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Optimal linear attitude estimators via geometric analysis

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Abstract: Three optimal linear attitude estimators are proposed for single-point real-time estimation of spacecraft attitude using a geometric approach. The final optimal attitude is represented by modified Rodrigues parameters (MRPs). After introducing incidental right-hand orthogonal coordinates for each pair of measured values, three error vectors are obtained by the use of dot or/and cross products. Corresponding optimality criteria are rigorously quadratic and unconstrained, which do not coincide with Wahba's constrained criterion. The singularity, which occurs when the principal angle is close to π, can be easily avoided by one proper rotation. Numerical simulations show that the proposed three optimal linear estimators can provide a precision comparable with those complying with the Wahba optimality definition, and have faster computational speed than the famous quaternion estimator (QUEST).

Key words: Linear attitude determination, Geometric analysis, Gibbs vector, Singularity avoidance
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1 Introduction

Wahba (1965) formulated the attitude determination problem using vector observations as a least-squares estimation problem which seeks the proper orthogonal matrix (attitude matrix or direction-cosine matrix (DCM)) $\boldsymbol{A}$ that minimizes the cost function:

$$J(\boldsymbol{A})=\frac{1}{2}\sum_{i=1}^{n}\xi_i \|\boldsymbol{b}_i-\boldsymbol{A}\boldsymbol{r}_i\|^2, \quad (1)$$

where the sequence $\boldsymbol{b}_i$ (i=1, 2, ⋯, n) of unit vectors are the results of measurements performed in vehicle Cartesian coordinates of the directions to known objects. The sequence $\boldsymbol{r}_i$ (i=1, 2, ⋯, n) of unit vectors are the corresponding unit vectors resolved in a reference Cartesian coordinate system and ξ_i (i=1, 2, ⋯, n) are a set of positive weights such that $\sum_{i=1}^{n}\xi_i=1$.

The Wahba problem has been solved in a number of ways which generally fall into two categories, single-frame (SF) algorithms and recursive algorithms. The method of SF algorithm is a set, and only this set, of simultaneous measurements at time k is used to estimate the quaternion. Cheng and Shuster (2007) analyzed numbers of SF algorithms with two classes. First, obtain the attitude profile matrix $\boldsymbol{B}$ or the Davenport matrix $\boldsymbol{K}$ by using algorithms of numerical linear algebra (Markley and Mortari, 1999; 2000). Such examples can be found by using a polar-decomposition method (Farrell *et al.*, 1966), the famous Davenport q-method (Shuster, 1978; Shuster and OH, 1981), and the singular value decomposition (SVD) algorithm (Markley, 1988). The other solution methods are a kind of iterative solutions which seek $\lambda_{\max}$, the maximum characteristic value of the Davenport matrix $\boldsymbol{K}$ by some methods especially designed for the Wahba problem. Examples of those methods can be referred to Shuster (1978)'s well-known quaternion estimator (QUEST), fast optimal attitude matrix (FOAM) (Markley, 1993), and estimators of the optimal quaternion (ESOQ, ESOQ2) algorithms (Mortari, 1997a; 1997b; 2000; Markley and Mortari,

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1999). Cheng and Shuster (2007) also provided detailed analysis of those algorithms mentioned above in respect of computational efficiency. Of the most interesting one is Mortari *et al*. (2000; 2007)'s optimal linear attitude estimator (OLAE) using Rodrigues (or Gibbs) vector, of which optimality criterion is based on Cayler transformation between the attitude matrix and Rodrigues vector. In summary, any SF attitude determination estimator is a batch estimator where the information contained in past measurements is lost.

The recursive forms of some estimators are proposed since they use the past data without requiring its storage, and allow real-time processing of new incoming observations. Shuster (1989) proposed an implementation of the Wahba problem for dynamic systems as a sequential filter and smoother, known as filter QUEST and smoother QUEST. Bar-Itzhack (1996) provided a recursive modification of QUEST algorithm for sequential attitude determination which is called REQUEST. The most recent research results in this field include: Choukroun *et al*. (2004) proposed an optimal-REQUEST, which is an optimal recursive time-varying estimator of the quaternion, Shuster (2009) analyzed the relationship of filter QUEST and recursive QUEST in detail, and Choukroun (2009) proposed a quaternion estimation using Kalman filtering of the vectorized $\boldsymbol{K}$-matrix.

Motivated by Mortari *et al.* (2000; 2007)'s work on linear attitude estimator, this paper proposes three optimal attitude estimation algorithms to solve the single-point attitude determination problem. This is achieved with the use of dot products and cross products which represent the attitude by Gibbs vector. Unlike Wahba (1965)'s optimality criterion, the three new optimality criteria, which are rigorously quadratic and unconstrained, have linear estimators. The final results of linear attitude estimators are represented by modified Rodrigues parameters (MRPs).

2 Optimal linear attitude estimators

2.1 Geometric analysis of rotation

Based on Euler's theorem (Angeles, 1988), which states that any rotation of a body (or coordinate system) with respect to another may be described by a single rotation through some angle about a single fixed axis, the direction cosine matrix $\boldsymbol{A}$ can be represented as

$$\boldsymbol{A}(\boldsymbol{a},\theta)=\cos\theta\boldsymbol{I}+(1-\cos\theta)\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}-\sin\theta\boldsymbol{a}^{\times}, \quad (2)$$

where $\boldsymbol{I}$ denotes the 3×3 identity matrix, $\boldsymbol{a}$ denotes the axis of rotation, θ denotes the angle of rotation, and $\boldsymbol{a}^{\times}$ denotes a skew-symmetric matrix generated by $\boldsymbol{a}$:

$$\boldsymbol{a}^{\times}=\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}. \quad (3)$$

According to the concept of attitude kinematics, the relationship between the body and reference vectors can be expressed as

$$\boldsymbol{b}_i=\boldsymbol{A}(\boldsymbol{a},\theta)\boldsymbol{r}_i. \quad (4)$$

Premultiplying Eq. (4) by $\boldsymbol{a}^{\mathrm{T}}$, it shows that

$$\begin{aligned}\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}_i&=\boldsymbol{a}^{\mathrm{T}}\boldsymbol{A}(\boldsymbol{a},\theta)\boldsymbol{r}_i\\&=\boldsymbol{a}^{\mathrm{T}}\left[\cos\theta\boldsymbol{I}+(1-\cos\theta)\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}-\sin\theta\boldsymbol{a}^{\times}\right]\boldsymbol{r}_i\\&=\boldsymbol{a}^{\mathrm{T}}\boldsymbol{r}_i.\end{aligned} \quad (5)$$

This means that the angle between $\boldsymbol{b}_i$ and $\boldsymbol{a}$ is equivalent to that between $\boldsymbol{r}_i$ and $\boldsymbol{a}$. Moreover, the dot product of $\boldsymbol{r}_i$ and $\boldsymbol{b}_i$ can be given by

$$\boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{b}_i=\boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{A}(\boldsymbol{a},\theta)\boldsymbol{r}_i=\cos\theta+(1-\cos\theta)(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{r}_i)^2. \quad (6)$$

Substituting Eq. (5) into Eq. (6), it is shown that

$$\begin{aligned}\cos\theta&=\frac{\boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{b}_i-(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{r}_i)^2}{1-(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{r}_i)^2}\\&=\frac{[\boldsymbol{b}_i-(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}_i)\boldsymbol{a}]^{\mathrm{T}}[\boldsymbol{r}_i-(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{r}_i)\boldsymbol{a}]}{\|\boldsymbol{b}_i-(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}_i)\boldsymbol{a}\|\cdot\|\boldsymbol{r}_i-(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{r}_i)\boldsymbol{a}\|}.\end{aligned} \quad (7)$$

This indicates that θ is just the angle between $[\boldsymbol{b}_i-(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}_i)\boldsymbol{a}]$ and $[\boldsymbol{r}_i-(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{r}_i)\boldsymbol{a}]$. Eqs. (5) and (7) implicated that $\boldsymbol{r}_i$ can be obtained by rotating the vector $\boldsymbol{b}_i$ through rotation angle θ about the axis $\boldsymbol{a}$ in a constant reference coordinate system, as illustrated in Fig. 1.

