Journal of Zhejiang University-SCIENCE A (Applied Physics & Engineering)
ISSN 1673-565X (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
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A study of optimal switching problem in limited-cycle with multiple periods*

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Received Oct. 28, 2010; Revision accepted Oct. 29, 2010; Crosschecked Oct. 29, 2010

Abstract: In a social system or production line, the restrictions of the cost and the due-time exist in each period. Generally, whether these restrictions are satisfied is dependent not only on the risks of this period, but also on the risks generated beforehand. We consider controlling the production process by switching the processing rate to a faster one at a given period. This paper deals with the optimal switching period to minimize the total expected cost of the production process. We first propose the optimal switching period model, and then the mathematic formulation of the total expectation is presented. Finally, the policy of optimal switching period is investigated in detail by numerical experiments.

Key words: Limited-cycle problem, Multi-period, Optimal switching period, Production seat system, Manufacturing line
doi:10.1631/jzus.A1001062 **Document code:** A **CLC number:** TP278

1 Introduction

In a multi-period system (e.g., a production line) where target processing time exists, idle and delay risks occur repeatedly for multiple periods. In such situations, a delay in one process may influence the delivery date of an entire process. In this paper, we discuss the minimum expected risk of the case mentioned above, where the risk depends on the previous situation and occurs repeatedly for multiple periods. There exists an object with some constraints (e.g., processing time with a target). These constraints produce a risk and the object occurs repeatedly for multiple periods. This kind of problem is called "a limited-cycle problem with multiple periods" (Yamamoto *et al*., 2006; 2010), which exists in production lines, time-bucket balancing, production seat systems, and so on (Matsui, 2008).

Under the condition of uncertainty, the result or efficiency of a period is often controlled not only by the risks of this period but also by the risks generated beforehand. Whether the process (period or seat) satisfies the due time (restriction) usually depends on the state of the process beforehand (Wight, 1974; Verzijl, 1976; Benders, 2002).

In particular, in the case of the risk, which depends on a situation of a past process (for instance, the case of a production line for a multi-period), how to assign machines, workers, or jobs to be the most efficient and economical is a problem (optimal assignment) in load/risk planning (Bergamashi *et al*., 1997; Swamidass, 2000).

Yamamoto *et al*. (2006; 2007) investigated the optimal arrangement to minimize the total expected risk of such a situation. They studied the optimal

* Project partially supported by the Ministry of Education, Science, Sports and Culture, and a Grant-in-Aid for Scientific Research (C), 20510130 in 2008, Japan

switching period to minimize the total expected cost of the production process. In this paper, first, the optimal switching problem is systematically classified and the corresponding model is presented. Next, the mathematical formulation of the total expectation for the above case is proposed. Finally, the policy of optimal switching period is investigated in detail by numerical experiments.

2 Review of the limited-cycle problem with multiple periods

Before describing a limited-cycle problem with multiple periods, we consider a single limited-cycle problem. If the production time of one period is T and the due time is Z, then the risks due to the length of the production time would occur, which are the risks by $T\leq Z$ and $T>Z$. In such situations, it can be noted that there is a trade-off problem in the two risks. This problem is shown as

$$\min_{z}\{\beta_1(Z)P(T\leq Z)+\beta_2(Z)P(T>Z)\}, \qquad (1)$$

where $\beta_1(Z)$ is the risk by $T\leq Z$, and $\beta_2(Z)$ is the risk by $T>Z$. This kind of problem is known as various problems of the reliability field, the problem of due-time restriction and the newsboy problem (a well-known stock problem) (Week, 1979). A detailed model of a single limited-cycle is shown in Matsui (2005).

This paper considers cases in which the above two risks not only occur in the single period, but also in multiple periods repeatedly. The problem of minimizing the expected risk in such a situation is a limited-cycle problem with multiple periods. The multi-period problem could be classified according to whether the periods are independent or not (Fig. 1).

