i}$ is given by

$$\overline{\boldsymbol{K}}_{\mathrm{c},i}(t)=\frac{1}{r_i}\boldsymbol{B}_i^{\mathrm{T}}\overline{\boldsymbol{P}}_{\mathrm{c},i}. \tag{25}$$

As is well known, this form of the control law is not truly optimal unless the original problem is recast in a stochastic framework, and the external excitation is defined as a white noise process. For a given structural system the control actions obtained by this simple approach do not depend on the characteristics of the external excitation and they are only governed by the selection of the matrix $\boldsymbol{Q}_i$ and r_i.

It is further noted that the ith generalized control force $u_i^*(t)$ in Eq.(24) linearly depends on the generalized state vector $\boldsymbol{z}_i^*(t)$, which consists of the generalized displacement and velocity responses. Therefore, Eq.(19) can be simplified as

$$u_i^*(t) = -c_1 q_i(t) - c_2 \dot{q}_i(t), \tag{26}$$

where c_1 and c_2 are constant feedback control parameters computed by performing matrix multiplication as given in Eq.(25). From a practical point of view, these two constants will generally be positive, since the generalized control force is generally applied in the direction opposite to the generalized displacement and velocity in order for the control system to be effective in controlling structural responses.

Once all of the generalized control forces $u_i^*(t)$ are obtained, the actual control force vector in the physical coordinate system can be computed as

$$\boldsymbol{U}(t) = (\boldsymbol{\Phi}^{\mathrm{T}})^{-1}\boldsymbol{U}^*(t). \tag{27}$$

TIME-DISCRETE SYSTEM ANALYSIS

Let $t_{k+1}=t$, $t_k=t_0$ and $\Delta t=t-t_0$, and represent the generalized earthquake excitation vector as well as the generalized control force vector at each time domain using Delta forcing function within the small time step Δt. Performing the integration in Eq.(14) gives

$$\boldsymbol{z}_{i,k+1}^* = \boldsymbol{F}_{\mathrm{s},i}\boldsymbol{z}_{i,k}^* + \boldsymbol{H}_{\mathrm{d},i}u_i^*(k) + \boldsymbol{G}_i F_{i,\ddot{X}_{\mathrm{g}}}^*(k), \tag{28}$$

where

$$\boldsymbol{F}_{\mathrm{s},i} = \mathrm{e}^{\boldsymbol{A}_i\Delta t},\ \boldsymbol{H}_{\mathrm{d},i} = \mathrm{e}^{\boldsymbol{A}_i\Delta t}\boldsymbol{B}_i\Delta t,\ \boldsymbol{G} = \mathrm{e}^{\boldsymbol{A}_i\Delta t}\boldsymbol{H}_i\Delta t, \tag{29}$$

and $F_{i,\ddot{X}_{\mathrm{g}}}^*(k)$ and $u_i^*(k)$ are the discretized forms of $F_{i,\ddot{X}_{\mathrm{g}}}^*(t)$ and $u_i^*(t)$, respectively.

GENERALIZED MODAL ENERGY FORMS

Once the control force vector is obtained, the various modal energy forms in the time-history can be calculated. The modal energy equation can be derived based on Eq.(8).

Similar to Eq.(7), define

$$\boldsymbol{q}_{\mathrm{g}}(t) = \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{X}_{\mathrm{g}}(t), \tag{30}$$

$$\boldsymbol{q}_{\mathrm{a}} = \boldsymbol{q} + \boldsymbol{q}_{\mathrm{g}}, \tag{31}$$

where $\boldsymbol{q}_{\mathrm{g}}(t)$ represents the generalized ground displacement vector caused by earthquake excitation, and $\boldsymbol{q}_{\mathrm{a}}$ represents the generalized absolute displacement vector. Then

$$\ddot{\boldsymbol{q}}_{\mathrm{g}} = \boldsymbol{\Phi}^{\mathrm{T}}\ddot{\boldsymbol{X}}_{\mathrm{g}}(t), \tag{32}$$

$$\ddot{\boldsymbol{q}}_{\mathrm{a}} = \ddot{\boldsymbol{q}} + \ddot{\boldsymbol{q}}_{\mathrm{g}}, \tag{33}$$

and then it follows from Eq.(8) that

$$\boldsymbol{M}^*\ddot{\boldsymbol{q}}_{\mathrm{a}}(t) + \boldsymbol{C}^*\dot{\boldsymbol{q}}(t) + \boldsymbol{K}^*\boldsymbol{q}(t) = \boldsymbol{U}^*(t). \tag{34}$$