For each pair of measured values $\{\boldsymbol{r}_i, \boldsymbol{b}_i\}$, we define a corresponding right-hand orthogonal coordinate system as follows:

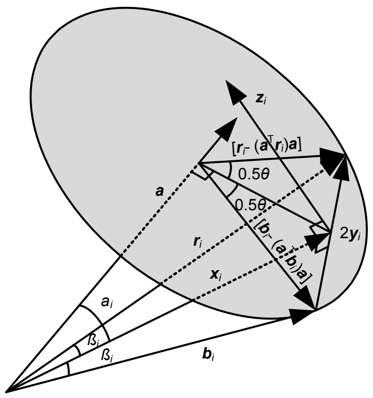

Fig. 1 Geometric relationship of rotation

$$\begin{aligned} \boldsymbol{x}_i &= (\boldsymbol{r}_i + \boldsymbol{b}_i)/2, \\ \boldsymbol{y}_i &= (\boldsymbol{r}_i - \boldsymbol{b}_i)/2, \\ \boldsymbol{z}_i &= \boldsymbol{x}_i^{\times} \boldsymbol{y}_i. \end{aligned} \tag{8}$$

The directions defined in Eq. (8) may not be unit vectors, and their magnitudes are given by

$$\begin{aligned} x_i &= |\boldsymbol{x}_i|, \\ y_i &= |\boldsymbol{y}_i|, \\ z_i &= |\boldsymbol{z}_i|. \end{aligned} \tag{9}$$

According to Eq. (5), we have $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{y}_i=0$, i.e., the axis of rotation $\boldsymbol{a}$ has non-zero components only along $\boldsymbol{x}_i$ and $\boldsymbol{z}_i$ coordinates. Thus $\boldsymbol{a}$ can be written into the following form:

$$\boldsymbol{a} = \cos\alpha_i \frac{\boldsymbol{x}_i}{x_i} + \sin\alpha_i \frac{\boldsymbol{z}_i}{z_i}, \tag{10}$$

where α_i denotes the angle between $\boldsymbol{x}_i$ and $\boldsymbol{a}$.

Note that the relationship among angles α_i, β_i and θ, as shown in Fig. 1, can be summarized as

$$\sin\alpha_i \tan\frac{\theta}{2} = \tan\beta_i = \frac{y_i}{x_i}, \tag{11}$$

where β_i denotes the angle between $\boldsymbol{x}_i$ and $\boldsymbol{b}_i$.

The Gibbs vector, which is widely used in attitude representation, is defined as

$$\boldsymbol{g} = \tan\frac{\theta}{2}\boldsymbol{a}. \tag{12}$$

Substituting Eqs. (10) and (11) into Eq. (12) yields

$$\boldsymbol{g} = \cot\alpha_i \frac{y_i}{x_i^2}\boldsymbol{x}_i + \frac{y_i}{x_i z_i}\boldsymbol{z}_i. \tag{13}$$

This means that $\boldsymbol{g}$ has known components projected in $\boldsymbol{y}_i$ coordinate and $\boldsymbol{z}_i$ coordinate. This property is very important and will be used in the following linear attitude estimators.

2.2 Three optimal linear attitude estimators

2.2.1 The first optimal linear attitude estimator

The first optimal linear attitude estimator (OLAE_1) can be obtained using dot products to eliminate the unknown angle α_i, from Eq. (13). Premultiplying Eq. (13) by $\boldsymbol{y}_i^{\mathrm{T}}$ and $x_i\boldsymbol{z}_i^{\mathrm{T}}$, respectively, the following can be developed:

$$\begin{cases} \boldsymbol{y}_i^{\mathrm{T}}\boldsymbol{g} = 0, \\ x_i\boldsymbol{z}_i^{\mathrm{T}}\boldsymbol{g} - y_i z_i = 0. \end{cases} \tag{14}$$

Eq. (14) cannot be satisfied by all measurements, due to the presence of sensor noise. In this study, it is supposed that the observed body vectors are given by

$$\tilde{\boldsymbol{b}}_i = \boldsymbol{b}_i + \boldsymbol{v}_i, \tag{15}$$

where $\boldsymbol{v}_i$ (i=1, 2, $\cdots$, n) is a zero-mean white-noise vector. Thus, the error vector for a given Gibbs vector $\boldsymbol{g}$ can be chosen as

$$\tilde{\boldsymbol{e}}_{1i} = \begin{bmatrix} \tilde{\boldsymbol{y}}_i^{\mathrm{T}}\boldsymbol{g} \\ \tilde{x}_i\tilde{\boldsymbol{z}}_i^{\mathrm{T}}\boldsymbol{g} - \tilde{y}_i\tilde{z}_i \end{bmatrix}, \tag{16}$$

where

$$\begin{aligned} \tilde{\boldsymbol{x}}_i &= (\boldsymbol{r}_i + \tilde{\boldsymbol{b}}_i)/2, & \tilde{x}_i &= |\tilde{\boldsymbol{x}}_i|, \\ \tilde{\boldsymbol{y}}_i &= (\boldsymbol{r}_i - \tilde{\boldsymbol{b}}_i)/2, & \tilde{y}_i &= |\tilde{\boldsymbol{y}}_i|, \\ \tilde{\boldsymbol{z}}_i &= \tilde{\boldsymbol{x}}_i^{\times}\tilde{\boldsymbol{y}}_i, & \tilde{z}_i &= |\tilde{\boldsymbol{z}}_i|. \end{aligned} \tag{17}$$

This allows us to introduce a new optimality criterion for spacecraft attitude. The optimal attitude estimate is defined as finding a Gibbs vector $\boldsymbol{g}$, which minimizes the following quadratic cost function:

$$J_1 = \frac{1}{2}\sum_{i=1}^{n}\xi_i \left\|\tilde{\boldsymbol{e}}_{1i}\right\|^2, \tag{18}$$

where ξ_i (i=1, 2, ⋯, n) denote a set of relative positive weights. Substituting Eqs. (16) and (17) into Eq. (18), the first optimality criterion expression can be obtained in terms of attitude and observations' vectors:

