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Optimal constrained multi-degree reduction of Bézier curves with explicit expressions based on divide and conquer*

Lian ZHOU[1,2], Guo-jin WANG[‡1,2]

(*[1]Institute of Computer Graphics and Image Processing; [2]State Key Lab of CAD & CG, Zhejiang University, Hangzhou 310027, China*)

E-mail: zhoulia5729@163.com; wanggj@zju.edu.cn

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Abstract: We decompose the problem of the optimal multi-degree reduction of Bézier curves with corners constraint into two simpler subproblems, namely making high order interpolations at the two endpoints without degree reduction, and doing optimal degree reduction without making high order interpolations at the two endpoints. Further, we convert the second subproblem into multi-degree reduction of Jacobi polynomials. Then, we can easily derive the optimal solution using orthonormality of Jacobi polynomials and the least square method of unequally accurate measurement. This method of 'divide and conquer' has several advantages including maintaining high continuity at the two endpoints of the curve, doing multi-degree reduction only once, using explicit approximation expressions, estimating error in advance, low time cost, and high precision. More importantly, it is not only deduced simply and directly, but also can be easily extended to the degree reduction of surfaces. Finally, we present two examples to demonstrate the effectiveness of our algorithm.

Key words: Bézier curves, Multi-degree reduction, Divide and conquer
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INTRODUCTION

Bézier curves are one of the main modeling tools in computer aided design/computer aided manufacture (CAD/CAM) systems. In many of these systems, including data exchange, data transfer, and data compression, degree reduction of Bézier curves is of great importance. Degree reduction is used to find a lower degree curve to approximate a parameter curve with a given degree, while keeping the error within a given tolerance. This is a topical nonlinear geometric approximating problem.

In the last twenty years, researchers have developed many methods for degree reduction of Bézier curves. These methods fall into two main groups. One includes discretization methods based on geometric information such as the control points of the original curve and its derivative vectors. Based on interpolation, convex linear combination, least squares, and Lagrange multiplier, these methods can achieve degree reduction using the inverse process of degree elevation (Forrest, 1972; Farin, 1983), approximate transformation (Dannenberg and Nowacki, 1985; Hoschek, 1987; Ahn *et al*., 2004), constrained optimization (Moore and Warren, 1991; Lodha and Warren, 1994), and perturbation of control points (Zheng and Wang, 2003). The other group includes algebraic methods based on basis conversion (Watson, 1980; Lachance, 1988; Eck, 1993; 1995; Chen and Wang, 2002; Zhang and Wang, 2005; Rababah *et al*., 2006; 2007). These methods focus on the approximating property of the original curve in different norm space such as L_1, L_2, and L_∞. However, these two groups of methods have some limitations. Following in-depth studies, researchers now generally agree that an ideal degree reduction algorithm should contain all the following seven features: (1) multi-degree reduction performed only once; (2) high continuity maintained

‡ Corresponding author
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at the two endpoints of the curves; (3) explicit approximation expression; (4) high precision; (5) low time cost; (6) error estimated in advance; (7) capability to be extended to degree reduction of surfaces. Here (1)~(4) are related to accuracy, and (5) and (6) are related to efficiency. In particular, (6) can prompt users to perform degree reduction of sub-curves when the error of degree reduction of the original curve is larger than the given tolerance, so as to avoid invalid results. However, existing methods do not have all the above characteristics. In particular, (7) is more important and harder to achieve for researchers because the interpolation conditions of Bézier surfaces are more complex than those of Bézier curves.

