Journal of Zhejiang University-SCIENCE A (Applied Physics & Engineering)
ISSN 1673-565X (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Nonlinear identification of electro-magnetic force model

R. SHABANI[†1], S. TARIVERDILO[2], H. SALARIEH[3]
([1]*Mechanical Engineering Department, Faculty of Engineering, Urmia University, Urmia, Iran*)
([2]*Civil Engineering Department, Faculty of Engineering, Urmia University, Urmia, Iran*)
([3]*Center of Excellence in Design, Robotics, and Automation (CEDRA), Sharif University of Technology, Tehran, Iran*)
[†]E-mail: r.shabani@urmia.ac.ir
Received Apr. 12, 2009; Revision accepted Sept. 3, 2009; Crosschecked Jan. 7, 2010

Abstract: Conventional attractive magnetic force models (proportional to the coil current squared and inversely proportional to the gap squared) cannot simulate the nonlinear responses of magnetic bearings in the presence of electromagnetic losses, flux leakage or saturation of iron. In this paper, based on results from an experimental set-up designed to study magnetic force, a novel parametric model is presented in the form of a nonlinear polynomial with unknown coefficients. The parameters of the proposed model are identified using the weighted residual method. Validations of the model identified were performed by comparing the results in time and frequency domains. The results show a good correlation between experiments and numerical simulations.

Key words: Identification, Nonlinear vibration, Magnetic bearing, Weighted residual method
doi:10.1631/jzus.A0900203   **Document code:** A   **CLC number:** O34

1 Introduction

Because of the nonlinearity of electromagnetic force, active magnetic bearings (AMB) show strong nonlinearity in their responses. This nonlinear behavior is especially important when high magnetic flux density is used or when large deflections of the rotor occur.

There are two common approaches to studying the nonlinear dynamics of AMBs. The first approach uses coupled Maxwell and kinetic equations (Moon and Pao, 1969; Sortore, 1990). These models are numerically exhaustive and are not able to capture all features of the systems dynamic. The second approach, identifying the input-output data, employs mathematical models to simulate the electromagnetic forces. Different identification methods, such as maximum likelihood (Gertler and Banyasz, 1974), least square (Hsia, 1976) and instrumental variable methods (Finigan and Rowe, 1976) are used mainly for identification of linear systems. However, other methods have been developed for identification of nonlinear systems (Ibanez, 1973; Beliveau, 1976). Yasuda *et al*. (1988) used the method of harmonic balance for identification of a nonlinear multi-degree of freedom system with simulated input-output data. Yi and Hedrick (1995) proposed an identification method based on a sliding observer and least square method to identify the parameters of a car suspension system. Bonisoli and Vigliani (2007) investigated a passive elasto-magnetic suspension system, and developed a polynomial form for passive repulsive magnetic force. Zhang and Roberts (1996) studied the identification of a nonlinear system in frequency domain.

As the electromagnetic force model is the main source of nonlinearity in magnetic suspension systems, several studies have investigated the identification of this model. Chang and Tung (1998) employed a polynomial form for electromagnetic force, and identified the parameters of their model using harmonic balance and least square methods. Their experimental set-up consisted of a symmetric rotor attached at one end to a flexible coupler and supported at the other end by four radial electromagnets. They used a summation form, while a multiplicative form works better (Shabani *et al.*, 2006). By augmenting unknown parameters to the state vectors, some researchers evaluated the Jacobian of the system equations at each increment, and used the extended Kalman filter (EKF) method to estimate the parameters of the augmented state vector (Athans *et al.*, 1968; Jazwinski, 1970; Julier and Uhlmann, 1997). Applying the EKF method, Alasty and Shabani (2006) identified a multiplicative electromagnetic force model.

Several researchers have investigated the nonlinear dynamics and control of systems based on closed loop control of magnetic levitation. Using a conventional electromagnetic force model and employing the perturbation method, Zhang and Zhan (2005) and Zhang *et al*. (2006) studied the nonlinear and chaotic response of magnetic bearings. Utilizing a proportional derivative (PD) controller, their formulation includes the quadratic and cubic nonlinearities of the resultant electromagnetic force. Ji and Hansen (2001) introduced a polynomial form for electromagnetic force models to simplify the process of controller design and analytically studied the dynamics and stability of the systems. Assuming small amplitude vibration and expanding the conventional model by taking into account the linear and cubic terms, they used a PD controller and a perturbation method to predict the stability of steady state solutions. Lu *et al*. (2008) designed a linear parameter-varying controller and investigated its performance and robustness experimentally. Using a conventional electromagnetic force model, Inayat-Hussain (2007) studied the effects of geometrical cross-coupling in the response of two degrees of freedom rotors supported by magnetic bearings. By varying the imbalances of the rotor and employing a PD controller, they investigated the emergence of chaos in the response. Other researchers used conventional electromagnetic force models to study the control of magnetic bearings by implementing nonlinear control methods such as feedback linearization (John *et al.*, 2003) and robust control methods (Ming-Jyi *et al.*, 2005; Gosiewski and Mystkowski, 2008). Because the electromagnetic force is the main source of nonlinearity in the system, the identification or introduction of a simple force model can be important in dynamic analysis and controller design. Jeng (2000) investigated the control of axial vibration of a rotor using electromagnetic force in an experimental set-up. He used a multi-layered neural network model to control the system without the need to consider the system dynamics. Schroder *et al.* (2001) studied an experimental set-up that included an electrical pump running on active magnetic bearings. They used a multi-objective genetic algorithm to optimize the online controller design.