$$\begin{aligned} J_1 &= \frac{1}{2}\sum_{i=1}^{n}\xi_i\Big[\boldsymbol{g}^{\mathrm{T}}(\tilde{\boldsymbol{y}}_i\tilde{\boldsymbol{y}}_i^{\mathrm{T}} + \tilde{x}_i^2\tilde{\boldsymbol{z}}_i\tilde{\boldsymbol{z}}_i^{\mathrm{T}})\boldsymbol{g} \\ &\quad - 2(\tilde{x}_i\tilde{y}_i\tilde{z}_i)\boldsymbol{g}^{\mathrm{T}}\tilde{\boldsymbol{z}}_i + \tilde{y}_i^2\tilde{z}_i^2\Big] \\ &= \frac{1}{16}\sum_{i=1}^{n}\xi_i\Big\{2\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{r}_i - \tilde{\boldsymbol{b}}_i)(\boldsymbol{r}_i - \tilde{\boldsymbol{b}}_i)^{\mathrm{T}}\boldsymbol{g} \\ &\quad - (1+\boldsymbol{r}_i^{\mathrm{T}}\tilde{\boldsymbol{b}}_i)\boldsymbol{g}^{\mathrm{T}}\tilde{\boldsymbol{b}}_i^{\times}\boldsymbol{r}_i\boldsymbol{r}_i^{\mathrm{T}}\tilde{\boldsymbol{b}}_i^{\times}\boldsymbol{g} - 2\Big[1-(\boldsymbol{r}_i^{\mathrm{T}}\tilde{\boldsymbol{b}}_i)^2\Big]\boldsymbol{g}^{\mathrm{T}}\tilde{\boldsymbol{b}}_i^{\times}\boldsymbol{r}_i \\ &\quad + (1-\boldsymbol{r}_i^{\mathrm{T}}\tilde{\boldsymbol{b}}_i)^2(1+\boldsymbol{r}_i^{\mathrm{T}}\tilde{\boldsymbol{b}}_i)\Big\}. \end{aligned} \tag{19}$$

Define

$$\begin{aligned} \tilde{\boldsymbol{M}}_1 &= \sum_{i=1}^{n}\xi_i\Big[2(\boldsymbol{r}_i - \tilde{\boldsymbol{b}}_i)(\boldsymbol{r}_i - \tilde{\boldsymbol{b}}_i)^{\mathrm{T}} - (1+\boldsymbol{r}_i^{\mathrm{T}}\tilde{\boldsymbol{b}}_i)\tilde{\boldsymbol{b}}_i^{\times}\boldsymbol{r}_i\boldsymbol{r}_i^{\mathrm{T}}\tilde{\boldsymbol{b}}_i^{\times}\Big], \\ \tilde{\boldsymbol{\upsilon}}_1 &= \sum_{i=1}^{n}\xi_i[1-(\boldsymbol{r}_i^{\mathrm{T}}\tilde{\boldsymbol{b}}_i)^2]\tilde{\boldsymbol{b}}_i^{\times}\boldsymbol{r}_i, \end{aligned} \tag{20}$$

and thus the first optimality criterion can be rewritten as

$$J_1 = \frac{1}{16}\Big[\boldsymbol{g}^{\mathrm{T}}\tilde{\boldsymbol{M}}_1\boldsymbol{g} - 2\boldsymbol{g}^{\mathrm{T}}\boldsymbol{\upsilon}_1 + (1-\boldsymbol{r}_i^{\mathrm{T}}\tilde{\boldsymbol{b}}_i)^2(1+\boldsymbol{r}_i^{\mathrm{T}}\tilde{\boldsymbol{b}}_i)\Big]. \tag{21}$$

Since this is an unconstrained minimization, stationarity conditions to minimize Eq. (21) yield

$$\frac{\partial J_1}{\partial \boldsymbol{g}} = \frac{1}{8}(\tilde{\boldsymbol{M}}_1\boldsymbol{g} - \tilde{\boldsymbol{\upsilon}}_1) = 0. \tag{22}$$

Note that $\tilde{\boldsymbol{M}}_1$, which is just the second derivative of the cost function J_1 with respect to $\boldsymbol{g}$, is a real symmetric positive definite matrix for $n\geq 2$. Therefore, the global minimum of J_1 can be given by

$$\tilde{\boldsymbol{M}}_1\hat{\boldsymbol{g}}_1 = \tilde{\boldsymbol{\upsilon}}_1, \tag{23}$$

where $\hat{\boldsymbol{g}}_1$ denotes the optimal Gibbs vector as the result of OLAE_1.

According to the relationship between the Gibbs vector and the MRP vector, the first optimal MRP vector $\hat{\boldsymbol{\sigma}}_1$ can be written as

$$\hat{\boldsymbol{\sigma}}_1 = \frac{1}{1+\sqrt{1+\hat{\boldsymbol{g}}_1^{\mathrm{T}}\hat{\boldsymbol{g}}_1}}\hat{\boldsymbol{g}}_1. \tag{24}$$

2.2.2 The second optimal linear attitude estimator

The second optimal linear attitude estimator (OLAE_2) can be achieved using cross products instead of dot products to remove the unknown angle α_i from Eq. (13). Premultiplying Eq. (13) by $\boldsymbol{x}_i^{\times}/x_i$, it is shown that

$$\boldsymbol{x}_i^{\times}\boldsymbol{g} + \boldsymbol{y}_i = 0. \tag{25}$$

With a similar method as used above, the error vector for a given Gibbs vector $\boldsymbol{g}$ can be defined as

$$\tilde{\boldsymbol{e}}_{2i} = \boldsymbol{x}_i^{\times}\boldsymbol{g} + \boldsymbol{y}_i, \tag{26}$$

and the second new optimality criterion can be given by

$$\begin{aligned} J_2 &= \frac{1}{2}\sum_{i=1}^{n}\xi_i\left\|\tilde{\boldsymbol{e}}_{2i}\right\|^2 \\ &= \frac{1}{2}\sum_{i=1}^{n}\xi_i(-\boldsymbol{g}^{\mathrm{T}}\tilde{\boldsymbol{x}}_i^{\times}\tilde{\boldsymbol{x}}_i^{\times}\boldsymbol{g} + 2\boldsymbol{g}^{\mathrm{T}}\tilde{\boldsymbol{x}}_i^{\times}\tilde{\boldsymbol{y}}_i + \tilde{y}_i^2) \\ &= \frac{1}{8}\sum_{i=1}^{n}\xi_i\Big[-\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{r}_i + \tilde{\boldsymbol{b}}_i)^{\times}(\boldsymbol{r}_i + \tilde{\boldsymbol{b}}_i)^{\times}\boldsymbol{g} \\ &\qquad -4\boldsymbol{g}^{\mathrm{T}}\tilde{\boldsymbol{b}}_i^{\times}\boldsymbol{r}_i + 2(1-\tilde{\boldsymbol{b}}_i^{\mathrm{T}}\boldsymbol{r}_i)\Big]. \end{aligned} \tag{27}$$

The optimal Gibbs vector $\hat{\boldsymbol{g}}_2$ and MRP vector $\hat{\boldsymbol{\sigma}}_2$ of OLAE_2 are obtained:

$$\tilde{\boldsymbol{M}}_2\hat{\boldsymbol{g}}_2 = \tilde{\boldsymbol{\upsilon}}_2,\ \ \hat{\boldsymbol{\sigma}}_2 = \frac{1}{1+\sqrt{1+\hat{\boldsymbol{g}}_2^{\mathrm{T}}\hat{\boldsymbol{g}}_2}}\hat{\boldsymbol{g}}_2, \tag{28}$$

where

$$\tilde{\boldsymbol{M}}_2 = -\sum_{i=1}^{n}\xi_i(\boldsymbol{r}_i + \tilde{\boldsymbol{b}}_i)^{\times}(\boldsymbol{r}_i + \tilde{\boldsymbol{b}}_i)^{\times},\ \tilde{\boldsymbol{\upsilon}}_2 = 2\sum_{i=1}^{n}\xi_i\tilde{\boldsymbol{b}}_i^{\times}\boldsymbol{r}_i. \tag{29}$$

Note that OLAE_2 is equivalent to Mortari *et al.* (2000; 2007)'s OLAE, because both of them have the same error vector and optimality criterion as shown in Eqs. (26) and (27), respectively.