The research summarized in this study was aimed at overcoming these limitations. Our basic approach was as follows. First, we divided the problem of the optimal degree reduction of a curve with high order interpolations at its two endpoints into two simpler problems, namely making high order interpolations at its two endpoints without performing degree reduction, and performing optimal degree reduction without making high order interpolations at its two endpoints. Second, we transformed the latter into Jacobi polynomial space, and then solved the problem using the orthonormality of Jacobi polynomials and the least square method of unequally accurate measurement. Our method greatly simplifies and clarifies the problem. It also has the potential to be extended to the degree reduction of Bézier surfaces because the control points of the surface arrayed in a 2D lattice form can be realigned into a 1D lattice form. During this realignment there is consistency in the mathematics and in the representation of symbols between the interpolation conditions of surface corners and boundaries, and those of curve endpoints. Thus, our algorithm can be expediently extended to the multi-degree reduction of surfaces. In this paper, we deduce the fundamental principles and use examples to evaluate our methods. The results are of significance to both researchers and the CAD/CAM industry. The generalization to surfaces will be discussed in another paper.

DESCRIPTION OF THE PROBLEM

Given control points $\{\boldsymbol{p}_i\}_{i=0}^{n}$, a degree n Bézier curve can be expressed as

$$\boldsymbol{P}_n(t)=\sum_{i=0}^{n}B_i^n(t)\boldsymbol{p}_i,\quad 0\le t\le 1, \tag{1}$$

where $B_i^n(t)$ are the Bernstein polynomial basis functions.

The multi-degree reduced approximation of the Bézier curve $\boldsymbol{P}_n(t)$ denoted as Eq.(1) with corners interpolations in the norm L_2 is to find another Bézier curve of degree m ($m<n$) expressed as

$$\boldsymbol{Q}_m(t)=\sum_{i=0}^{m}B_i^m(t)\boldsymbol{q}_i,\ 0\le t\le 1,$$

such that the distance between these two curves in the norm L_2 reaches the minimum, i.e.,

$$\|\boldsymbol{P}_n(t)-\boldsymbol{Q}_m(t)\|_{L_2}=\left(\int_0^1\|\boldsymbol{P}_n(t)-\boldsymbol{Q}_m(t)\|^2\,\mathrm{d}t\right)^{1/2}=\min,$$

and moreover, the prescribed (r, s)-order continuous conditions should also be preserved at the two endpoints of the two curves. Thus, there exist two non-negative integers r and s, such that the following equations hold:

$$\begin{cases}\boldsymbol{P}_n^{(k)}(0)=\boldsymbol{Q}_m^{(k)}(0), & k=0,1,\cdots,r,\\ \boldsymbol{P}_n^{(l)}(1)=\boldsymbol{Q}_m^{(l)}(1), & l=0,1,\cdots,s.\end{cases} \tag{2}$$

PREPARATION AND NOTATION

This paper applies two properties of Jacobi polynomials. The first is from (Borwein and Erdelyi, 1995) and the second can be derived easily from (Sunwoo, 2005).

Property 1 (Borwein and Erdelyi, 1995) Jacobi polynomials $J_n(x)$ of degree n are orthogonal to each other when their weight functions are constant 1. That is,

$$\int_{-1}^{1}J_n(x)J_m(x)\mathrm{d}x=\begin{cases}0, & n\ne m,\\ \delta_n=2/(2n+1), & n=m.\end{cases}$$

Property 2 (Sunwoo, 2005) Jacobi polynomials $\{J_i(2t-1)\}_{i=0}^{n}$ ($0\le t\le 1$) and Bernstein polynomials $\{B_i^n(t)\}_{i=0}^{n}$ ($0\le t\le 1$) have the following relationship:

$$B_k^n(t) = \sum_{i=0}^{n} J_i(2t-1)a_{i,k}, \quad k = 0,1,\cdots,n,$$

where

$$a_{i,k} = \frac{2k+1}{n+k+1}\binom{n}{i}\sum_{l=0}^{k}(-1)^{k+l}\binom{k}{l}\binom{k}{k-l}\bigg/\binom{n+k}{i+l}.$$