Integrating both sides of Eq.(34) over the path of structural response gives

$$\int_0^{t_k}\ddot{\boldsymbol{q}}_{\mathrm{a}}^{\mathrm{T}}\boldsymbol{M}^*\mathrm{d}\boldsymbol{q} + \int_0^{t_k}\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{C}^*\mathrm{d}\boldsymbol{q} + \int_0^{t_k}\boldsymbol{q}^{\mathrm{T}}\boldsymbol{K}^*\mathrm{d}\boldsymbol{q} = \int_0^{t_k}\boldsymbol{U}^{*\mathrm{T}}\mathrm{d}\boldsymbol{q}. \tag{35}$$

Substituting $\mathrm{d}\boldsymbol{q}=\mathrm{d}\boldsymbol{q}_{\mathrm{a}}-\mathrm{d}\boldsymbol{q}_g$ into then Eq.(35) yields

$$\begin{aligned}\int_0^{t_k}\ddot{\boldsymbol{q}}_{\mathrm{a}}^{\mathrm{T}}\boldsymbol{M}^*\mathrm{d}\boldsymbol{q}_{\mathrm{a}} + \int_0^{t_k}\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{C}^*\mathrm{d}\boldsymbol{q} + \int_0^{t_k}\boldsymbol{q}^{\mathrm{T}}\boldsymbol{K}^*\mathrm{d}\boldsymbol{q} - \int_0^{t_k}\boldsymbol{U}^{*\mathrm{T}}\mathrm{d}\boldsymbol{q} \\ = \int_0^{t_k}\ddot{\boldsymbol{q}}_{\mathrm{a}}^{\mathrm{T}}\boldsymbol{M}^*\mathrm{d}\boldsymbol{q}_{\mathrm{g}}.\end{aligned} \tag{36}$$

The first three terms on the left-hand side of Eq.(36) are

$$\int_0^{t_k}\ddot{\boldsymbol{q}}_{\mathrm{a}}^{\mathrm{T}}\boldsymbol{M}^*\mathrm{d}\boldsymbol{q}_{\mathrm{a}} = \frac{1}{2}\dot{\boldsymbol{q}}_{\mathrm{a}}^{\mathrm{T}}(t_k)\boldsymbol{M}^*\dot{\boldsymbol{q}}_{\mathrm{a}}(t_k) - \frac{1}{2}\dot{\boldsymbol{q}}_{\mathrm{a}}^{\mathrm{T}}(0)\boldsymbol{M}^*\dot{\boldsymbol{q}}_{\mathrm{a}}(0), \tag{37}$$

$$\int_0^{t_k} \dot{\boldsymbol{q}}^{\mathrm{T}} \boldsymbol{C}^* \mathrm{d}\boldsymbol{q} = \int_0^{t_k} \dot{\boldsymbol{q}}^{\mathrm{T}} \boldsymbol{C}^* \dot{\boldsymbol{q}} \mathrm{d}t, \tag{38}$$

$$\int_0^{t_k} \boldsymbol{q}^{\mathrm{T}} \boldsymbol{K}^* \mathrm{d}\boldsymbol{q} = \frac{1}{2}\boldsymbol{q}^{\mathrm{T}}(t_k)\boldsymbol{K}^*\boldsymbol{q}(t_k) - \frac{1}{2}\boldsymbol{q}^{\mathrm{T}}(0)\boldsymbol{K}^*\boldsymbol{q}(0), \tag{39}$$

where $\dot{\boldsymbol{q}}_{\mathrm{a}}$ is the generalized absolute velocity vector. The structure is assumed rest at the instant when the earthquake ground motion begins, and therefore both $\dot{\boldsymbol{q}}_{\mathrm{a}}(0)$ and $\boldsymbol{q}(0)$ are zero. Substituting Eqs.(37)~(39) into Eq.(36) gives