2.2.3 The third optimal linear attitude estimator

The third optimal linear attitude estimator (OLAE$_3$) can be developed using both dot products and cross products to remove α_i from Eq. (13). The corresponding error vector $\tilde{\boldsymbol{e}}_{3i}$ can be selected as

$$\tilde{\boldsymbol{e}}_{3i} = \begin{bmatrix} \tilde{\boldsymbol{e}}_{1i} \\ \tilde{\boldsymbol{e}}_{2i} \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{y}}_i^{\mathrm{T}} \boldsymbol{g} \\ \tilde{x}_i \tilde{\boldsymbol{z}}_i^{\mathrm{T}} \boldsymbol{g} - \tilde{y}_i \tilde{z}_i \\ \boldsymbol{x}_i^{\times} \boldsymbol{g} + \boldsymbol{y}_i \end{bmatrix}, \tag{30}$$

and the third new optimality criterion is given by

$$J_3 = \frac{1}{2} \sum_{i=1}^{n} \xi_i \left\| \tilde{\boldsymbol{e}}_{3i} \right\|^2 = J_1 + J_2. \tag{31}$$

Substituting Eqs. (21) and (27) into Eq. (31), the third optimality criterion expression can be obtained in terms of attitude and observations' vectors:

$$\begin{aligned} J_3 = \frac{1}{16} \sum_{i=1}^{n} \xi_i \Big\{ & 4(\boldsymbol{g}^{\mathrm{T}} \boldsymbol{g})(1 - \boldsymbol{r}_i^{\mathrm{T}} \tilde{\boldsymbol{b}}_i) \boldsymbol{I} - 4 \boldsymbol{g}^{\mathrm{T}} (\boldsymbol{r}_i^{\times} \tilde{\boldsymbol{b}}_i^{\times} + \tilde{\boldsymbol{b}}_i^{\times} \boldsymbol{r}_i^{\times}) \boldsymbol{g} \\ & - \boldsymbol{g}^{\mathrm{T}} (1 + \boldsymbol{r}_i^{\mathrm{T}} \tilde{\boldsymbol{b}}_i) \tilde{\boldsymbol{b}}_i^{\times} \boldsymbol{r}_i \boldsymbol{r}_i^{\mathrm{T}} \tilde{\boldsymbol{b}}_i^{\times} \boldsymbol{g} - 2[5 - (\boldsymbol{r}_i^{\mathrm{T}} \tilde{\boldsymbol{b}}_i)^2] \boldsymbol{g}^{\mathrm{T}} \tilde{\boldsymbol{b}}_i^{\times} \boldsymbol{r}_i \\ & + (1 - \boldsymbol{r}_i^{\mathrm{T}} \tilde{\boldsymbol{b}}_i)[5 - (\boldsymbol{r}_i^{\mathrm{T}} \tilde{\boldsymbol{b}}_i)^2] \Big\}. \end{aligned} \tag{32}$$

The optimal Gibbs vector $\hat{\boldsymbol{g}}_3$ and MRP vector $\hat{\boldsymbol{\sigma}}_3$ of OLAE$_3$ are obtained:

$$\tilde{\boldsymbol{M}}_3 \hat{\boldsymbol{g}}_3 = \tilde{\boldsymbol{v}}_3, \quad \hat{\boldsymbol{\sigma}}_3 = \frac{1}{1 + \sqrt{1 + \hat{\boldsymbol{g}}_3^{\mathrm{T}} \hat{\boldsymbol{g}}_3}} \hat{\boldsymbol{g}}_3, \tag{33}$$

where

$$\begin{aligned} \tilde{\boldsymbol{M}}_3 &= \tilde{\boldsymbol{M}}_1 + 2\tilde{\boldsymbol{M}}_2 \\ &= \sum_{i=1}^{n} \xi_i \Big[4(1 - \boldsymbol{r}_i^{\mathrm{T}} \tilde{\boldsymbol{b}}_i) \boldsymbol{I} - 4(\boldsymbol{r}_i^{\times} \tilde{\boldsymbol{b}}_i^{\times} + \tilde{\boldsymbol{b}}_i^{\times} \boldsymbol{r}_i^{\times}) \\ &\qquad - (1 + \boldsymbol{r}_i^{\mathrm{T}} \tilde{\boldsymbol{b}}_i) \tilde{\boldsymbol{b}}_i^{\times} \boldsymbol{r}_i \boldsymbol{r}_i^{\mathrm{T}} \tilde{\boldsymbol{b}}_i^{\times} \Big], \\ \tilde{\boldsymbol{v}}_3 &= \tilde{\boldsymbol{v}}_1 + 2\tilde{\boldsymbol{v}}_2 = \sum_{i=1}^{n} \xi_i [5 - (\boldsymbol{r}_i^{\mathrm{T}} \tilde{\boldsymbol{b}}_i)^2] \boldsymbol{g}^{\mathrm{T}} \tilde{\boldsymbol{b}}_i^{\times} \boldsymbol{r}_i. \end{aligned} \tag{34}$$

2.3 Singularity avoidance

Due to the use of Gibbs vectors in Eqs. (23), (28), and (33), the singularity occurs in the three proposed OLAEs algorithms. In this study, we adopt the method of one prior rotation of reference directions, as similarly introduced in (Shuster and OH, 1981) for the QUEST algorithm to avoid the singularity associated with π rotation about the principal axis.

When the principal angle is π, the principal axis $\boldsymbol{a}$ locals in the plane which can be expanded from measured values $\boldsymbol{r}_i$ and $\boldsymbol{b}_i$. Note that if a new reference direction $\boldsymbol{r}_i^*$ is used instead of the old one $\boldsymbol{r}_i$, which is defined as

$$\boldsymbol{r}_i^* = \boldsymbol{A}(\boldsymbol{a}^*, \pi) \boldsymbol{r}_i = 2(\boldsymbol{a}^{*\mathrm{T}} \boldsymbol{r}_i) \boldsymbol{a}^* - \boldsymbol{r}_i, \tag{35}$$

where $\boldsymbol{a}^*$ denotes a proper unit vector, then the singularity can be easily avoided in the new computation process.

In order to avoid potential singularity, a practical implementation of such unit vector is

$$\boldsymbol{a}^* = \begin{cases} \sum_{i=1}^{n} \xi_i \dfrac{\tilde{\boldsymbol{x}}_i}{\tilde{x}_i}, & \tilde{x}_1 \ge \tilde{y}_1, \\ \sum_{i=1}^{n} \xi_i \dfrac{\tilde{\boldsymbol{y}}_i}{\tilde{y}_i}, & \text{otherwise}. \end{cases} \tag{36}$$

Therefore, it yields

$$\boldsymbol{b}_i = \boldsymbol{A} \boldsymbol{r}_i = (\boldsymbol{A} \boldsymbol{A}^{*\mathrm{T}}) \boldsymbol{A}^* \boldsymbol{r}_i, \tag{37}$$

where $\boldsymbol{A}^* = \boldsymbol{A}(\boldsymbol{a}^*, \pi)$.