By introducing transfer matrix

$$\boldsymbol{A}_{(n+1)\times(n+1)}^n = (a_{i,k})_{(n+1)\times(n+1)} = \begin{pmatrix} \boldsymbol{A}_{(m+1)\times(n+1)}^n \\ \boldsymbol{A}_{(n-m)\times(n+1)}^n \end{pmatrix}, \qquad (3)$$

and basis matrices

$$\boldsymbol{B}_n = [B_0^n(t), B_1^n(t), \cdots, B_n^n(t)],$$
$$\boldsymbol{J}_n = [J_0(2t-1), J_1(2t-1), \cdots, J_n(2t-1)],$$

Property 2 can be rewritten as

$$\boldsymbol{B}_n = \boldsymbol{J}_n \boldsymbol{A}_{(n+1)\times(n+1)}^n. \qquad (4)$$

For convenience of application in the following section, we again define two weight matrices:

$$\boldsymbol{V}_{(m+1)\times(m+1)} = (v_{i,j})_{(m+1)\times(m+1)},$$
$$\boldsymbol{K}_{(n-m)\times(n-m)} = (k_{i,j})_{(n-m)\times(n-m)},$$

where

$$v_{i,j} = \begin{cases} \delta_i, & i = j, \\ 0, & i \neq j, \end{cases} \quad i, j = 0,1,\cdots,m,$$

$$k_{i,j} = \begin{cases} \delta_{m+1+i}, & i = j, \\ 0, & i \neq j, \end{cases} \quad i, j = 0,1,\cdots,n-m-1.$$

DECOMPOSITION AND SIMPLIFICATION OF DEGREE REDUCTION

Decomposition of the degree reduction

Based on the interpolation condition Eq.(2) that the curve $\boldsymbol{P}_n(t)$ denoted as Eq.(1) should satisfy, we deduce the following lemma according to (Chen and Wang, 2002):

Lemma 1 The necessary and sufficient condition for interpolation condition Eq.(2) is

$$(\boldsymbol{q}_0, \boldsymbol{q}_1, \cdots, \boldsymbol{q}_r) = (\boldsymbol{p}_0, \boldsymbol{p}_1, \cdots, \boldsymbol{p}_r)\boldsymbol{M}_{(r+1)\times(r+1)},$$
$$(\boldsymbol{q}_{m-s}, \boldsymbol{q}_{m-s+1}, \cdots, \boldsymbol{q}_m) = (\boldsymbol{p}_{n-s}, \boldsymbol{p}_{n-s+1}, \cdots, \boldsymbol{p}_n)\boldsymbol{N}_{(s+1)\times(s+1)},$$

where

$$\boldsymbol{M}_{(r+1)\times(r+1)} = \begin{pmatrix} b_{0,0} & b_{0,1} & \cdots & b_{0,r} \\ 0 & b_{1,1} & \cdots & b_{1,r} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b_{r,r} \end{pmatrix}^{-1},$$

$$\boldsymbol{N}_{(s+1)\times(s+1)} = \begin{pmatrix} b_{m-s,n-s} & 0 & \cdots & 0 \\ b_{m-s+1,n-s} & b_{m-s+1,n-s+1} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ b_{m,n-s} & b_{m,n-s+1} & \cdots & b_{m,n} \end{pmatrix}^{-1},$$

$$b_{i,j} = \binom{m}{i}\binom{n-m}{j-i}\bigg/\binom{n}{j}.$$

Now we divide the control points of the degree-reduced curve into two parts. One part consists of the control points that should satisfy the high order interpolation condition without degree reduction (described in this subsection). The second part can be obtained by using the control points of the first part and the condition of unconstrained optimal multi-degree reduction (described in the next subsection). The idiographic category rule is as follows.