This paper introduces a novel parametric model, in the form of a nonlinear Mathieu-Duffing equation with unknown coefficients, to identify the system parameters of an experimental set-up. The experimental set-up was similar to that used by Alasty and Shabani (2006) with some additional input-output data reported for the first time. A Galerkin weighted residual method is then employed to identify the parameters of the proposed model. Finally, simulation and experimental results are compared. The parametric model used leads to a simpler model and gives a better correlation with experimental results when compared with the model of Alasty and Shabani (2006).

2 Experimental set-up and results

The experimental set-up (Fig. 1), included a symmetric rigid rotor, rigidly attached to a cantilever load cell, which acted as a rigid beam with an elastic joint at the clamped end. The rotor was disturbed by one electromagnet of a four-pole magnetic bearing with DC and harmonic AC voltages. The sensor read the rotor displacement as the system was excited with bias voltage of V_{DC}=0.558 V and harmonic AC voltage of V_0=0.5 V. The excitation was provided by a power amplifier used as a *V*/*I* converter with a gain of 0.1 A/V, and a saturation limit of 3 A. Limiting the forcing voltage to 10 V and with this gain ratio, there was no current saturation in the system. Further details on the experimental set-up are given by Alasty and Shabani (2006), who used a non-contact

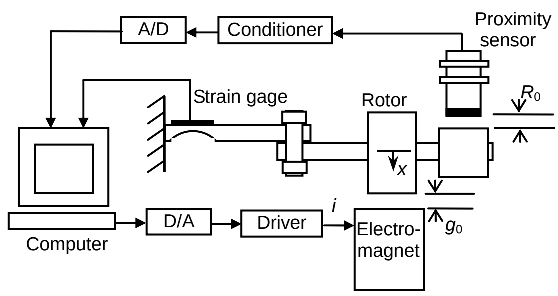

Fig. 1 A schematic diagram of the experimental system

proximity displacement sensor with sensitivity of 0.3 V/mm to measure the rotor displacement near the electromagnet location. However, some of the experimental results that show a superharmonic and chaotic response in a narrow frequency band are presented for the first time.

Figs. 2a and 2c depict the time history of the rotor steady state vibration and its Fourier transform for an excitation frequency of 2 Hz. x is the rotor displacement and $D(x)$ is the rotor velocity. Fourier transforms of the response show that the response was rich in the excitation frequency together with its 2nd, 3rd and 4th multiplications. Initial judgment of the existence of chaotic behavior was based on the Poincare map and power spectrum plots. However, presence of chaos in the response was also investigated by calculating the Lyapunov exponents from experimental time histories (Wolf *et al*., 1985). The largest Lyapunov exponent "λ" for an excitation frequency of 2 Hz is shown in Fig. 2d, where stabilization of the exponent at $\lambda=-0.35$ confirms that the response was periodic.

A superharmonic response was also observed for an excitation frequency of 3 Hz (Fig. 3). Some small deviations of response from the periodic case can be observed in the phase plane (Fig. 3b). The largest associated Lyapunov exponent then stabilized at $\lambda=-0.1$ (Fig. 3d).

The periodic response lost its stability and changed to a chaotic one as the excitation frequency increased to 3.5 Hz (Fig. 4). At this excitation frequency, multiplication of forcing frequency over a broad band was observed (Fig. 4c). Figs. 4b and 4d show the phase plane and Poincare map of the response. At this excitation frequency, the complex structure of the Poincare map (Fig. 4d) and positive value of the largest Lyapunov exponent (Fig. 4e) confirm that the response was chaotic. The emergence of chaotic response is also shown in Fig. 5, where the rotor was excited with a frequency of 3.6 Hz.

By increasing the excitation frequency the chaotic responses disappeared until 9 Hz was reached. At a narrow band around this frequency, the chaotic response re-emerged (Fig. 6). The fast Fourier