With a similar method as used above, the new Gibbs vector $\hat{\boldsymbol{g}}_j^*$ and the new MRP vector $\hat{\boldsymbol{\sigma}}_j^*$ can be obtained:

$$\begin{aligned} \hat{\boldsymbol{g}}_j^* &= \tilde{\boldsymbol{M}}_j^{*-1} \tilde{\boldsymbol{v}}_j^*, \quad j = 1, 2, 3, \\ \hat{\boldsymbol{\sigma}}_j^* &= \frac{1}{1 + \sqrt{1 + \hat{\boldsymbol{g}}_j^{*\mathrm{T}} \hat{\boldsymbol{g}}_j^*}} \hat{\boldsymbol{g}}_j^*, \quad j = 1, 2, 3, \end{aligned} \tag{38}$$

where $\tilde{\boldsymbol{M}}_j^*$ and $\tilde{\boldsymbol{v}}_j^*$ have the same forms of $\tilde{\boldsymbol{M}}_j$ and $\tilde{\boldsymbol{v}}_j$ only replacing $\boldsymbol{r}_i$ by $\boldsymbol{r}_i^*$, respectively.

Inspecting the form of Eq. (37), it shows that

$$\boldsymbol{A}(\hat{\boldsymbol{\sigma}}_j, \pi) = \boldsymbol{A}(\hat{\boldsymbol{\sigma}}_j^*, \pi) \boldsymbol{A}(\boldsymbol{a}^*, \pi). \tag{39}$$

Using the relationship of DCM and MRP vector yields

$$\hat{\boldsymbol{\sigma}}_j = \hat{\boldsymbol{\sigma}}_j^* \otimes (-\boldsymbol{a}^*)^{-1} = \frac{(1-\boldsymbol{a}^{*\mathrm{T}}\boldsymbol{a}^*)\hat{\boldsymbol{\sigma}}_j^* + (1-\hat{\boldsymbol{\sigma}}_j^{*\mathrm{T}}\hat{\boldsymbol{\sigma}}_j^*)\boldsymbol{a}^* - 2\hat{\boldsymbol{\sigma}}_j^{*\times}\boldsymbol{a}^*}{1-2\hat{\boldsymbol{\sigma}}_j^{*\mathrm{T}}\boldsymbol{a}^* + \hat{\boldsymbol{\sigma}}_j^{*\mathrm{T}}\hat{\boldsymbol{\sigma}}_j^*\boldsymbol{a}^{*\mathrm{T}}\boldsymbol{a}^*}. \quad (40)$$

Therefore, the singularity, which occurs in the three proposed OLAEs, can be easily avoided by applying one proper rotation.

In fact, the determinant of $\tilde{\boldsymbol{M}}_j$ varies with the principal angle θ. In order to improve the precision of OLAE$_j$ (j=1, 2, 3) estimate, the method which is used to avoid singularity can be employed more widely. That is, if $\det\tilde{\boldsymbol{M}}_j < \det\tilde{\boldsymbol{M}}_j^*$, Eqs. (36), (38), and (40) are adopted to calculate the optimal MRP vectors $\hat{\boldsymbol{\sigma}}_j$ (j=1, 2, 3). Thus the OLAEs algorithms without singularity can be achieved as follows.

OLAEs algorithms

Step 0: Select a proper OLAE$_j$ (j=1, 2, 3).

Step 1: Define the incidental directions as Eq. (17).

Step 2: Calculate the unit vector and new reference directions as Eqs. (35) and (36).

Step 3: Define the second incidental directions as follows:

$$\tilde{\boldsymbol{x}}_i^* = (\boldsymbol{r}_i^* + \tilde{\boldsymbol{b}}_i)/2, \quad \tilde{x}_i^* = |\tilde{\boldsymbol{x}}_i^*|,$$
$$\tilde{\boldsymbol{y}}_i^* = (\boldsymbol{r}_i^* - \tilde{\boldsymbol{b}}_i)/2, \quad \tilde{y}_i^* = |\tilde{\boldsymbol{y}}_i^*|,$$
$$\tilde{\boldsymbol{z}}_i^* = \tilde{\boldsymbol{x}}_i^{*\times}\tilde{\boldsymbol{y}}_i^*, \quad \tilde{z}_i^* = |\tilde{\boldsymbol{z}}_i^*|.$$

Step 4: Calculate matrices $\tilde{\boldsymbol{M}}_j$ and $\tilde{\boldsymbol{M}}_j^*$, and vectors $\tilde{\boldsymbol{v}}_j$ and $\tilde{\boldsymbol{v}}_j^*$.

Step 5: Obtain the final optimal estimate according to the comparison of $\det\tilde{\boldsymbol{M}}_j$ and $\det\tilde{\boldsymbol{M}}_j^*$. If $\det\tilde{\boldsymbol{M}}_j \ge \det\tilde{\boldsymbol{M}}_j^*$, then

$$\hat{\boldsymbol{g}}_j = \tilde{\boldsymbol{M}}_j^{-1}\tilde{\boldsymbol{v}}_j,$$
$$\hat{\boldsymbol{\sigma}}_j = \frac{1}{1+\sqrt{1+\hat{\boldsymbol{g}}_j^{\mathrm{T}}\hat{\boldsymbol{g}}_j}}\hat{\boldsymbol{g}}_j,$$

else do as Eqs. (38) and (40).

Consider a special case that there are n (n=3) measurements with equal weights along the reference axes. Therefore, $\boldsymbol{r}_i$ takes the three values $[1, 0, 0]^{\mathrm{T}}$, $[0, 1, 0]^{\mathrm{T}}$, and $[0, 0, 1]^{\mathrm{T}}$. The determinants of matrices $\tilde{\boldsymbol{M}}_j$ and $\tilde{\boldsymbol{M}}_j^*$ (j=1, 2, 3) are plotted in Fig. 2. It is shown that OLAE$_2$ and OLAE$_3$ have higher precisions than that of OLAE$_1$, because of their absolute nonsingularity.

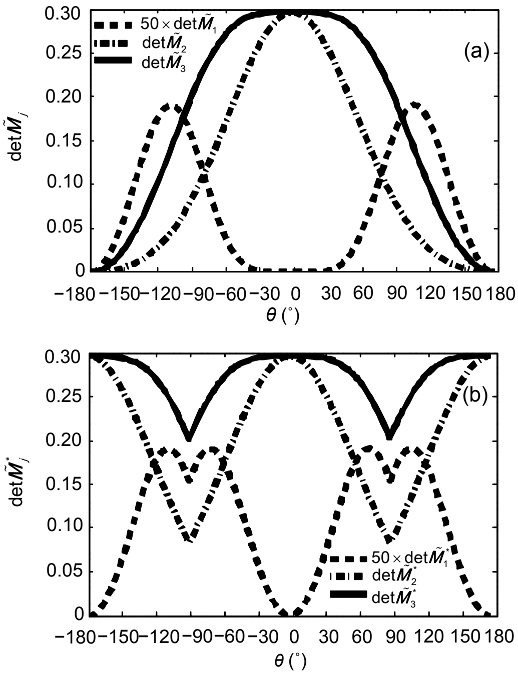

Fig. 2 Determinant plots for $\tilde{\boldsymbol{M}}_j$ with (a) and without (b) singularity