Denote

$$\tilde{\boldsymbol{Q}}_m = [\boldsymbol{q}_0, \boldsymbol{q}_1, \cdots, \boldsymbol{q}_r, \boldsymbol{0}, \cdots, \boldsymbol{0}, \boldsymbol{q}_{m-s}, \boldsymbol{q}_{m-s+1}, \cdots, \boldsymbol{q}_m],$$
$$\bar{\boldsymbol{Q}}_m = [\boldsymbol{0}, \cdots, \boldsymbol{0}, \boldsymbol{q}_{r+1}, \boldsymbol{q}_{r+2}, \cdots, \boldsymbol{q}_{m-s-1}, \boldsymbol{0}, \cdots, \boldsymbol{0}],$$
$$\hat{\boldsymbol{Q}}_{m-r-s-2} = [\boldsymbol{q}_{r+1}, \boldsymbol{q}_{r+2}, \cdots, \boldsymbol{q}_{m-s-1}],$$

and introduce the endpoints constrained matrix

$$\boldsymbol{H}_{(n+1)\times(m+1)} = \begin{pmatrix} \boldsymbol{M}_{(r+1)\times(r+1)} & & \boldsymbol{0}_{(n-s)\times(s+1)} \\ & \boldsymbol{0}_{(n+1)\times(m-r-s-1)} & \\ \boldsymbol{0}_{(n-r)\times(r+1)} & & \boldsymbol{N}_{(s+1)\times(s+1)} \end{pmatrix},$$

and the control points matrix

$$\boldsymbol{P}_n = [\boldsymbol{p}_0, \boldsymbol{p}_1, \cdots, \boldsymbol{p}_n].$$

Then, according to Lemma 1, it is obvious that

$$\tilde{\boldsymbol{Q}}_m = \boldsymbol{P}_n \boldsymbol{H}_{(n+1)\times(m+1)}. \qquad (5)$$

Transferring from constrained multi-degree reduction into unconstrained multi-degree reduction and the solution

Through the preparation above, the problem of the optimal degree reduction of a curve with high order interpolations at two endpoints is divided into two simpler ones: making high order interpolations at two endpoints without doing degree reduction, and doing optimal degree reduction without making high order interpolations at two endpoints. The former problem has been easily solved, so we are left to solve the latter. The key to resolving the problem lies in transforming the original curve into Jacobi polynomial space to solve the residual control points $\{\boldsymbol{q}_i\}_{i=r+1}^{m-s-1}$ of the degree-reduced curve. Applying Eqs.(3)~(5), we have

$$\begin{aligned}
&\boldsymbol{P}_n(t)-\boldsymbol{Q}_m(t)=\boldsymbol{B}_n\boldsymbol{P}_n^{\mathrm{T}}-\boldsymbol{B}_m(\tilde{\boldsymbol{Q}}_m+\bar{\boldsymbol{Q}}_m)^{\mathrm{T}}\\
&=\boldsymbol{J}_n\boldsymbol{A}^n_{(n+1)\times(n+1)}\boldsymbol{P}_n^{\mathrm{T}}-\boldsymbol{J}_m\boldsymbol{A}^m_{(m+1)\times(m+1)}\left(\boldsymbol{P}_n\boldsymbol{H}_{(n+1)\times(m+1)}+\bar{\boldsymbol{Q}}_m\right)^{\mathrm{T}}\\
&=\boldsymbol{J}_n\left(\begin{pmatrix}\boldsymbol{A}^n_{(m+1)\times(n+1)}\\ \boldsymbol{A}^n_{(n-m)\times(n+1)}\end{pmatrix}-\begin{pmatrix}\boldsymbol{A}^m_{(m+1)\times(m+1)}\boldsymbol{H}^{\mathrm{T}}_{(n+1)\times(m+1)}\\ \boldsymbol{0}\end{pmatrix}\right)\boldsymbol{P}_n^{\mathrm{T}}\\
&\quad-\boldsymbol{J}_m\boldsymbol{A}^m_{(m+1)\times(m+1)}\bar{\boldsymbol{Q}}_m^{\mathrm{T}}\\
&=\boldsymbol{J}_m\boldsymbol{L}_{(m+1)\times1}+[J_{(m+1)}(2t-1),\ J_{m+2}(2t-1),\ldots,\ J_n(2t-1)]\\
&\quad\cdot\boldsymbol{A}^n_{(n-m)\times(n+1)}\boldsymbol{P}_n^{\mathrm{T}},
\end{aligned}$$