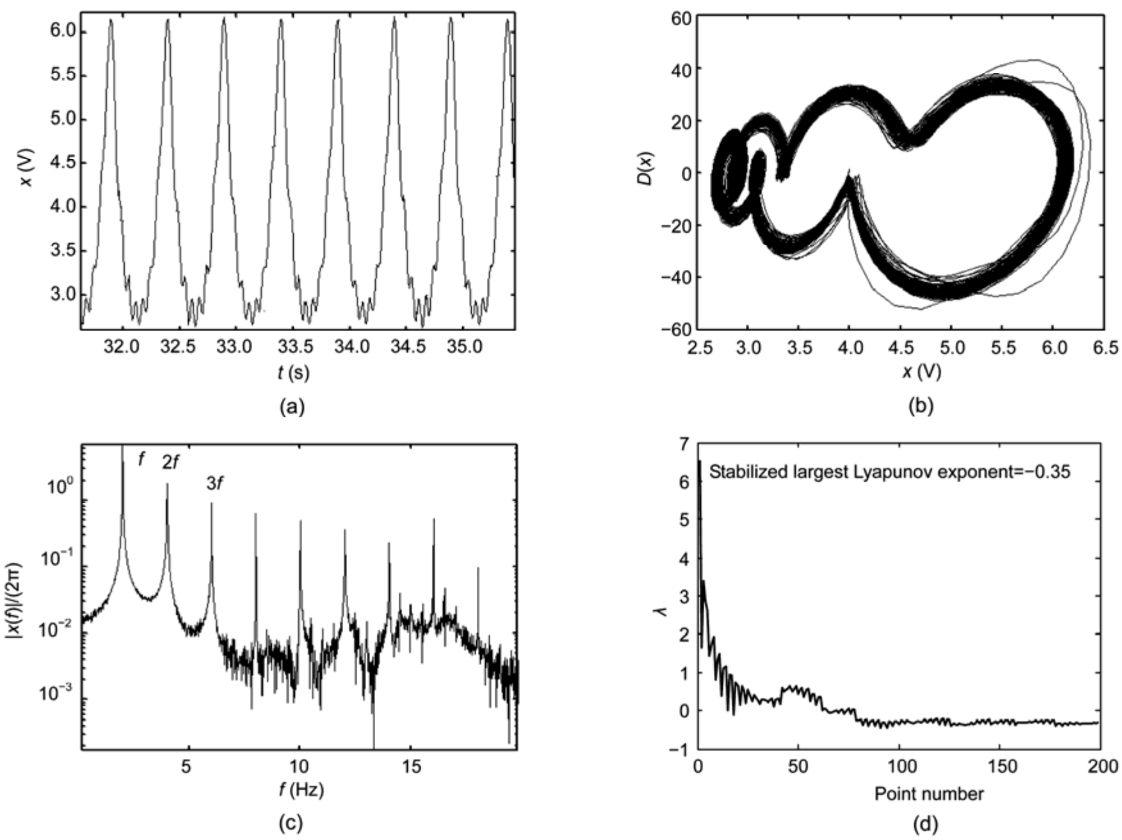

Fig. 2 Experimental results for sinusoidal excitation of U=0.558+0.5sin(2πft) and f=2 Hz
(a) Time history; (b) Phase plane; (c) Fourier transform; (d) Largest Lyapunov exponent

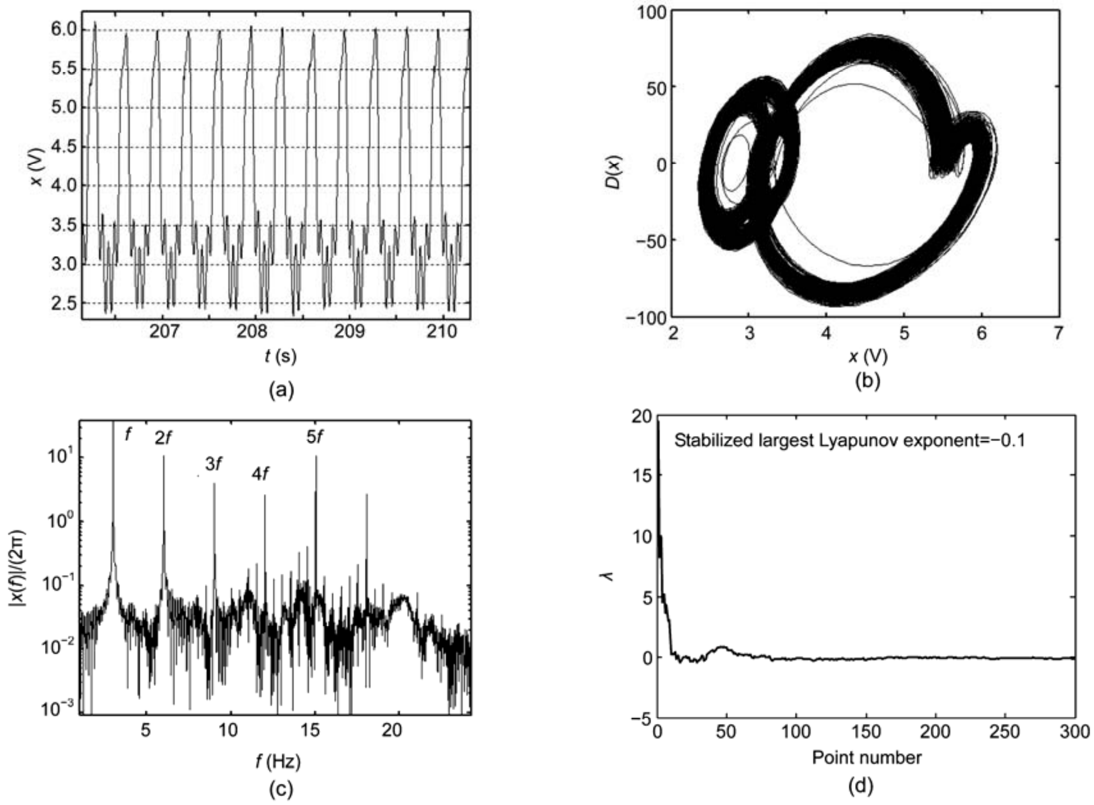

Fig. 3 Experimental results for excitation input of U=0.558+0.5sin(2πft) and f=3 Hz
(a) Time history; (b) Phase plane; (c) Fourier transform; (d) Largest Lyapunov exponent

