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Stationary response of stochastically excited nonlinear systems with continuous-time Markov jump*

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Abstract: An approximate method for predicting the stationary response of stochastically excited nonlinear systems with continuous-time Markov jump is proposed. By using the stochastic averaging method, the original system is reduced to one governed by a 1D averaged Itô equation for the total energy with the Markov jump process as parameter. A Fokker-Planck-Kolmogorov (FPK) equation is then deduced, from which the approximate stationary probability density of the response of the original system is obtained for different jump rules. To illustrate the effectiveness of the proposed method, a stochastically excited Markov jump Duffing system is worked out in detail.

Key words: Nonlinear system; Continuous-time Markov jump; Stochastic excitation; Stochastic averaging
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1 Introduction

The operation of complex dynamical systems is often accompanied by abrupt changes in their configurations caused by component or interconnection failure, or by the onset of environmental disturbance. When these sudden changes in the operating rules occur in accordance with a Markov process, the associated stochastic system is referred to as a continuous-time Markov jump system (MJS). MJSs have many applications in a variety of fields, including air vehicles (Stoica and Yaesh, 2002), economics (do Val and Basar, 1999), power systems (Ugrinovskii and Pota, 2005), satellite dynamics (Meskin and Khorasani, 2009), and communication systems (Abdollahi and Khorasani, 2011).

Since Kats and Krasovskii (1960), Krasovskii and Lidskii (1961) first introduced MJSs, considerable attention has been devoted to the analysis and synthesis of MJSs (Costa *et al*., 2006; 2013). Necessary and sufficient conditions for moment stability were obtained by means of an explicit formula for the corresponding Lyapunov exponent for a piecewise deterministic jump linear system (Mariton, 1988). Kushner (1967) applied the 'almost sure stability' concept to jump linear systems. Krasovskii and Lidskii (1961) studied the linear quadratic regulator (LQR) control of Markov jump linear systems. Sworder (1969) solved the optimal control problem with finite time horizon using the maximum principle. The ergodic control problem of MJS is studied based on the dynamic programming principle (Ghosh *et al*., 1997). However, previous study on MJSs mainly focused on stability and optimal control (Ji and Chizeck, 1992; Natache and Vilma, 2004; Luo, 2006; Huang and Nguang, 2008). Little

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effort has been given to studying the response of MJSs, especially for stochastically excited nonlinear MJSs. Development of the methodology for analyzing nonlinear MJS is thus much deserving.

In this paper, a method for predicting the stationary response of stochastically excited nonlinear systems with continuous-time Markov jump is proposed. In the case of small transition rate, an averaged Itô equation governing the energy envelope with the Markov jump process as parameter is first derived using stochastic averaging (Zhu and Lin, 1991; Zhu, 2006; Huan *et al*., 2008). The associated Fokker-Planck-Kolmogorov (FPK) equation is then set up, from which the approximate stationary probability density of the response of the original system with different jump rules is finally obtained.

In this study, first, the nonlinear MJS subjected to stochastic excitation is introduced, and the stochastic averaging method is applied to the system. Then, the associated FPK equation is set up. Finally, the validity and accuracy of the proposed method are demonstrated by using the Markov jump Duffing oscillator subjected to Gaussian white noise, wherein the detailed calculation is provided and comparison with direct simulation of the original system is made.

2 Formulation of problem

Consider a single-degree-of-freedom (SDOF) stochastically excited nonlinear system with continuous-time Markov jump:

$$\begin{aligned}&\ddot{x}+g(x)=\varepsilon f(x,\dot{x},s(t))+\varepsilon^{1/2}h_k(x,s(t))W_k(t),\\&x(t_0)=x_0,\ \ s(t_0)=s_0,\end{aligned}\tag{1}$$

where $g(x)$ is the nonlinear restoring force, ε is a small parameter, $\varepsilon f(x,\dot{x},s(t))$ denotes light Markov jump damping force, $\varepsilon^{1/2}h_k(x, s(t))$ (k=1, 2, …, m) represent the Markov jump amplitudes of weakly external and/or parametric stochastic excitations, and $W_k(t)$ are the independent Gaussian white noises with zero means and intensities $2D_k$. The repeated index k implies the summation over its range, i.e.,

$$h_k(x,s(t))W_k(t)=\sum_{k=1}^{m}h_k(x,s(t))W_k(t).$$

$s(t)$ is a continuous-time Markov jump process which takes discrete values in a given finite set S={1, 2, …, l}. Each $s\in S$ denotes the mode in which the system operates. The transition probability between the modes is

$$P\{s(t+\Delta t)=j\mid s(t)=i\}=\begin{cases}\lambda_{ij}\Delta t+o(\Delta t), & i\neq j,\\ 1+\lambda_{ii}\Delta t+o(\Delta t), & i=j,\end{cases}\tag{2}$$