where

$$\boldsymbol{L}_{(m+1)\times1}=(l_i)_{(m+1)\times1}=\boldsymbol{U}_{(m+1)\times1}-\boldsymbol{F}\hat{\boldsymbol{Q}}^{\mathrm{T}}_{m-r-s-2},$$

$$\boldsymbol{U}_{(m+1)\times1}=\left(\boldsymbol{A}^n_{(m+1)\times(n+1)}-\boldsymbol{A}^m_{(m+1)\times(m+1)}\boldsymbol{H}^{\mathrm{T}}_{(n+1)\times(m+1)}\right)\boldsymbol{P}_n^{\mathrm{T}},$$

and $\boldsymbol{F}=\boldsymbol{F}^m_{(m+1)\times(m-r-s-1)}$ is the submatrix formed from the $(r+2)$th column to the $(m-s)$th column of the matrix

$$\boldsymbol{A}^m_{(m+1)\times(m+1)}=[\boldsymbol{F}^m_{(m+1)\times(r+1)},\boldsymbol{F}^m_{(m+1)\times(m-r-s-1)},\boldsymbol{F}^m_{(m+1)\times(s+1)}].$$

Next we solve the only unknown quantity $\hat{\boldsymbol{Q}}_{m-r-s-2}$. This is a simple problem of optimal degree reduction without corners constrained. According to Property 1, it is clear that $\|\boldsymbol{P}_n(t)-\boldsymbol{Q}_m(t)\|_{L_2}$ reaches the minimum if and only if

$$\boldsymbol{L}^{\mathrm{T}}_{(m+1)\times1}\boldsymbol{V}_{(m+1)\times(m+1)}\boldsymbol{L}_{(m+1)\times1}=\min.$$

That is,

$$\sum_{i=0}^{m}v_il_i^2=\min.$$

Now we use the least square method of unequally accurate measurement to find the solution. For a minimum of $\sum_{i=0}^{m}v_il_i^2$, it is necessary that the derivatives of $\sum_{i=0}^{m}v_il_i^2$ with respect to the elements of the vector $\hat{\boldsymbol{Q}}_{m-r-s-2}$ be zero, i.e.,

$$\frac{\partial\left(\sum_{i=0}^{m}v_il_i^2\right)}{\partial\boldsymbol{q}_j}=\boldsymbol{0},\quad j=r+1,\ r+2,\ \ldots,\ m-s-1,$$

Solve the normal equations

$$\boldsymbol{F}^{\mathrm{T}}\boldsymbol{V}_{(m+1)\times(m+1)}\boldsymbol{L}_{(m+1)\times1}=0.$$

Substitute $\boldsymbol{L}_{(m+1)\times1}=\boldsymbol{U}_{(m+1)\times1}-\boldsymbol{F}\hat{\boldsymbol{Q}}_{m-r-s-2}$ into the above expression,

$$\boldsymbol{F}^{\mathrm{T}}\boldsymbol{V}_{(m+1)\times(m+1)}\boldsymbol{F}\hat{\boldsymbol{Q}}_{m-r-s-2}=\boldsymbol{F}^{\mathrm{T}}\boldsymbol{V}_{(m+1)\times(m+1)}\boldsymbol{U}_{(m+1)\times1},$$

and then,

$$\hat{\boldsymbol{Q}}_{m-r-s-2}=\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{V}_{(m+1)\times(m+1)}\boldsymbol{F}\right)^{-1}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{V}_{(m+1)\times(m+1)}\boldsymbol{U}_{(m+1)\times1}.$$

Remark 1 $\forall\boldsymbol{x}\in\mathbb{R}^{m-r-s-1}$ $(\boldsymbol{x}\neq\boldsymbol{0})$, $(\boldsymbol{Fx})^{\mathrm{T}}\boldsymbol{VFx}>0$, so $\boldsymbol{F}^{\mathrm{T}}\boldsymbol{VF}$ is a real symmetric positive definite matrix and it is invertible.