$$\begin{aligned} &\frac{1}{2}\dot{\boldsymbol{q}}_{\mathrm{a}}^{\mathrm{T}}(t_k)\boldsymbol{M}^*\dot{\boldsymbol{q}}_{\mathrm{a}}(t_k) + \int_0^{t_k} \dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{C}^*\dot{\boldsymbol{q}}\mathrm{d}t \\ &+ \frac{1}{2}\boldsymbol{q}^{\mathrm{T}}(t_k)\boldsymbol{K}^*\boldsymbol{q}(t_k) - \int_0^{t_k} \boldsymbol{U}^{*\mathrm{T}}\mathrm{d}\boldsymbol{q} = \int_0^{t_k} \ddot{\boldsymbol{q}}_{\mathrm{a}}^{\mathrm{T}}\boldsymbol{M}^*\mathrm{d}\boldsymbol{q}_{\mathrm{g}}. \end{aligned} \tag{40}$$

This equation shows the different forms of modal energy in the structure that is formulated in generalized forms. These generalized energy forms include:

(1) Kinetic energy (KE): The first term on the left side of Eq.(40) represents the kinetic energy. Since the generalized mass matrix is a positive-definite matrix, the kinetic energy will always be positive. It can also be formulated as

$$KE = \frac{1}{2}\sum_{i=1}^{n} m_i^* \dot{q}_{\mathrm{a},i}^2 = \frac{1}{2}\dot{\boldsymbol{Y}}^{\mathrm{T}}\boldsymbol{M}\dot{\boldsymbol{Y}}, \tag{41}$$

where $\dot{\boldsymbol{Y}}$ represents the absolute velocity corresponding to the physical coordinate system, and $\dot{\boldsymbol{Y}} = \dot{\boldsymbol{X}} + \dot{\boldsymbol{X}}_{\mathrm{g}}$.

(2) Damping energy (DE): The second term on the left side of Eq.(40) represents the total damping energy dissipated through the time duration. Since the damping matrix is a positive-definite matrix, the integrand is always a positive value. Therefore, more energy will be accumulated and dissipated as time progresses. It can also be formulated as

$$DE = \sum_{i=1}^{n} c_i^* \int_0^{t_k} \dot{q}_i^2 \mathrm{d}t = \int_0^{t_k} \dot{\boldsymbol{X}}^{\mathrm{T}}\boldsymbol{C}\dot{\boldsymbol{X}}\mathrm{d}t. \tag{42}$$

(3) Strain energy (SE): The third term on the left side of Eq.(40) represents the strain energy, which is the total amount of elastic energy stored in the structure. Since the stiffness matrix is a positive definite matrix, the strain energy will always be positive. Similarly, it can also be formulated as

$$SE = \frac{1}{2}\sum_{i=1}^{n} k_i^* q_i^2 = \frac{1}{2}\boldsymbol{X}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{X}. \tag{43}$$

(4) Control energy (CE): The fourth term on the left side of Eq.(40) represents the control energy, which is the accumulative energy due to application of the control forces. Although this term has a negative sign in front of it, the control energy is actually positive because the control force negatively depends on the displacement and velocity responses as given in Eq.(26). To visualize this, substituting Eq.(26) into this fourth term gives

$$\begin{aligned} CE &= -\int_0^{t_k} \boldsymbol{U}^{*\mathrm{T}}\mathrm{d}\boldsymbol{q} = \int_0^{t_k} (\boldsymbol{C}_1\boldsymbol{q} + \boldsymbol{C}_2\dot{\boldsymbol{q}})^{\mathrm{T}}\mathrm{d}\boldsymbol{q} \\ &= \frac{1}{2}\boldsymbol{q}^{\mathrm{T}}\boldsymbol{C}_1^{\mathrm{T}}\boldsymbol{q} + \int_0^{t_k} \dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{C}_2^{\mathrm{T}}\dot{\boldsymbol{q}}\mathrm{d}t. \end{aligned} \tag{44}$$