where $o(\Delta t)$ is such that $\lim_{\Delta t\to 0}o(\Delta t)/\Delta t=0$, $P\{s(t+\Delta t)=j\mid s(t)=i\}$ represents the probability that the system takes the mode j at time $t+\Delta t$ given that it has the mode i at time t. $\lambda_{ij}>0$ for $i\neq j$, is the transition rate from mode i to mode j and (Wonham, 1970)

$$\lambda_{ii}=-\sum_{j=1,j\neq i}^{l}\lambda_{ij}.\tag{3}$$

Eq. (1) can be used, for instance, to model a class of linear or nonlinear systems whose random changes in their structures may be a consequence of abrupt phenomena such as component and/or interconnection failure. Our primary concern here is the stationary response of Eq. (1).

3 Averaged equation

In this paper, $s(t)$ is supposed to be ergodic and independent of system state. To apply the stochastic averaging method to Eq. (1), the Markov jump process is first assumed to be fixed at $s(t)=i\in S$. That means, the system operates in the mode i and no jump occurs. In this case, for simplicity, $f(\cdot, s(t))$ and $h_k(\cdot, s(t))$ are denoted by $f(i)(\cdot)$ and $h_k^{(i)}(\cdot)$, respectively. The equivalent Itô equations of Eq. (1) with fixed $s(t)=i$ are (Ji and Chizeck, 1992)

$$\begin{aligned}&\mathrm{d}Q=P\mathrm{d}t,\\&\mathrm{d}P=[-g(Q)+\varepsilon f^{(i)}(Q,P)]\mathrm{d}t+\varepsilon^{1/2}\sigma_k^{(i)}(Q)\mathrm{d}B_k(t),\end{aligned}\tag{4}$$

where $Q=x$ and $P=\dot{x}$, $B_k(t)$ are independent unit Wiener processes, and $\sigma_k^{(i)}(Q)=\sqrt{2D_k}h_k^{(i)}(Q)$. The Hamiltonian H (total energy) of the Hamiltonian system associated with Eq. (4) is

$$H(Q,P)=\frac{P^2}{2}+\int_0^Q g(u)\mathrm{d}u. \tag{5}$$

Since H is a function of Q and P, the Itô equation for H can be derived from Eq. (4) by using the Itô differential rule as follows:

$$\begin{aligned}\mathrm{d}H=&\varepsilon\left[f^{(i)}(Q,P)\frac{\partial H}{\partial P}+\frac{1}{2}\sigma_k^{(i)}(Q)\sigma_k^{(i)}(Q)\frac{\partial^2 H}{\partial P^2}\right]\mathrm{d}t\\&+\varepsilon^{1/2}\frac{\partial H}{\partial P}\sigma_k^{(i)}(Q)\mathrm{d}B_k(t).\end{aligned} \tag{6}$$

Introduce transformation from Q and P to Q and H. Then, the system is governed by the first equation of Eq. (4) and Eq. (6). In the case of light damping and weak excitations, the Hamiltonian H in Eq. (6) is a slowly varying process while the generalized displacement Q in Eq. (4) is a rapidly varying process. According to a theorem due to Khasminskii (1968), H approaches to a diffusion process as $\varepsilon\to 0$. Since the slowly varying process is essential for describing the long-term behavior of the system, the stochastic averaging method is used to average out the rapidly varying process and to yield the following averaged Itô equation for a slowly varying process H:

$$\mathrm{d}H=\varepsilon m^{(i)}(H)\mathrm{d}t+\varepsilon^{1/2}\bar{\sigma}^{(i)}(H)\mathrm{d}B(t), \tag{7}$$

where the drift coefficient $m^{(i)}(H)$ and diffusion coefficient $\bar{\sigma}^{(i)}(H)$ are given by (Zhu and Lin, 1991)

$$\begin{aligned}m^{(i)}(H)=&\frac{1}{T(H)}\int_\Omega\left[f^{(i)}(q,p)\frac{\partial H}{\partial p}\right.\\&\left.+\frac{1}{2}\sigma_k^{(i)}(q)\sigma_k^{(i)}(q)\frac{\partial^2 H}{\partial p^2}\right]\left(\frac{\partial H}{\partial p}\right)^{-1}\mathrm{d}q,\\ \{\bar{\sigma}^{(i)}(H)\}^2=&\frac{1}{T(H)}\int_\Omega\left[\sigma_k^{(i)}(q)\right.\\&\left.\times\sigma_k^{(i)}(q)\left(\frac{\partial H}{\partial p}\right)^2\right]\left(\frac{\partial H}{\partial p}\right)^{-1}\mathrm{d}q.\end{aligned} \tag{8}$$

The region of integration is $\Omega=\{q\mid H(q,0)\le H\}$, and

$$T(H)=\int_\Omega\left(\frac{\partial H}{\partial p}\right)^{-1}\mathrm{d}q. \tag{9}$$