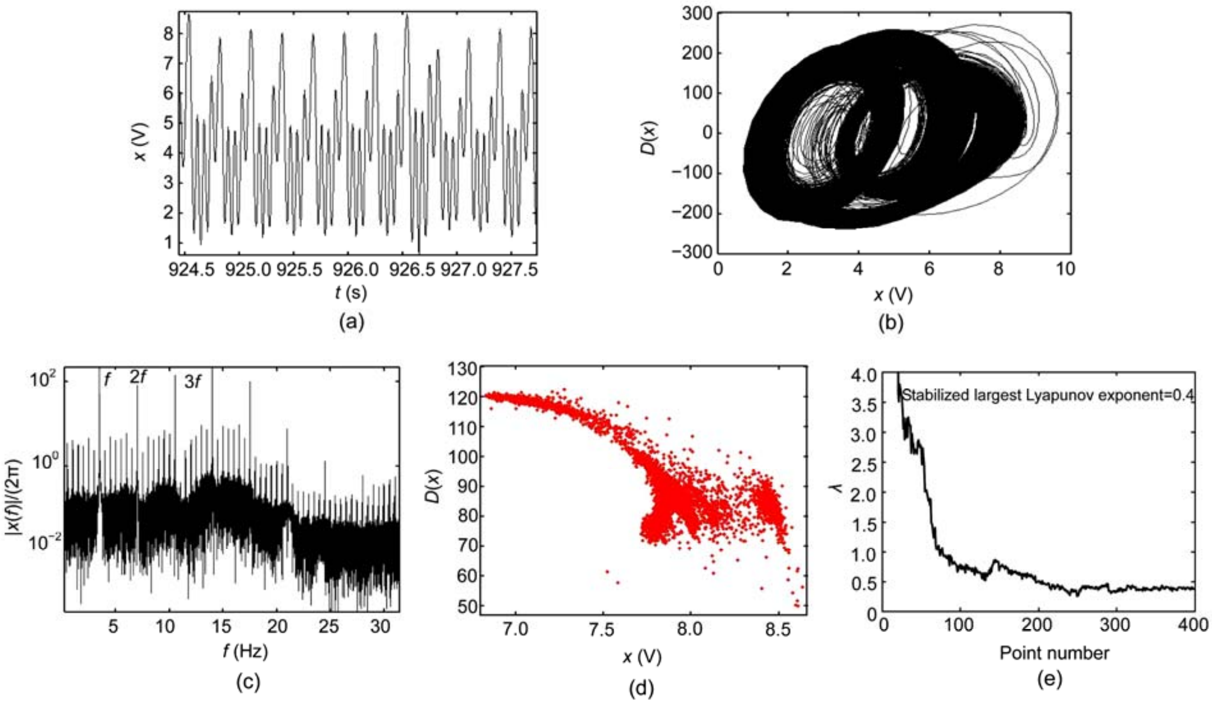

Fig. 4 Experimental results for excitation input of U=0.558+0.5sin(2πft) and f=3.5 Hz
(a) Time history; (b) Phase plane; (c) Fourier transform; (d) Poincare map; (e) Largest Lyapunov exponent

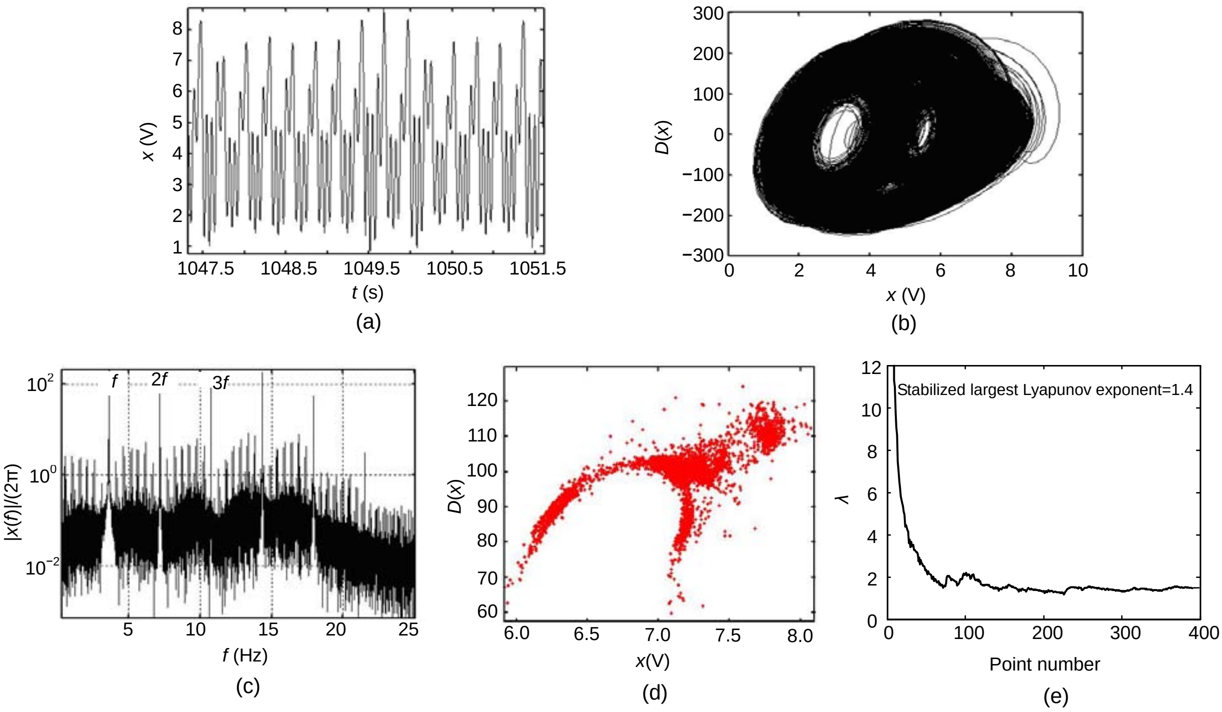

Fig. 5 Experimental results for excitation input of U=0.558+0.5sin(2πft) and f=3.6 Hz
(a) Time history; (b) Phase plane; (c) Fourier transform; (d) Poincare map; (e) Largest Lyapunov exponent