2.4 Covariance analysis

A detailed covariance analysis is presented to quantify the approximation error of OLAEs estimates. Matrices $\tilde{\boldsymbol{M}}_j$ and vectors $\tilde{\boldsymbol{v}}_j$ (j=1, 2, 3), which are defined in Eqs. (20), (29), and (34), can be rewritten (to first order in $\boldsymbol{v}_i$) as

$$\tilde{\boldsymbol{M}}_j \approx \boldsymbol{M}_j + \Delta\boldsymbol{M}_j, \quad \tilde{\boldsymbol{v}}_j \approx \boldsymbol{v}_j + \Delta\boldsymbol{v}_j, \quad (41)$$

where

$$\begin{cases} \boldsymbol{M}_1 = \sum_{i=1}^{n}\xi_i\left[2(\boldsymbol{r}_i-\boldsymbol{b}_i)(\boldsymbol{r}_i-\boldsymbol{b}_i)^{\mathrm{T}} - (1+\boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{b}_i)\boldsymbol{b}_i^{\times}\boldsymbol{r}_i\boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{b}_i^{\times}\right], \\ \Delta\boldsymbol{M}_1 = -\sum_{i=1}^{n}\xi_i\left[2(\boldsymbol{r}_i-\boldsymbol{b}_i)\boldsymbol{v}_i^{\mathrm{T}} + \boldsymbol{v}_i(\boldsymbol{r}_i-\boldsymbol{b}_i)^{\mathrm{T}} + (\boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{v}_i)\boldsymbol{b}_i^{\times}\boldsymbol{r}_i\boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{b}_i^{\times} + (1+\boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{b}_i)(\boldsymbol{v}_i^{\times}\boldsymbol{r}_i\boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{b}_i^{\times} + \boldsymbol{b}_i^{\times}\boldsymbol{r}_i\boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{v}_i^{\times})\right], \\ \boldsymbol{v}_1 = \sum_{i=1}^{n}\xi_i[1-(\boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{b}_i)^2]\boldsymbol{b}_i^{\times}\boldsymbol{r}_i, \\ \Delta\boldsymbol{v}_1 = -\sum_{i=1}^{n}\xi_i\left\{2(\boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{b}_i)\boldsymbol{b}_i^{\times}\boldsymbol{r}_i\boldsymbol{r}_i^{\mathrm{T}} + [1-(\boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{b}_i)^2]\boldsymbol{r}_i^{\times}\right\}\boldsymbol{v}_i, \end{cases} \quad (42)$$

$$\begin{cases} \boldsymbol{M}_2 = -\sum_{i=1}^{n} \xi_i (\boldsymbol{r}_i + \boldsymbol{b}_i)^\times (\boldsymbol{r}_i + \boldsymbol{b}_i)^\times, \\ \Delta\boldsymbol{M}_2 = -\sum_{i=1}^{n} \xi_i [\boldsymbol{v}_i^\times (\boldsymbol{r}_i + \boldsymbol{b}_i)^\times + (\boldsymbol{r}_i + \boldsymbol{b}_i)^\times \boldsymbol{v}_i^\times], \\ \boldsymbol{\upsilon}_2 = 2\sum_{i=1}^{n} \xi_i \boldsymbol{b}_i^\times \boldsymbol{r}_i, \\ \Delta\boldsymbol{\upsilon}_2 = 2\sum_{i=1}^{n} \xi_i \boldsymbol{v}_i^\times \boldsymbol{r}_i, \end{cases} \tag{43}$$

$$\begin{cases} \boldsymbol{M}_3 = \boldsymbol{M}_1 + 2\boldsymbol{M}_2, \\ \Delta\boldsymbol{M}_3 = \Delta\boldsymbol{M}_1 + 2\Delta\boldsymbol{M}_2, \\ \boldsymbol{\upsilon}_3 = \boldsymbol{\upsilon}_1 + 2\boldsymbol{\upsilon}_2, \\ \Delta\boldsymbol{\upsilon}_3 = \Delta\boldsymbol{\upsilon}_1 + 2\Delta\boldsymbol{\upsilon}_2. \end{cases} \tag{44}$$

The exact Gibbs vector $\boldsymbol{g}$ is given by

$$\boldsymbol{g} = \boldsymbol{M}_j^{-1}\boldsymbol{\upsilon}_j,\ j = 1, 2, 3. \tag{45}$$

Thus the OLAE$_j$ estimate $\hat{\boldsymbol{g}}_j$ can be rewritten (to first order in $\boldsymbol{v}_i$) as

$$\begin{aligned} \hat{\boldsymbol{g}}_j &= \tilde{\boldsymbol{M}}_j^{-1}\tilde{\boldsymbol{\upsilon}}_j \\ &= (\boldsymbol{M}_j + \Delta\boldsymbol{M}_j)^{-1}(\boldsymbol{\upsilon}_j + \Delta\boldsymbol{\upsilon}_j) \\ &\approx (\boldsymbol{M}_j^{-1} - \boldsymbol{M}_j^{-1}\Delta\boldsymbol{M}_j\boldsymbol{M}_j^{-1})(\boldsymbol{\upsilon}_j + \Delta\boldsymbol{\upsilon}_j) \\ &\approx \boldsymbol{M}_j^{-1}\boldsymbol{\upsilon}_j + \boldsymbol{M}_j^{-1}\Delta\boldsymbol{\upsilon}_j - \boldsymbol{M}_j^{-1}\Delta\boldsymbol{M}_j\boldsymbol{M}_j^{-1}\boldsymbol{\upsilon}_j \\ &= \boldsymbol{g} + \boldsymbol{M}_j^{-1}(\Delta\boldsymbol{\upsilon}_j - \Delta\boldsymbol{M}_j\boldsymbol{g}). \end{aligned} \tag{46}$$

The error angle vector of the OLAE$_j$ is given (to first order in $\boldsymbol{v}_i$) by

$$\begin{aligned} \delta\boldsymbol{\theta}_j &= 2\hat{\boldsymbol{g}}_j \otimes \boldsymbol{g}^{-1} \\ &= 2\frac{\hat{\boldsymbol{g}}_j - \boldsymbol{g} + \hat{\boldsymbol{g}}_j^\times\boldsymbol{g}}{1 + \hat{\boldsymbol{g}}_j^{\mathrm{T}}\boldsymbol{g}} \\ &\approx \frac{2}{1 + \boldsymbol{g}^{\mathrm{T}}\boldsymbol{g}}(\boldsymbol{I} - \boldsymbol{g}^\times)\boldsymbol{M}_j^{-1}(\Delta\boldsymbol{\upsilon}_j - \Delta\boldsymbol{M}_j\boldsymbol{g}). \end{aligned} \tag{47}$$

Therefore, the covariance matrix $\boldsymbol{P}_{\theta_j\theta_j} = E\{\delta\boldsymbol{\theta}_j\delta\boldsymbol{\theta}_j^{\mathrm{T}}\}$ has the following expression:

$$\boldsymbol{P}_{\theta_j\theta_j} = \frac{4}{(1 + \boldsymbol{g}^{\mathrm{T}}\boldsymbol{g})^2}(\boldsymbol{I} - \boldsymbol{g}^\times)\boldsymbol{M}_j^{-1}\boldsymbol{Q}_j\boldsymbol{M}_j^{-\mathrm{T}}(\boldsymbol{I} + \boldsymbol{g}^\times),\quad j = 1, 2, 3, \tag{48}$$

$$\boldsymbol{Q}_j = E\left\{(\Delta\boldsymbol{\upsilon}_j - \Delta\boldsymbol{M}_j\boldsymbol{g})(\Delta\boldsymbol{\upsilon}_j - \Delta\boldsymbol{M}_j\boldsymbol{g})^{\mathrm{T}}\right\}, \tag{49}$$

where $E\{\cdot\}$ denotes the expectation operation.