For the original jump equation (1), there are l (l is the total number of the modes of the original system) averaged equations like Eq. (7). In the case of small transition rate, the original jump system can be approximately substituted by an averaged equation of the form of Eq. (7) with the Markov jump process as parameter:

$$\mathrm{d}H=\varepsilon m(H,s)\mathrm{d}t+\varepsilon^{1/2}\bar{\sigma}(H,s)\mathrm{d}B(t), \tag{10}$$

where $m(H, s)$ and $\bar{\sigma}(H,s)$ denote the drift and diffusion coefficients, respectively. The expressions are similar to those in Eq. (8) with $f^{(i)}(q, p)$ and $\sigma_k^{(i)}(q)$ replaced by $f(q, p, s)$ and $\sigma_k(q, s)$, respectively.

4 Stationary response

Recall from Eq. (2) that $P(s, t+\Delta t \mid r, t)$ represents the probability that the system takes the mode s at time $t+\Delta t$ under the condition that it has the mode r at time t. In order that $s(t+\Delta t)=s$, the system either remains in the mode s or it jumps to mode s from mode r (r=1, 2, …, $s-1$, $s+1$, …, l) in the time interval $[t, t+\Delta t]$. Based on the averaged equation, Eq. (7), the following FPK equation can be deduced (see Appendix A for detail):

$$\begin{aligned}\frac{\partial}{\partial t}p(H,s,t)=&-\frac{\partial}{\partial H}[m(H,s)p(H,s,t)]\\&+\frac{1}{2}\frac{\partial^2}{\partial H^2}\left[\{\bar{\sigma}(H,s)\}^2p(H,s,t)\right]-\sum_{\substack{r=1\\r\neq s}}^{l}\left[\lambda_{sr}p(H,s,t)\right.\\&\left.-\lambda_{rs}\int_0^\infty p(H',r,t)q(H,s,t\mid H',r,t)\mathrm{d}H'\right],\end{aligned} \tag{11}$$

where $p(H, s, t)$ is the probability density of the total energy H with the Markov jump process s as a parameter. The initial condition is

$$p(H,s,0)=p(H_0,s), \tag{12}$$

and boundary conditions are

$$\begin{aligned}&p(0,s,t)=\text{finite},\\&p(H,s,t)|_{H\to\infty}\to 0,\quad \frac{\partial p(H,s,t)}{\partial H}|_{H\to\infty}\to 0.\end{aligned} \tag{13}$$

The conditional probability density $q(H, s, t \mid H', r, t)$ in Eq. (11) can be specified according to the physical meaning of the real system. In the independent jump case, $q(H, s, t \mid H', r, t)$ is assumed to have the following form (Wu, 2007; Fang *et al.*, 2012):

$$q(H, s, t \mid H', r, t) = \delta(H - H'). \tag{14}$$

Substituting Eq. (14) into Eq. (11) and completing the integral yields the following FPK equation:

$$\begin{aligned} \frac{\partial}{\partial t} p(H, s, t) = &-\frac{\partial}{\partial H}[m(H, s)p(H, s, t)] \\ &+\frac{1}{2}\frac{\partial^2}{\partial H^2}\left[\left\{\bar{\sigma}(H, s)\right\}^2 p(H, s, t)\right] \\ &-\sum_{\substack{r=1 \\ r\neq s}}^{l}[\lambda_{sr} p(H, s, t) - \lambda_{rs} p(H, r, t)]. \end{aligned} \tag{15}$$

The FPK equation (15) does not admit an easy solution, analytically or numerically. Fortunately, in practical application we are more interested in the stationary solution of FPK equation (15). In this case, FPK equation (15) is simplified by letting $\partial p/\partial t=0$. Then, the joint stationary probability density $p(H, s)$ is obtained readily from solving Eq. (15) using the finite difference method. The stationary probability density $p(H)$ can be obtained from $p(H, s)$ as follows (see Appendix B for detail):

$$p(H) = \sum_{s\in S} p(H, s). \tag{16}$$

The marginal stationary probability density $p(q)$ of the generalized displacement is then obtained as

$$p(q) = \int_{-\infty}^{\infty} p(q, p)\mathrm{d}p, \tag{17}$$

where

$$p(q, p) = \frac{p(H)}{T(H)}\Big|_{H=H(q,\, p)} \tag{18}$$

is the stationary joint probability density of the displacement and velocity.

Note that rigorous analysis of the error in the stochastic averaging has not been reported in the open literature. Thus, the quantification of the error of the proposed method can be made only by comparing with direct simulation.