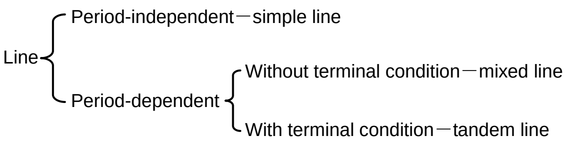

Fig. 1 Classification of multi limited-cycle problems

For this problem, one result is the general form of production rate and waiting time by a station-centered approach as discussed in Matsui *et al.* (1977) and Matsui (2004). The explicit form is obvious and consists of the product form in the period-independent case, such as a single line, but it is untraceable in the period-dependent case such as a mixed or tandem line. The mixed line has an absorbing barrier, but the tandem line has a reflective barrier at the end. This study presents a cost approach for the latter.

3 Optimal switching model

3.1 Optimal switching problem

In the multi-period production process, one process delay may influence the delivery date of an entire process. We consider controlling the production process by switching the processing rate to a faster one at a given point (time or period). The optimal switching problem occurs when the processing rate should be switched to minimize the total expected cost. In this study, we propose the optimal switching time to minimize the total expected cost of the production process. In our proposed model, when the switching time is kT, if process k is complete at time kT, the processing rate is not changed; and, if process k is not yet finished at kT, the processing rate is changed at the next process. Fig. 2 shows an example of a switching time model in multiple periods.

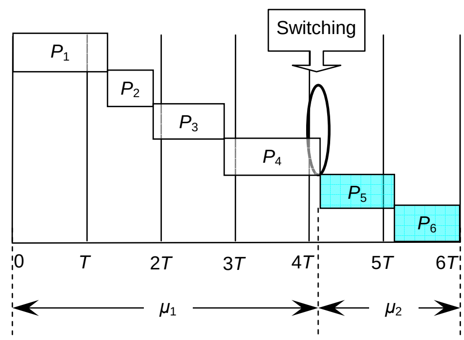

Fig. 2 Example of a switching time model

Period number n=6, switching time kT=4T, μ_1 is the usual processing rate, and μ_2 is the emergency processing rate

3.2 Assumptions and notations

The optimal switching model for the multi limited-cycle is considered based on the following assumptions:

1. One product is made by an entire process with n processes.

2. For i=1, 2, …, n, the production time of process i is denoted by T_i which is assumed to be exponentially distributed and statistically independent. The usual processing rate is μ_1, and the emergency processing rate is μ_2.

3. For i=1, 2, …, n, the target production time of process i is denoted by U_i, and the due time of the entire process (n periods) is U_n.

4. kT is the switching time.

5. The cost per unit time ($C_s^{(1)}$) occurs when a process is executed before the target production time of the process.

6. The cost per unit time ($C_p^{(1)}$) occurs when a process is executed after the target production time of the process.

7. When $X_i=\sum_{l=1}^{i}T_l > U_n$, the delay cost C_p occurs.

8. When $X_i=\sum_{l=1}^{i}T_l < U_n$, the idle cost C_s occurs.

Some notations are also defined: for i=1, 2, …, n; $C(T_1, T_2, \ldots, T_n)$ is the total cost of the production process; $C(i)$ is the production cost of period i; T_i is the production time of period i; X_i is the production time of i periods ($X_i=\sum_{l=1}^{i}T_l$); $P(X_n>U_n)$ is the probability of delay; and, $P(X_n\le U_n)$ is the probability of idle.

3.3 Problem formulation

From Assumptions 1–7 mentioned in Section 3.2, we can easily obtain Eq. (2):

$$E\left[C(k;T_1,T_2,\ldots,T_n)\right]=\sum_{i=1}^{n}E\left[C(i)\right]+C_pP\left\{\sum_{l=1}^{n}T_l>U_n\right\}+C_sP\left\{\sum_{l=1}^{n}T_l\le U_n\right\},\tag{2}$$

where $E[C(i)]$ is the expected cost of period i, $C_pP\left\{\sum_{l=1}^{n}T_l>U_n\right\}$ is the delayed expected cost, and $C_sP\left\{\sum_{l=1}^{n}T_l\le U_n\right\}$ is the idle expected cost.