Generally, matrices $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ contain positive terms, and thus pre-multiplied and post-multiplied by the generalized displacement and velocity vectors give positive values most of the time. Since both the first term and the integrand of the second term on the right side of Eq.(44) are positive, the control energy will generally be positive. Note that the second term on the right side of Eq.(44) is very similar to the damping energy, which is accumulated and dissipated throughout the entire duration of the earthquake. The control energy can also be formulated as

$$CE = -\sum_{i=1}^{n} \int_0^{t_k} u_i^* \mathrm{d}q_i = -\int_0^{t_k} \boldsymbol{F}_{\mathrm{c}}^{\mathrm{T}}\boldsymbol{D}^{\mathrm{T}}\mathrm{d}\boldsymbol{X}. \tag{45}$$

(5) Input energy (IE): The term on the right side of Eq.(40) represents the input energy, and it is always positive because it is equal to the sum of kinetic, damping, strain and control energies. However, input energy does not always increase with time, because, as presented in Eqs.(30), (32) and (40), the changes in ground displacement may be in the direction opposite to the absolute acceleration at certain time steps. When this happens, the earthquake acts as a reactive force to help the structure in stabilizing itself. The input energy can also be formulated as

$$IE=\sum_{i=1}^{n} m_i^* \int_0^{t_k} \ddot{q}_{a,i} \mathrm{d}q_{g,i} = \int_0^{t_k} \ddot{\boldsymbol{Y}}^{\mathrm{T}} \boldsymbol{M} \mathrm{d}\boldsymbol{X}_g. \quad (46)$$

In summary, Eq.(40) can also be written as

$$KE + DE + SE + CE = IE. \quad (47)$$

NUMERICAL ANALYSIS

A practical 20-storey framed structural model is adopted as an example to demonstrate the control efficiency of the proposed control strategy. The model is an assembly of Grade Q345 box-shaped steel columns and H-shaped steel beams (Table 1). Fig.1 shows the elevation view of the model with controllers installed schematically. The model has a uniform storey height of 3.2 m, so the elevation of the roof is 64 m. The dead load on each floor is 3.2 kN/m^2, and the total mass of each floor is about 66.5 t. The first five natural frequencies of the structure are 3.293 s, 9.691 s, 16.094 s, 22.426 s and 28.600 s, respectively. The Rayleigh damping is used, i.e., $\boldsymbol{C}=\alpha\boldsymbol{M}+\beta\boldsymbol{K}$, where α=0.2456 and β=0.0077.

Time-history analysis of the original structural model without applying control systems is first carried out using KOBE-EW (KOBE-East-Westward) earthquake. Based on the time-history responses and energy forms obtained, the root mean square (RMS) values of the displacements of each storey and modal energies corresponding to each mode shape are generated (Figs.2~3). As shown in Fig.2, the RMS value of the displacements of each storey increases as the height of the stories goes up, due to the drift superposition effect of the lower storeys. From Fig.3, it can be seen that the first modal energy contributes the most RMS value to the total RMS of all modal energies. Numerical results based on time-history analysis show that the maximal value of the first modal energy reaches 226.9 kJ, and the second modal energy reaches 68.3 kJ, while the summation of the other modal energies corresponding to the rest 18 modes is 4.9 kJ. Therefore, almost all the modal energy of this structural model is concentrated on the first two modes.

Following this result, the structural model is further analyzed using an ensemble of 9 more earth-

Table 1 Dimensions of structural members

Member	Storey	Dimension (mm×mm×mm)
Side columns	1~6	Box 450×450×32
	7~12	Box 450×450×28
	13~20	Box 450×450×24
Middle columns	1~7	Box 450×450×36
	8~14	Box 450×450×32
	15~20	Box 450×450×28
Beams	1~10	H600×250×25×12
	11~20	H600×250×20×12
Bracings	1~10	H220×220×16×9.5
	11~20	H240×240×17×10

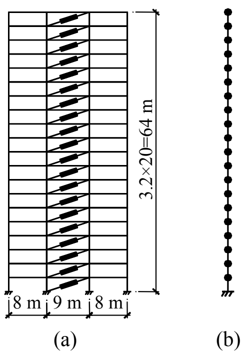

Fig.1 20-storey building (a) and its lumped mass model (b)