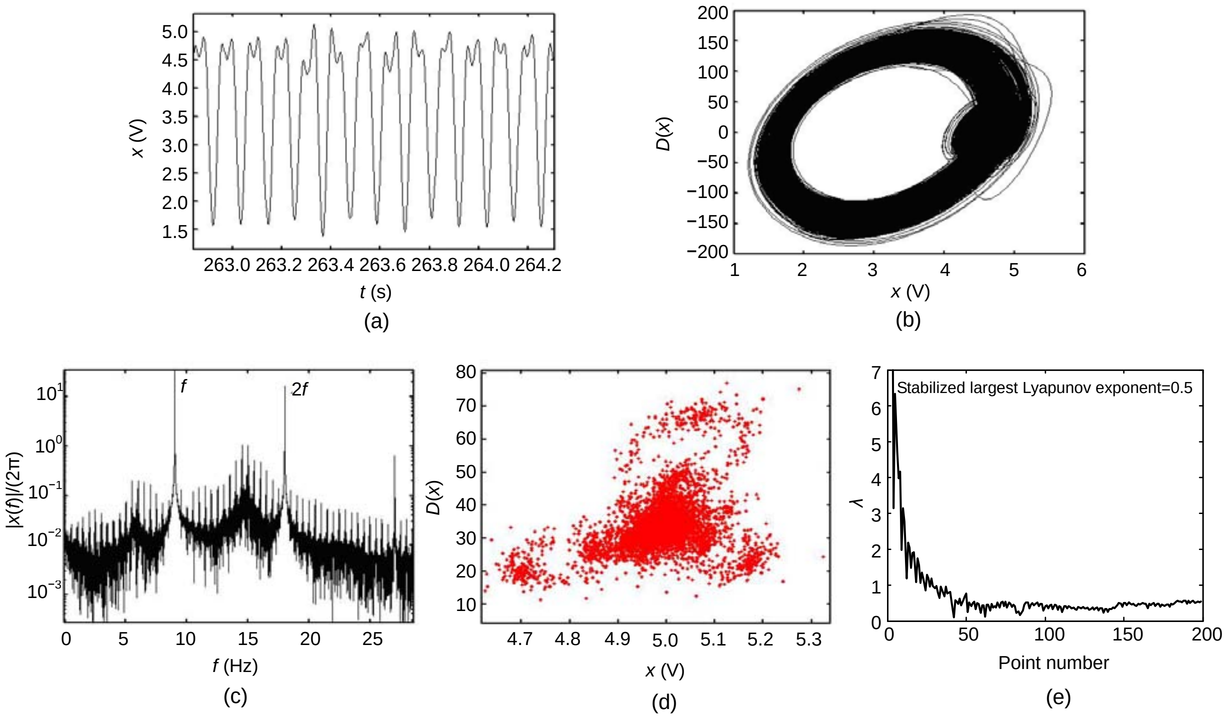

Fig. 6 Experimental results for excitation input of U=0.558+0.5sin(2πft) and f=9 Hz
(a) Time history; (b) Phase plane; (c) Fourier transform; (d) Poincare map; (e) Largest Lyapunov exponent

transform, Poincare map, and the largest Lyapunov exponent (Figs. 6c~6e, respectively), provide the evidence for this finding. By further increasing the frequency up to 20 Hz—the maximum frequency considered in this study—there was no sign of irregularity or chaotic response.

With the emphasis being on capturing the global dynamics of the system, a polynomial form was used to identify the electromagnetic force. Based on the system response, the structure of the model is proposed and its parameters are determined using the Galerkin method.

3 System identification and model verification

The weighted residual method is a sound and robust method for identification of nonlinear systems. After selecting a nonlinear parametric model, function of coil current and air gap, the weighted residual method is used at each frequency to identify the parameters of the model. The general form of differential equations of the system is assumed as

$$x^{(n)} = f(x, \dot{x}, ..., x^{(n-1)}, u_1, u_2, ..., u_m), \quad (1)$$

where u_i and $x^{(i)}$ are the inputs and the state variables of system, respectively, and f is a function that is to be identified. Assuming a superlative form for this function, we have

$$\widehat{f} = \sum_{k=1}^{N} \alpha_k g_k (x, \dot{x}, ..., x^{(n-1)}, u_1, u_2, ..., u_n), \quad (2)$$

where α_k denote unknown constant coefficients and g_k are interpolation functions of inputs and state variables. We find α_k by minimizing the following residual function:

$$\begin{aligned} R(x, \dot{x}, ..., x^{(n)}, u_1, u_2, ..., u_n) \\ = x^{(n)} - \sum \alpha_k g_k (x, \dot{x}, ..., x^{(n-1)}, u_1, u_2, ..., u_n) \quad (3) \\ = f - \widehat{f}. \end{aligned}$$

Using the interpolation functions also as weight functions (the Galerkin method), the result is the following algebraic equations in terms of α_k:

$$\int g_j R \mathrm{d}t = \int x^{(n)} g_j \mathrm{d}t - \sum \alpha_k \int g_k g_j \mathrm{d}t = 0, \quad j, k \in \mathbb{N}_+. \quad (4)$$

In the mathematical formulation of magnetic bearing systems, the voltage balance equation must be considered in addition to the dynamics of the rotor. In this system, the exciting voltage $V_{\mathrm{DC}}+V_0\sin(\omega t)$, is balanced by the voltage drop owing to the resistance and inductive reactance of the coil. The induced voltage in the coil is proportional to the number of turns (n) and the time rate of change of flux (φ), i.e.,

$$n\frac{\mathrm{d}\varphi(I, x)}{\mathrm{d}t} + RI = F_\varphi + RI = K_1 + K_2 \sin(\omega t), \quad (5)$$

where I, R, K_1, K_2, F_φ are the coil current, coil resistance, coefficients of input voltage and voltage drop owing to inductance and the back electromotive force (EMF), respectively. With frequencies lower than the natural frequency of the system (16.6 Hz), variation of φ does not affect the voltage balances in the coil. Therefore, the relationship between coil current and voltage drop is static, and can be incorporated in the electromagnetic force mathematical form as a coefficient (Chang and Tung, 1998; Alasty and Shabani, 2006). Consequently the governing equation of the system can be written as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_{\mathrm{m}}[(g_0 - x), I], \quad (6)$$

where f_{m}, I, ω_n and ζ are the electromagnetic force, coil current, natural frequency and damping ratio, respectively, and g_0 is initial gap between the electromagnet and the rotor. Considering the response of the experimental set-up and limitations, such as saturation of the iron, electromagnetic losses, and flux leakage, the electromagnetic model cannot have the conventional form $kI^2/(g_0-x)^2$ (Chang and Tung, 1998). Deriving superharmonics of the system using a conventional electromagnetic force model usually requires modeling of saturation effect, where the source of this saturation could, for instance, be current saturation (Ji and Hansen, 2001). However, these conventional models are able to model subharmonics and chaos in the system (Ji and Hansen, 2001). The main contribution of this study is to propose a nonlinear electromagnetic force model that is able to capture the global nonlinear dynamics of the system. Based on the experimental results, the structure of this model is proposed and its parameters are identified using the Galerkin weighted residual method.