For i=1, 2, …, k,

$$\begin{aligned}C(i)&=C_p^{(1)}S_i+C_s^{(1)}(T_i-S_i)\\&=(C_p^{(1)}-C_s^{(1)})S_i+C_s^{(1)}T_i,\end{aligned}\tag{3}$$

then

$$E[C(i)]=E[(C_p^{(1)}-C_s^{(1)})S_i+C_s^{(1)}T_i].\tag{4}$$

For i=k+1, k+2, …, n,

$$C(i)=\begin{cases}(C_p^{(1)}-C_s^{(1)})S_i+C_s^{(1)}T_i, & \sum_{i=1}^{k}T_i\le kT,\\(C_p^{(2)}-C_s^{(2)})S_i+C_s^{(2)}T_i, & \sum_{i=1}^{k}T_i> kT,\end{cases}\tag{5}$$

then

$$\begin{aligned}E[C(i)]=&(C_p^{(1)}-C_s^{(1)})E\left[S_i\,\middle|\,\sum_{i=1}^{k}T_i\le U_k\right]P\left\{\sum_{i=1}^{k}T_i\le U_k\right\}\\&+(C_p^{(2)}-C_s^{(2)})E\left[S_i\,\middle|\,\sum_{i=1}^{k}T_i> U_k\right]P\left\{\sum_{i=1}^{k}T_i> U_k\right\}\\&+\frac{C_s^{(1)}}{\mu_1}P\left\{\sum_{i=1}^{k}T_i\le U_k\right\}+\frac{C_s^{(2)}}{\mu_2}P\left\{\sum_{i=1}^{k}T_i> U_k\right\}.\end{aligned}\tag{6}$$

From Assumption 3 and the relationship between exponential distribution and Poisson distribution, the items in Eq. (2) are given in Eqs. (7)–(9):

$$\begin{aligned}\sum_{i=1}^{n}E[C(i)]=&\frac{k}{\mu_1}C_s^{(1)}+\frac{n-k}{\mu_1}C_s^{(1)}\left(1-\sum_{l=0}^{k-1}\frac{(\mu_1U_k)^l}{l!}\mathrm{e}^{-\mu_1U_k}\right)+\frac{n-k}{\mu_2}C_s^{(2)}\sum_{l=0}^{k-1}\frac{(\mu_1U_k)^l}{l!}\mathrm{e}^{-\mu_1U_k}\\&+\frac{(C_p^{(1)}-C_s^{(1)})}{\mu_1}\left(\sum_{i=1}^{k}\sum_{l=0}^{i-1}\frac{(\mu_1U_i)^l}{l!}\mathrm{e}^{-\mu_1U_i}+\sum_{i=k+1}^{n}\mathrm{e}^{-\mu_1U_i}\sum_{l=k}^{i-1}\sum_{l_1=k}^{l}\frac{(\mu_1U_k)^{l_1}}{l_1!}\frac{(\mu_1(U_i-U_k))^{l-l_1}}{(l-l_1)!}\right)\\&+\frac{(C_p^{(2)}-C_s^{(2)})}{\mu_2}\left(\sum_{i=k+1}^{n}\sum_{l=k}^{i-1}\int_{U_k}^{U_i}\frac{(\mu_1x)^{k-1}}{(k-1)!}\mathrm{e}^{-\mu_1x}\frac{\{\mu_2(U_i-x)\}^{l-k}}{(l-k)!}\mathrm{e}^{-\mu_2(U_i-x)}\mu_1\mathrm{d}x+\sum_{i=k+1}^{n}\sum_{l=0}^{k-1}\frac{(\mu_1U_i)^l}{l!}\mathrm{e}^{-\mu_1U_i}\right),\end{aligned}\tag{7}$$

$$P\left\{\sum_{l=1}^{n}T_l>U_n\right\}=\mathrm{e}^{-\mu_1U_n}\sum_{l_1=k}^{n-1}\sum_{l_2=0}^{n-l_1-1}\frac{(\mu_1U_k)^{l_1}\left(\mu_1(U_n-U_k)\right)^{l_2}}{l_1!l_2!}$$
$$+\frac{\mu_1^k\mathrm{e}^{-\mu_2U_n}}{(k-1)!}\sum_{l=0}^{n-k-1}\int_{U_k}^{U_n}\frac{(x_k)^{k-1}\left(\mu_2(U_n-x_k)\right)^l}{l!}\mathrm{e}^{-(\mu_1-\mu_2)x_k}\mathrm{d}x_k$$
$$+\sum_{l=0}^{k-1}\frac{(\mu_1U_n)^l}{l!}\mathrm{e}^{-\mu_1U_n}, \tag{8}$$