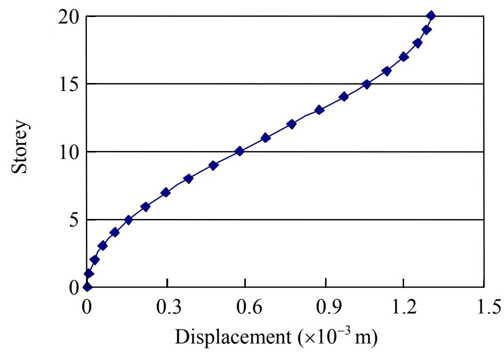

Fig.2 RMS of displacements of each storey of the 20-storey frame subjected to KOBE-EW earthquake

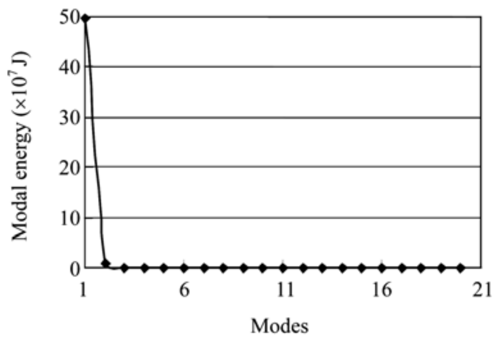

Fig.3 RMS of modal energies corresponding to each mode of the 20-storey frame subjected to KOBE-EW earthquake

quake time-histories. Similar results are obtained. The modal energy in the first two mode shapes comprises more than 99% of the total modal energies subjected to all the earthquake excitations. Therefore, it is possible to minimize the structural response as well as the various energy forms in the structure by minimizing the modal energy corresponding to the first two mode shapes.

Now assume the MDOF structural model be decomposed into 20 generalized SDOF systems as formulated in Eq.(8), and only the first two generalized SDOF models be considered, with one generalized control actuator installed in each SDOF system. Using the LQR control algorithm, and let the weighting matrices be of the form

$$\boldsymbol{Q}_1=\frac{1}{m_1^*}\begin{bmatrix}k_1^* & 0\\ 0 & m_1^*\end{bmatrix},\quad r_1=5.0\times10^{-2}, \tag{48}$$

for control of the first generalized SDOF system, and

$$\boldsymbol{Q}_2=\frac{1}{m_2^*}\begin{bmatrix}k_2^* & 0\\ 0 & m_2^*\end{bmatrix},\quad r_2=2.0\times10^{-2}, \tag{49}$$

for control of the second generalized SDOF system, respectively. The reason for selecting these matrices is to represent the cost function in LQR control as some forms of energy, where the term involving the $\boldsymbol{Q}$ matrix represents the modal energy forms of the structure and the term involving the $\boldsymbol{R}$ matrix represents the energy in the control system. The control gains for the two generalized SDOF systems are $\bar{\boldsymbol{K}}_{c,1}=[-1.148\ \ -3.851]$ and $\bar{\boldsymbol{K}}_{c,2}=[-22.342\ \ -8.810]$, respectively. Constant linear state feedback as formulated in Eq.(24) is applied.

The corresponding control force vector at time step k in the physical coordinate system can be

$$\boldsymbol{U}(k)=(\boldsymbol{\Phi}^{\mathrm{T}})^{-1}[u_1^*(k)\quad u_2^*(k)\quad 0\quad \cdots\quad 0]^{\mathrm{T}}. \tag{50}$$

The structure is subjected to the magnified El Centro earthquake, which is obtained by multiplying with scale factor of 1.5. Fig.4 shows comparisons of the modal energy time-histories corresponding to the first two generalized SDOF systems for the cases with and without control. The control force time-histories on the 9th and the 10th floors of the frame structure are shown as typical results in Fig.5.