Assume that the measurement model for observations is given by

$$E\{\boldsymbol{v}_i\boldsymbol{v}_j^{\mathrm{T}}\} = \begin{cases} \sigma_i^2\boldsymbol{I}, & i = j, \\ \boldsymbol{0}, & i \neq j, \end{cases} \tag{50}$$

where σ_i denotes the variance of the ith sensor noise, and the relative weights are chosen to minimize the original loss function, which leads to

$$\xi_i = \sigma_{\mathrm{tot}}^2 / \sigma_i^2,\ i = 1, 2, \ldots, n, \tag{51}$$

where

$$\sigma_{\mathrm{tot}}^{-2} = \sum_{i=1}^{n}\sigma_i^{-2}. \tag{52}$$

Thus, it yields

$$\Delta\boldsymbol{\upsilon}_j - \Delta\boldsymbol{M}_j\boldsymbol{g} = \sum_{i=1}^{n}\xi_i\boldsymbol{G}_{ji}\boldsymbol{v}_i. \tag{53}$$

Substituting Eqs. (42), (43), and (44) into Eq. (53), $\boldsymbol{G}_{ji}$ (j=1, 2, 3) are given by

$$\begin{aligned} \boldsymbol{G}_{1i} &= -2(\boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{b}_i)\boldsymbol{b}_i^\times\boldsymbol{r}_i\boldsymbol{r}_i^{\mathrm{T}} - [1 - (\boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{b}_i)^2]\boldsymbol{r}_i^\times \\ &\quad + 2(\boldsymbol{r}_i - \boldsymbol{b}_i)\boldsymbol{g}^{\mathrm{T}} + (\boldsymbol{r}_i - \boldsymbol{b}_i)^{\mathrm{T}}\boldsymbol{g}\boldsymbol{I} + \boldsymbol{b}_i^\times\boldsymbol{r}_i\boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{b}_i^\times\boldsymbol{g}\boldsymbol{r}_i^{\mathrm{T}} \\ &\quad - (1 + \boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{b}_i)[(\boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{b}_i^\times\boldsymbol{g})\boldsymbol{r}_i^\times + \boldsymbol{b}_i^\times\boldsymbol{r}_i\boldsymbol{r}_i^{\mathrm{T}}\boldsymbol{g}^\times], \\ \boldsymbol{G}_{2i} &= -\left\{2\boldsymbol{r}_i^\times + [(\boldsymbol{r}_i + \boldsymbol{b}_i)^\times\boldsymbol{g}]^\times + (\boldsymbol{r}_i + \boldsymbol{b}_i)^\times\boldsymbol{g}^\times\right\}, \\ \boldsymbol{G}_{3i} &= \boldsymbol{G}_{1i} + 2\boldsymbol{G}_{2i}. \end{aligned} \tag{54}$$

Therefore, $\boldsymbol{Q}_j$ and the covariance matrix $\boldsymbol{P}_{\theta_j\theta_j}$ can be rewritten as

$$\boldsymbol{Q}_j = E\left\{\sum_{i=1}^{n}\xi_j^2\boldsymbol{G}_{ji}\boldsymbol{v}_i\boldsymbol{v}_i^{\mathrm{T}}\boldsymbol{G}_{ji}^{\mathrm{T}}\right\} = \sigma_{\mathrm{tot}}^2\sum_{i=1}^{n}\boldsymbol{G}_{ji}\boldsymbol{G}_{ji}^{\mathrm{T}}, \tag{55}$$

$$\begin{aligned} \boldsymbol{P}_{\theta_j\theta_j} &= \frac{1}{(1 + \boldsymbol{g}^{\mathrm{T}}\boldsymbol{g})^2}\sigma_{\mathrm{tot}}^2(\boldsymbol{I} - \boldsymbol{g}^\times)\boldsymbol{M}_j^{-1} \cdot \\ &\quad \left(\sum_{i=1}^{n}\boldsymbol{G}_{ji}\boldsymbol{G}_{ji}^{\mathrm{T}}\right)\boldsymbol{M}_j^{-\mathrm{T}}(\boldsymbol{I} + \boldsymbol{g}^\times). \end{aligned} \tag{56}$$

Especially, for $\boldsymbol{g}$=$\boldsymbol{0}$, it can be obtained that $\boldsymbol{G}_{2i} = -2\boldsymbol{r}_i^\times$, and $\boldsymbol{P}_{\theta_2\theta_2} = 4\sigma_{\mathrm{tot}}^2\boldsymbol{M}_2^{-1}\sum_{i=1}^{n}(\boldsymbol{I} - \boldsymbol{r}_i\boldsymbol{r}_i^{\mathrm{T}})\boldsymbol{M}_2^{-\mathrm{T}}$.

3 Numerical simulations

The accuracy of the proposed OLAEs estimate compared with QUEST is quantified here with a Monte Carlo numerical analysis. The accuracy achieved by OLAE$_j$ estimator is analyzed by error vector for angles of rotation defined in Eq. (47). Simulation data are generated almost identically to Mortari *et al.* (2007) in this section, which is summaried as follows: Measurement data and the attitudes are N=1000, randomly produced using sensor noise with 10^{-3} rad (1σ), the true attitude is $\boldsymbol{g}$=[1, 1, 1]$^{\mathrm{T}}$ and the observed directions are [1, 0, 0]$^{\mathrm{T}}$, [0, 1, 0]$^{\mathrm{T}}$, and [0, 0, 1]$^{\mathrm{T}}$. The results of these tests are plotted in Fig. 3. It shows that OLAEs provide attitude accuracy almost identical to that of QUEST, and the computed estimates are compatible with the covariance analysis presented in the previous section.

Furthermore, the robustness of the proposed algorithms is tested and verified with respect to attitude principal angle and deviation of sensor noise. Define noise amplification factor ε_j, which represents the ratio of the deviation of attitude error $\sigma_{\delta\theta_j}$ and the deviation of sensor noise σ_{noise} of the jth estimator, as follows:

$$\varepsilon_j = \frac{\sigma_{\delta\theta_j}}{\sigma_{\text{noise}}},$$

$$\sigma_{\delta\theta_j} = \frac{1}{N}\sum_{k=1}^{N}\delta\theta_{jk},$$

where N=10 000 denotes the number of sets of measurements in simulations, and $\delta\theta_{jk}$ denotes the kth attitude error of OLAE$_j$:

$$\delta\theta_{jk} = 2\left\|\hat{\boldsymbol{g}}_{jk} \otimes \boldsymbol{g}^{-1}\right\| = 2\frac{\left\|\hat{\boldsymbol{g}}_{jk} - \boldsymbol{g} + \hat{\boldsymbol{g}}_{jk}^{\times}\boldsymbol{g}\right\|}{1+\hat{\boldsymbol{g}}_{jk}^{\mathrm{T}}\boldsymbol{g}},$$

where $\hat{\boldsymbol{g}}_{jk}$ denotes the kth attitude estimate of the jth estimator.