All that remains is to combine the control points of the degree-reduced curve. Denote

$$\bar{\boldsymbol{G}}_{(m+1)\times(n+1)}=\begin{pmatrix}\boldsymbol{0}_{(r+1)\times(n+1)}\\ \hat{\boldsymbol{G}}_{(m-r-s-1)\times(n+1)}\\ \boldsymbol{0}_{(s+1)\times(n+1)}\end{pmatrix},$$

where

$$\begin{aligned}\hat{\boldsymbol{G}}_{(m-r-s-1)\times(n+1)}&=\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{V}_{(m+1)\times(m+1)}\boldsymbol{F}\right)^{-1}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{V}_{(m+1)\times(m+1)}\\&\quad\cdot\left(\boldsymbol{A}^n_{(m+1)\times(n+1)}-\boldsymbol{A}^m_{(m+1)\times(m+1)}\boldsymbol{H}^{\mathrm{T}}_{(n+1)\times(m+1)}\right).\end{aligned}$$

Then

$$\bar{\boldsymbol{Q}}_m^{\mathrm{T}}=\bar{\boldsymbol{G}}_{(m+1)\times(n+1)}\boldsymbol{P}_n^{\mathrm{T}}.$$

Finally, denote

$$\boldsymbol{W}_{(m+1)\times(n+1)}=\boldsymbol{H}^{\mathrm{T}}_{(n+1)\times(m+1)}+\bar{\boldsymbol{G}}_{(m+1)\times(n+1)}.$$

Then observing Eq.(5), we can express the control points of the degree-reduced curve in explicit form as

$$\boldsymbol{Q}_m^{\mathrm{T}} = \boldsymbol{W}_{(m+1)\times(n+1)}\boldsymbol{P}_n^{\mathrm{T}}.$$

Remark 2 Endpoints constrained matrix $\boldsymbol{H}_{(n+1)\times(m+1)}$, basis transfer matrix $\boldsymbol{A}^m_{(m+1)\times(m+1)}$, and weight matrix $\boldsymbol{V}_{(m+1)\times(m+1)}$ can be calculated beforehand, and stored in a database. Therefore, the method is less time consuming.

Remark 3 When $r=s=-1$, the problem is converted to unconstrained degree reduction of the curve. Here, the control points of the degree-reduced curve are

$$\boldsymbol{Q}_m^{\mathrm{T}} = \left(\boldsymbol{A}^m_{(m+1)\times(m+1)}\right)^{-1}\boldsymbol{A}^n_{(m+1)\times(n+1)}\boldsymbol{P}_n^{\mathrm{T}}.$$

Approximating error

Here, we give the explicit expression of the approximating error to show how it is predicted. Denote

$$\boldsymbol{D}_{(m+1)\times(n+1)} = \boldsymbol{A}^n_{(m+1)\times(n+1)} - \boldsymbol{A}^m_{(m+1)\times(m+1)}\boldsymbol{H}^{\mathrm{T}}_{(n+1)\times(m+1)} - \boldsymbol{A}^m_{(m+1)\times(m+1)}\bar{\boldsymbol{G}}_{(m+1)\times(n+1)}.$$