As can be inferred from the experimental results reported by Alasty and Shabani (2006) and additional results presented here, the response contains 2nd, 3rd and 4th multiplications of the excitation frequency, so it seems that superharmonic resonance occurs in the system. Therefore, the governing equations of the system should have cubic and quadratic terms. In nonlinear systems with cubic and quadratic terms, superharmonic resonances occur only in specific excitation frequencies (external resonances). Existence of superharmonic response in a wide spectrum of the excitation frequencies shows that the considered electromagnetic force must contain nonlinearities other than cubic and quadratic terms. The frequency characteristic of the response implies that the system equations must be in a form similar to a Mathieu equation, in which the response frequency is a function of the excitation frequency. In addition, the model should be a function of the current, in such a way that it vanishes when the excitation current tends to zero (Shabani *et al.*, 2006). Based on these considerations, the following polynomial form for electromagnetic force f_m is considered:

$$f_m[(g_0 - x), I] = b_0 + b_1 I + b_2 I^2 + b_3 I x_1 + b_4 I x_1^2, \quad x_1 = g_0 - x, \tag{7}$$

where b_i (i=0,1,…,4) denote the unknown coefficients.

Substituting Eq. (7) into Eq. (6), we obtain the Mathieu-Duffing type nonlinear equation:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = a_0 + a_1 I + a_2 I^2 + a_3 I x + a_4 I x^2, \tag{8}$$

where the coefficients a_i (i=0,1,…,4) should be determined by an identification method. With measured coil current I, rotor displacement x, and numerical calculations of velocity and accelerations of rotor in various exciting frequencies, the unknown coefficients a_i are identified (Table 1). The coefficients are frequency dependent. Numerical simulations show that small variations in these coefficients have no large impact on the system response. Therefore, it is possible to interpolate the coefficients between the mentioned frequencies.

The identified model was verified by comparing the time histories and Fourier transforms of the outputs of the identified model against the experimental results. In Fig. 7, the model and experimental system

Table 1 Coefficients a_i of the model, identified at different frequencies (×10^6)

f (Hz)	a_i				
	a_0 (m/s^2)	a_1 (m/(s^2·A))	a_2 (m/(s^2·A^2))	a_3 (1/(s^2·A))	a_4 (1/(s^2·A·m))
1.0	0.0218	−0.0463	−0.2062	0.0549	−0.0024
1.5	0.0233	−0.0385	−0.1144	0.0399	−0.0012
2.0	0.0270	−0.0441	0.0394	0.0210	0.0003
2.5	0.0270	−0.0267	0.0807	0.0097	0.0013
3.0	0.0255	0.0042	0.0653	0.0003	0.0023
3.5	0.0316	−0.0580	0.1895	0.0040	0.0017
4.0	0.0238	0.0113	0.0130	0.0063	0.0017
4.5	0.0304	−0.0650	0.1530	0.0143	0.0005
5.0	0.0431	−0.1530	0.3704	0.0036	0.0017
9.0	−0.0757	0.9712	−1.2515	−0.0816	0.0092

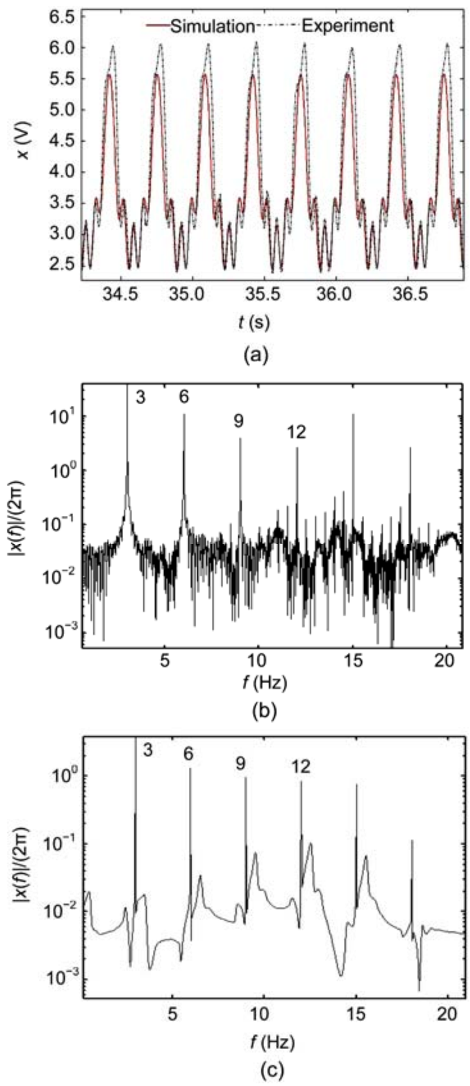

Fig. 7 Comparison of simulation results with experimental measurements for the input of U=0.5+0.5sin(2πft) in f=3 Hz
(a) Comparison of time histories; (b) Fourier transform of the experiment; (c) Fourier transform of the identified model

outputs are compared for an exciting frequency of 3 Hz. The amplitudes of vibration for the estimated model and the experiment are about the same and the dominant frequencies of the system are identified accurately. This verification was also performed at excitation frequencies of 3.5 and 9 Hz (Figs. 8 and 9, respectively).

Although, in this experiment, the low natural frequency of the rotor imposes a limit on the operating and identification frequency of the system, these results confirm that the proposed model can be used in other magnetic levitation systems with different operational frequencies, with new coefficients to be identified.