In the first case, the true attitude $\boldsymbol{g} = \tan(\theta/2)[1/\sqrt{3},\ 1/\sqrt{3},\ 1/\sqrt{3}]^{\mathrm{T}}$ varies with the principal angle θ, and the measured observed directions are corrupted by Gaussian white noise with zero-mean, standard deviation $\sigma_{\text{noise}}=10^{-3}$ rad. In order to analyze the evolution of the noise amplification

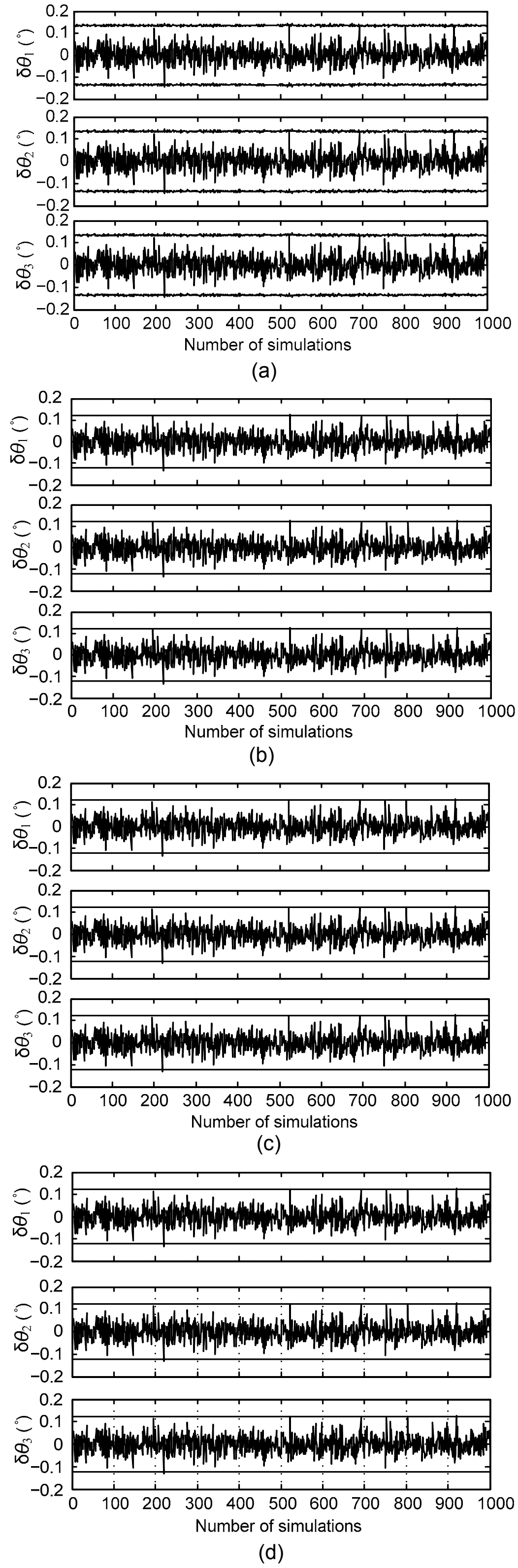

Fig. 3 Attitude accuracy plots
(a) OLAE$_1$ estimate error; (b) OLAE$_2$ estimate error; (c) OLAE$_3$ estimate error; (d) QUEST estimate error

factor, sensor noises are fully identical in each sample of principal angle. The results of ε_j (j=1, 2, 3, 4, j=4 for QUEST algorithm), are plotted in Fig. 4a. It is shown that

1. The precision of $OLAE_3$ is almost identical to that of QUEST, and the differences between them are less than 0.089% and can be negligible for practical purposes.

2. The attitude accuracy of $OLAE_1$ or $OLAE_2$ is a little lower than that of $OLAE_3$ or QUEST, and the performance of $OLAE_1$ is comparatively the lowest amongst these four estimators.

3. When the principal angle is close to 0 or π, $\det \tilde{M}_1 \approx 0$ and $\det \tilde{M}_1^* \approx 0$, singularity occurs in calculating the estimated Gibbs vector of $OLAE_1$, and consequently its accuracy error increases abruptly in these two cases.

4. According to the results of simulation data, the differences among three proposed algorithms and the traditional QUEST algorithm are less than 2.5% in most cases.

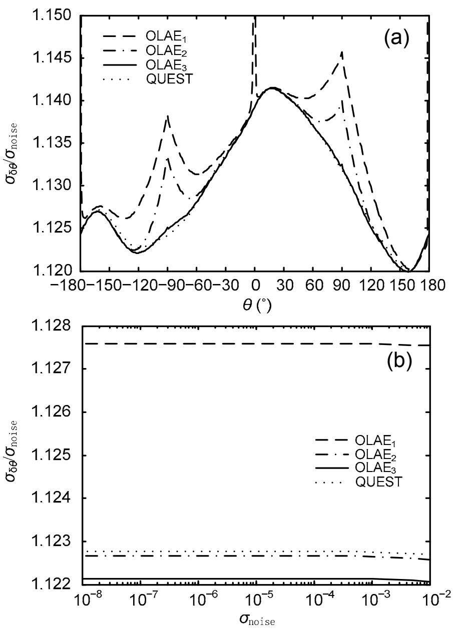

Fig. 4 Measurement noise intensity

(a) Along principal angle (σ_{noise}=10^{-3} rad); (b) Along deviation of sensor noise (θ=−120°)

In the second case, the true attitude is a constant vector $\boldsymbol{g}$=−[1, 1, 1]$^{\mathrm{T}}$, while the error of the observed directions is Gaussian white noise with zero-mean, standard deviation between 10^{-2} to 10^{-8} rad. As shown in Fig. 4b, noise amplification factors ε_j (j=1, 2, 3, 4) become slowly decreasing functions with respect to σ_{noise}, while decrements are less than 0.033% and can also be negligible.

The comparisons of computational speeds of the four different algorithms are illustrated in Fig. 5. The number of random tests is N=10 000. Elapsed times of $OLAE_1$, $OLAE_2$, $OLAE_3$, and QUEST are 2.768, 2.688, 2.776, and 3.495 s, respectively. It is shown that $OLAE_j$ (j=1, 2, 3) need less multiplications than QUEST algorithm.

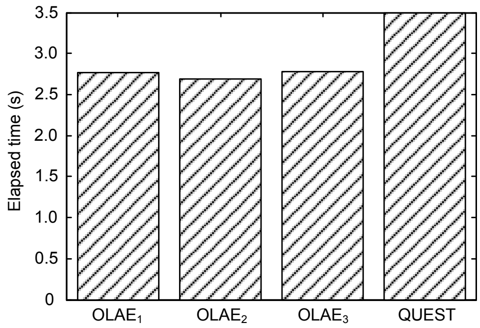

Fig. 5 Histogram of elapsed time

4 Conclusions

This paper presents a new approach for spacecraft attitude estimation via geometric analysis and proposes three OLAEs with results parameterized by MRPs. The geometry of attitude rotation shows that the Gibbs vector has known components along y and z directions in incidental right-hand orthogonal coordinates which can be induced by any pair of measured values. Thus, the unknown component along x direction can be eliminated using dot or/and cross products. Three new error vectors are introduced, and three associated optimality criteria are developed in terms of attitude and observations' vectors. These criteria are rigorously quadratic and unconstrained, which do not apply to Wahba's constrained criterion. The optimal attitude estimate in terms of Gibbs vector can be expressed by linear equations. To avoid singularity, which occurs when the principal angle is close to π, one proper rotation is adopted. Comparison with the Wahba compliant QUEST algorithm shows that the proposed OLAEs have faster computational speeds, because they do not need to calculate the determinant and adjoint of a 3×3 matrix and the maximum root of a quartic equation. Numerical simulations are employed to test and verify the performance of OLAEs using the Monte Carlo approach. The accuracy provided by OLAEs is comparable with that of QUEST algorithm in most cases, and thus the differences among these estimators can be negligible for practical purposes. Relatively, the order of priority of performance is $OLAE_3$>QUEST≈$OLAE_2$>$OLAE_1$. Although OLAEs provide high precision in attitude determination, they do not include the information from all past measurements. Thus, recursive OLAEs using the concept of Kalman filter may significantly enhance the precision of attitude determination.

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