Then the approximating error is as follows:

$$\begin{aligned}\varepsilon &= \left\|\boldsymbol{P}_n(t)-\boldsymbol{Q}_m(t)\right\|_{L_2} = \left(\int_0^1 \left\|\boldsymbol{P}_n(t)-\boldsymbol{Q}_m(t)\right\|^2 \mathrm{d}t\right)^{1/2} \\ &= \frac{\sqrt{2}}{2}\left(\left\|\boldsymbol{P}_n\boldsymbol{D}^{\mathrm{T}}_{(m+1)\times(n+1)}\boldsymbol{V}_{(m+1)\times(m+1)}\boldsymbol{D}_{(m+1)\times(n+1)}\boldsymbol{P}_n^{\mathrm{T}}\right\|^2_{l_2} \right. \\ &\quad \left. + \left\|\boldsymbol{P}_n\left(\boldsymbol{A}^n_{(n-m)\times(n+1)}\right)^{\mathrm{T}}\boldsymbol{K}_{(n-m)\times(n-m)}\boldsymbol{A}^n_{(n-m)\times(n+1)}\boldsymbol{P}_n^{\mathrm{T}}\right\|^2_{l_2}\right)^{1/2}.\end{aligned}$$

EXAMPLE ANALYSES

Finally two examples are presented.

Example 1 For a given Bernstein polynomial of degree 4 with the control points (1, 2, 4, 3, 2), we are to find its best degree-reduced Bernstein polynomial of degree 3, keeping the (0, 0)-order endpoints interpolations. Applying the algorithm of this paper, the control points of the degree-reduced curve are (1, 17/6, 23/6, 2). The approximating error is 0.0527. The result is given in Fig.1.

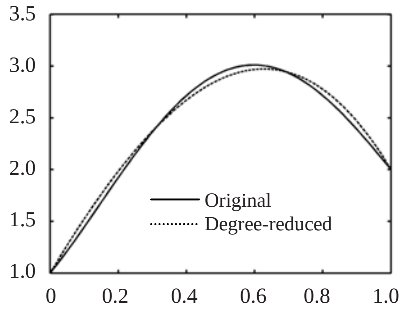

Fig.1 Bernstein polynomial of degree 4 and its best one-degree-reduced Bernstein polynomial keeping the endpoints (*r*, *s*)=(0, 0) order interpolations

Example 2 For a given Bernstein polynomial of degree 6 with the control points (0, 1, 4, 3, 2, 1, 0), we are to find its best degree-reduced Bernstein polynomial of degree 4, keeping the (1, 0)-order endpoints interpolations. Applying the algorithm of this paper, the control points of the degree-reduced curve are (0, 1.5000, 5.6061, 0.5545, 0). The approximating error is 0.1077. The approximating effect is very good. Fig.2 shows the result.

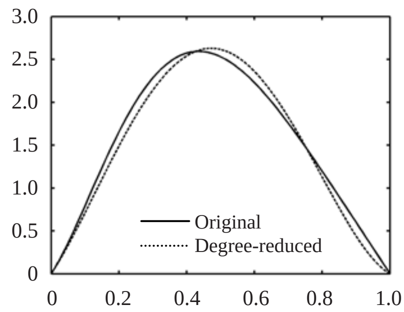

Fig.2 Bernstein polynomial of degree 6 and its best two-degree-reduced Bernstein polynomial keeping the endpoints (*r*, *s*)=(1, 0) order interpolations

CONCLUSION

In this paper we have introduced a new framework for multi-degree reduction of Bézier curves with endpoints continuity and have obtained the optimal approximation in L_2-norm. This paper differs from previous work because of its novel method: divide and conquer. That is, it divides the problem of the optimal degree reduction of a curve with high order interpolations at its two endpoints into two simpler problems: making high order interpolations at its two endpoints without doing degree reduction, and doing unconstrained optimal degree reduction. So it greatly reduces the difficulty of the problem. Our method can be easily generalized to the optimal multi-degree reduction of tensor product surfaces or triangular surfaces. This method also avoids stepwise computing for multi-degree reduction so that the computing time can be reduced. The two examples, with the approximating errors being 0.0527 and 0.1077, respectively, show that our method is effective in multi-degree reduction of Bézier curves.

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