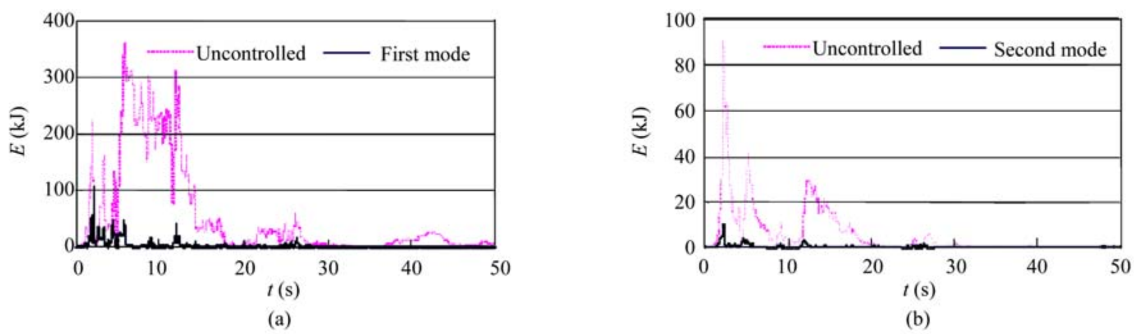

Fig.4 Comparison of the first two modal energies of the generalized SDOF systems with and without control. (a) First mode; (b) Second mode

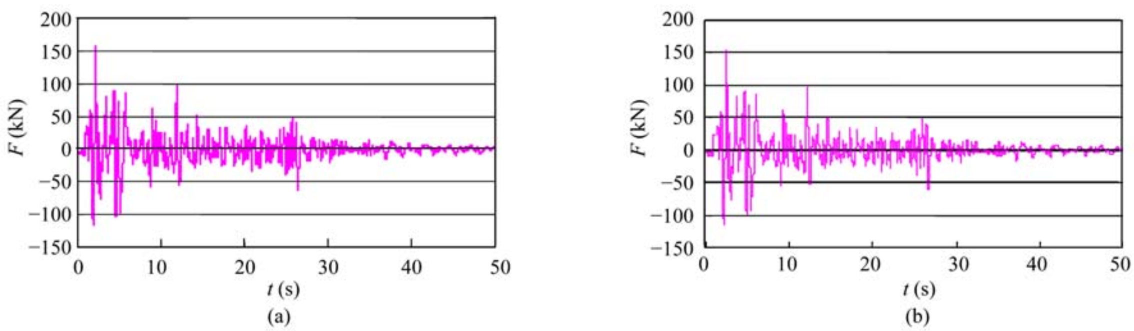

Fig.5 Control force time histories of the 9th (a) and 10th (b) storeys of the framed structure

As seen from these results, by applying control to the two generalized SDOF systems, the modal energy forms of both systems can be greatly reduced, and the MDOF system structural displacement and velocity can also be significantly reduced. Since both the total displacements and velocities at each floor decrease when generalized modal space control forces are applied to both generalized SDOF systems, less structural damage would be caused during severe earthquakes, which is the expected result in actual structural control applications.

Fig.6 shows a comparison of the roof time-history responses that include displacement, velocity and acceleration responses for the cases of two-mode control, five-mode control and no control, respectively. These figures show that the controlled structural responses can be greatly reduced if appropriate control parameters are selected. However, the roof responses are approximately the same for the two-mode control scheme and the five-mode one. Little improvement of the controlled structural responses using five-mode control is observed even though similar control forces are employed. This shows that augmentation of additional generalized control force corresponding to the modes that contain little modal energy does little help to improve the controlled structural performance.