It should be noted that in addition to the limitations in the frequency range, the dynamics of the system were studied only in the vertical direction. When the rotor is rotating, interaction of adjacent electromagnets is significant. In such conditions the magnetic forces of the electromagnets in the model

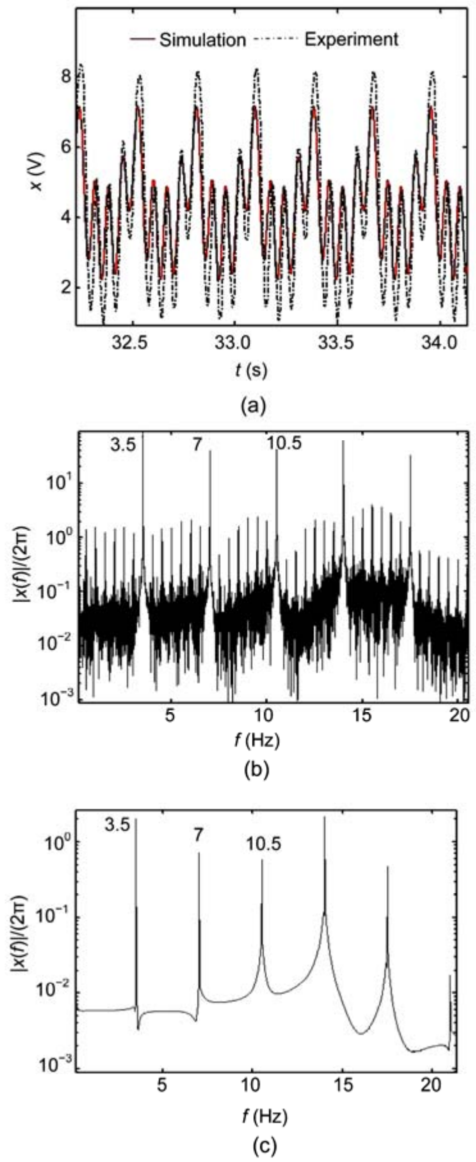

Fig. 8 Comparison of simulation results with experimental measurements for the input of U=0.5+0.5sin(2πft) in f=3.5 Hz

(a) Comparison of time histories; (b) Fourier transform of the experiment; (c) Fourier transform of the identified model

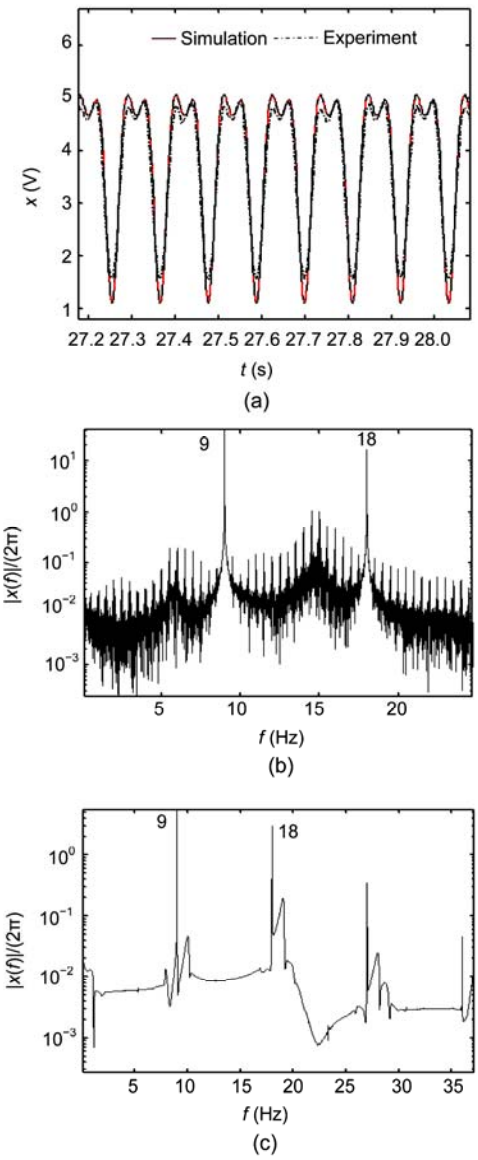

Fig. 9 Comparison of simulation results with experimental measurements for the input of U=0.5+0.5sin(2πft) in f=9 Hz

(a) Comparison of time histories; (b) Fourier transform of the experiment; (c) Fourier transform of the identified model should be a function of the current and the air gaps in both vertical and horizontal directions. In these cases the proposed identification method is still applicable. However, the proposed model requires some modification for identification of the chaotic dynamics appearing in a narrow band of excitation frequencies.

4 Conclusion

Harmonic excitation of the experimental set-up revealed that the system was rich in nonlinear dynamics. This was shown by the presence of superharmonics and, in some situations, chaotic response. Instead of solving the full set of kinetic and Maxwell equations, a proper parametric nonlinear model which is a function of air gap and coil current is proposed for electromagnetic force and then the weighted residual method is used to identify the model parameters. Time history and Fourier transforms of the identified model and the experimental system outputs were compared for verification of the identified model. These comparisons indicated that the proposed model is quite reliable.

References

Alasty, A., Shabani, R., 2006. Nonlinear parametric identification of magnetic bearings. *Mechatronics*, **16**(8):451-459. [doi:10.1016/j.mechatronics.2006.03.008]

Athans, M., Wishner, R.P., Bertolini, A.B., 1968. Suboptimal state estimation for continuous-time nonlinear system from discrete noisy measurements. *IEEE Transactions on Automatic Control*, **13**(5):504-514. [doi:10.1109/TAC.1968.1098986]

Beliveau, J.G., 1976. Identification of viscous damping in structures from model information. *ASME, Journal of Applied Mechanics*, **43**:335-339.