$$P\left\{\sum_{l=1}^{n}T_l\le U_n\right\}$$
$$=\mathrm{e}^{-\mu_1U_n}\sum_{l_1=k}^{\infty}\sum_{l_2=\max(n-l_1,0)}^{\infty}\mu_1^{l_1+l_2}\frac{U_k^{l_1}}{l_1!}\frac{(U_n-U_k)^{l_2}}{l_2!}$$
$$+\frac{\mu_1^k\mathrm{e}^{-\mu_2U_n}}{(k-1)!}\sum_{l=n-k}^{\infty}\int_{U_k}^{U_n}\frac{(x_k)^{k-1}(\mu_2(U_n-x_k))^l}{l!}\mathrm{e}^{-(\mu_1-\mu_2)x_k}\mathrm{d}x_k. \tag{9}$$

4 Experimental consideration

In this section, we obtain the optimal switching time to minimize the total expected cost by numerical experiments, where $C_s^{(1)}=1$, $C_p^{(1)}=2$, $C_s^{(2)}=3$, $C_p^{(2)}=6$, T=5, n=10, C_s=1, and C_p=200.

4.1 Behavior of the optimal switching time by the usual processing rate

Table 1 shows the behavior of the optimal switching time by changing the usual processing rate. It can be noted that the rate should be switched to a faster one ahead of time when the usual processing speed is slow. This is because the delay cost can be decreased.

Table 1 Behavior of the optimal switching time by changing the usual processing rate (μ_2=0.6)

Switching period (T)	Usual processing rate (μ_1)				
	0.1	0.2	0.3	0.4	0.5
1	205.732	133.340	74.614	54.799	45.920
2	**204.118**	128.443	71.638	51.398	42.767
3	223.568	**126.771**	69.310	49.338	41.399
4	252.501	127.851	67.674	48.127	40.804
5	283.311	131.702	66.692	47.428	40.545
6	311.389	138.180	**66.354**	47.044	40.433
7	334.845	146.821	66.663	46.865	40.387
8	353.647	156.830	67.601	**46.821**	40.371
9	368.644	167.009	69.064	46.870	**40.367**
10	380.541	174.092	70.356	46.942	40.368

It can be noted from the bold numbers that when usual processing rates are 0.1, 0.2, 0.3, 0.4, and 0.5, the optimal switching times are 2T, 3T, 6T, 8T, and 9T, respectively

4.2 Behavior of the optimal switching time by the emergency processing rate

Table 2 shows the behavior of the optimal switching time by changing the emergency processing rate. It can be noted that the rate should be switched to a faster one ahead of time when the emergency processing speed is slow.

Table 2 Behavior of the optimal switching time by changing the emergency processing rate (μ_1=0.2)

Switching period (T)	Emergency processing rate (μ_2)				
	0.3	0.4	0.5	0.8	1.0
1	177.874	152.347	140.433	124.839	119.807
2	177.871	149.745	136.224	119.513	114.375
3	177.411	**149.305**	**135.010**	**117.574**	**112.446**
4	177.128	150.694	136.348	118.379	113.183
5	177.049	153.608	140.056	122.197	116.960
6	177.071	157.676	145.815	129.204	124.160
7	177.035	162.443	153.097	139.160	134.739
8	176.748	167.373	161.163	151.320	148.036
9	175.998	171.824	169.022	164.327	162.647
10	**174.092**	174.092	174.092	174.092	174.092

It can be noted from the bold numbers that when usual processing rates are 0.3, 0.4, 0.5, 0.8, and 1.0, the optimal switching times are 10T, 3T, 3T, 3T, and 3T, respectively

5 Conclusions

In this paper, we considered an optimal switching problem to minimize the total expected cost in limited-cycle with multiple periods and investigated behaviors of the optimal switching period. For explaining or applying to more problems in real world, we should consider analytical procedure for optimal switching periods and limited-cycle problems with more than one switching periods, as our future works.

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