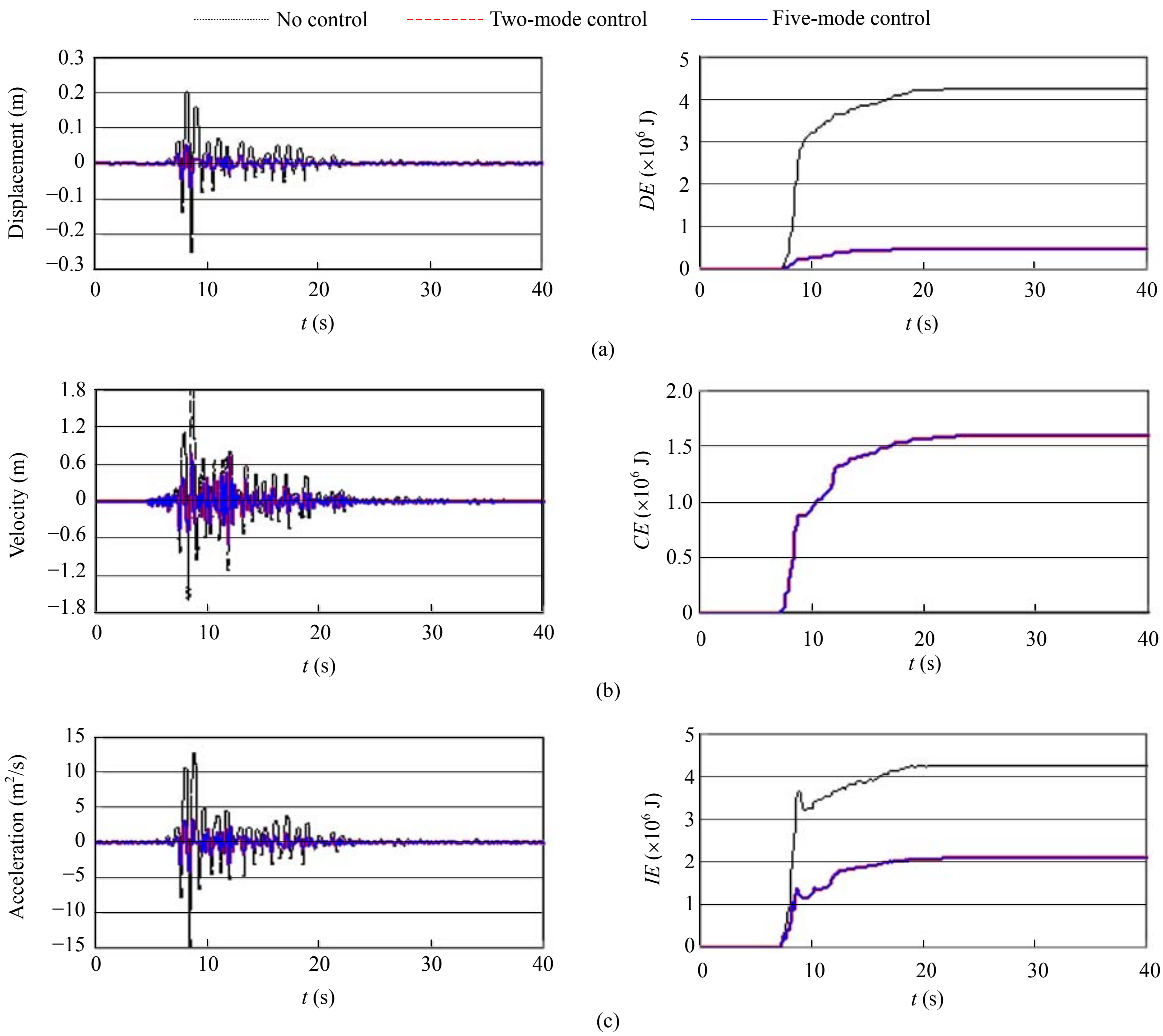

Fig.6 Comparison of the roof time-history responses that include displacement (a), velocity (b) and acceleration (c) responses for the cases of two-mode control, five-mode control and no control

CONCLUSION

This paper introduces a new control strategy based on modal energy criterion to minimize the controlled structural responses using the modal control algorithm. The main advantage of this approach is an MDOF structural control problem can be decomposed into a simple one that deals with control of SDOF structural models in the generalized modal space coordinate systems. Moreover, by skipping the generalized SDOF systems that contain only a small amount of modal energy, computational efforts can be greatly reduced. The various modal energy forms are derived by definition of the generalized absolute displacement vector. Through numerical study on a 20-storey frame structural model, the following conclusions are drawn:

The control gain in each modal space SDOF control can be formulated as the feedback of the states in the modal space based on LQR control algorithm. By comparing the numerical results, it is shown that the proposed control strategy based on modal energy is very effective in reducing the structural responses as well as in consuming a large amount of modal energy.

Finally, the numerical results of the control strategy using two-mode control are compared with those of the five-mode control scheme. It is found that while three more generalized control forces in the modal space are augmented, little improvement is observed in structural responses using five-mode control. For such tiny benefit to improving the controlled structural performance, it is unnecessary to add these additional generalized control forces corresponding to the modes that contain little modal energy.

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