Bonisoli, E., Vigliani, A., 2007. Identification techniques applied to a passive elasto-magnetic suspension. *Mechanical Systems and Signal Processing*, **21**(3):1479-1488. [doi:10.1016/j.ymssp.2006.05.009]

Chang, S.C., Tung, P.C., 1998. Identification of a non-linear electromagnetic system: An experimental study. *Journal of Sound and Vibration*, **214**(5):853-871. [doi:10.1006/jsvi.1998.1590]

Finigan, B.M., Rowe, I.H., 1974. Strongly consistent parameter estimation by the introduction of strong instrumental variables. *IEEE Transaction on Automatic Control*, **19**(6):825-830. [doi:10.1109/TAC.1974.1100719]

Gertler, J., Banyasz, C., 1974. A recursive (on-line) maximum likelihood identification method. *IEEE Transaction on Automatic Control*, **19**(6):816-820. [doi:10.1109/TAC.1974.1100722]

Gosiewski, Z., Mystkowski, A., 2008. Robust control of active magnetic suspension: Analytical and experimental results. *Mechanical Systems and Signal Processing*, **22**(6):1297-1303. [doi:10.1016/j.ymssp.2007.08.005]

Hsia, T.C., 1976. Least square algorithm for system identification. *IEEE Transaction on Automatic Control*, **21**(1):104-108. [doi:10.1109/TAC.1976.1101132]

Hung, J.Y., Albritton, N.G., Xia, F., 2003. Nonlinear control of a magnetic bearing system. *Mechatronics*, **13**(6):621-637. [doi:10.1016/S0957-4158(02)00034-X]

Ibanez, P., 1973. Identification of dynamic parameters of linear and nonlinear structural models from experimental data. *Nuclear Engineering and Design*, **25**:30-41. [doi:10.1016/0029-5493(73)90059-9]

Inayat-Hussain, J.I., 2007. Chaos via torus breakdown in the vibration response of a rigid rotor supported by active magnetic bearings. *Chaos, Solitons and Fractals*, **31**(4):912-927. [doi:10.1016/j.chaos.2005.10.039]

Jang, M.J., Chen, C.L., Tsao, Y.M., 2005. Sliding mode control for active magnetic bearing system with flexible rotor. *Journal of the Franklin Institute*, **342**(4):401-419. [doi:10.1016/j.jfranklin.2005.01.006]

Jazwinski, A.H., 1970. Stochastic Processes and Filtering Theory. Academic Press, New York.

Jeng, J.T., 2000. Nonlinear adaptive inverse control for the magnetic bearing system. *Journal of Magnetism and Magnetic Materials*, **209**(1-3):186-188. [doi:10.1016/S03048853(99)00683-6]

Ji, J.C., Hansen, C.H., 2001. Non-linear oscillations of a rotor in active magnetic bearings. *Journal of Sound and Vibration*, **240**(4):599-612. [doi:10.1006/jsvi.2000.3257]

Julier, S.J., Uhlmann, J.K., 1997. A New Extension of the Kalman Filter to Nonlinear Systems. The 11th International Symposium on Aerospace/Defense Sensing, Simulation and Control, Orlando, FL.

Lu, B., Choi, H., Buckner, G.D., Tammi, K., 2008. Linear parameter-varying techniques for control of a magnetic bearing system. *Control Engineering Practice*, **16**(10):1161-1172. [doi:10.1016/j.conengprac.2008.01.002]

Moon, F.C., Pao, Y.H., 1969. Vibration and dynamic instability of a beam-plate in a transverse magnetic field. *ASME Transactions, Journal of Applied Mechanics*, **36**:92-100.

Schroder, P., Green, B., Grumm, N., Fleming, P.J., 2001. Online evolution of robust control systems: an industrial active magnetic bearing application. *Control Engineering Practice*, **9**(1):37-49. [doi:10.1016/S0967-0661(00)00087-3]

Shabani, R., Salarieh, H., Alasty, A., 2006. Comments on "Identification of a nonlinear electromagnetic system: An experimental study". *Journal of Sound and Vibrations*, **290**(3-5):1333-1334.

Sortore, C.K., 1990. Permanent Magnet Biased Magnetic Bearing Design, Construction and Testing. Proceedings 2nd Internatioal Symposium on Magnetic Bearings, Tokyo, Japan, p.175-182.

Wolf, A., Swift, J.B., Swinney, H.L., Vastano, J.A., 1985.

Determining Lyapunov exponents from a time series. *Physica*, **16D**:285-317.
Yasuda, K., Kawamura, S., Watanabe, S., 1988. Identification of non-linear multi-degree-of-freedom systems (Presentation of an identification technique). *JSME International Journal Series III*, **31**:8-14.
Yi, K., Hedrick, K., 1995. Observer-based identification of nonlinear system parameters. *Journal of Dynamic Systems Measurement and Control*, **117**:175-182. [doi:10.1115/1.2835177]
Zhang, J.X., Roberts, J.B., 1996. A frequency domain parametric identification method for studying the non-linear performance of squeeze-film dampers. *Journal of Sound and Vibration*, **189**:173-191. [doi:10.1006/jsvi.1996.0014]
Zhang, W., Zhan, X.P., 2005. Periodic and chaotic motions of a rotor-active magnetic bearing with quadratic and cubic terms and time-varying stiffness. *Nonlinear Dynamics*, **41**(4):331-359. [doi:10.1007/s11071-005-7959-2]
Zhang, W., Yao, M.H., Zhan, X.P., 2006. Multi-pulse chaotic motions of a rotor-active magnetic bearing system with time-varying stiffness. *Chaos, Solitons and Fractals*, **27**(1):175-186. [doi:10.1016/j.chaos.2005.04.003]