5 Numerical example

To demonstrate the validity and accuracy of the proposed method, consider a stochastically excited Duffing oscillator with independent Markov jump process as parameter and governed by the equation:

$$\ddot{x} + \omega^2 x + \alpha x^3 = \beta(s(t))\dot{x} + h(s(t))\xi(t), \tag{19}$$

where ω and α are constants, $\beta(s(t))$ is the Markov jump coefficient of linear damping, $h(s(t))$ is the Markov jump amplitude of external random excitation, and $\xi(t)$ is the Gaussian white noise with zero mean and intensity $2D$. $s(t)$ is a continuous-time Markov jump process with the transition probability defined in Eq. (2). $s(t)$ takes discrete values in a given finite set $S=\{1, 2, \ldots, l\}$.

Following the steps in Eqs. (4)–(10), the original Eq. (19) can be approximated by the following averaged Itô system with the Markov jump process as parameter:

$$\mathrm{d}H = m(H, s)\mathrm{d}t + \bar{\sigma}(H, s)\mathrm{d}B(t), \tag{20}$$

where

$$\begin{aligned} &m(H, s=i) = m^{(i)}(H) = \left\{h^{(i)}\right\}^2 D - \beta^{(i)}G(H), \\ &\left\{\bar{\sigma}(H, s=i)\right\}^2 = \left\{\sigma^{(i)}(H)\right\}^2 = 2\left\{h^{(i)}\right\}^2 DG(H), \\ &G(H) = \frac{4}{T(H)}\int_0^{A(H)} (2H - \omega^2 q^2 - \alpha q^4/2)^{1/2}\mathrm{d}q, \\ &T(H) = 4\int_0^{A(H)} (2H - \omega^2 q^2 - \alpha q^4/2)^{-1/2}\mathrm{d}q, \\ &A(H) = \left[\frac{\left(\omega + 4\alpha H\right)^{1/2} - \omega^2}{\alpha}\right]^{1/2}, \\ &i \in S. \end{aligned} \tag{21}$$

The associated averaged FPK equation is of the form of Eq. (15) with $m(H, s)$ and $\bar{\sigma}(H, s)$ given by Eq. (21). The stationary probability density $p(H, s)$ can be obtained by solving the FPK equation (15) with $\partial p/\partial t=0$ numerically. The stationary probability densities $p(q)$ and $p(q, p)$ are then determined using Eqs. (17) and (18).

5.1 Two-mode system

In this case, l=2 and S={1, 2}. Some numerical results are obtained as shown in Figs. 1–3 for system parameters ω=1.0, α=1.0, D=0.02, $\beta^{(1)}$=0.02, $\beta^{(2)}$=0.04, $h^{(1)}$=2.0, and $h^{(2)}$=1.0. Prescribe the transition rate λ_{ij} by a transition matrix Λ=[λ_{ij}]. Three special cases with

$$\Lambda_1=\begin{bmatrix}-2 & 2\\ 2 & -2\end{bmatrix},\ \Lambda_2=\begin{bmatrix}-1 & 1\\ 2 & -2\end{bmatrix},\ \Lambda_3=\begin{bmatrix}-2 & 2\\ 1 & -1\end{bmatrix} \quad (22)$$

are considered. The stationary probability density $p(q)$ of displacement is shown in Fig. 1. Also plotted are the probability densities when the Markov jump process is fixed at either $s(t)$=1 or $s(t)$=2. Obviously, the system operating in the mode $s(t)$=2 has larger damping coefficients and/or smaller amplitude of stochastic excitation than that operating in the mode $s(t)$=1. Thus, the probability density $p(q)$ near the equilibrium position is higher, which means the system spends more time near equilibrium in the mode $s(t)$=2 than that in the mode $s(t)$=1. The value around q=0 will decrease as Λ cycles through $\Lambda=\Lambda_3$, $\Lambda=\Lambda_1$, and $\Lambda=\Lambda_2$.

The lines in Fig. 1 are obtained from solving the averaged FPK equation (15) while the dots are obtained by direct simulation of Eq. (19). The joint probability densities $p(q, p)$ of the displacement and velocity obtained from Eqs. (19) and (15) are shown in Figs. 2a and 2b, respectively. It can be seen that the analytical results agree well with those from digital simulation of the original Eq. (19), which

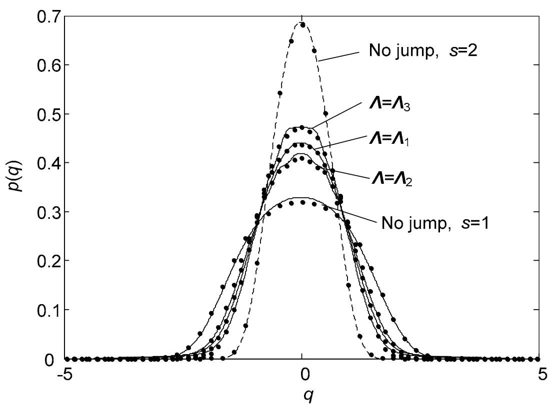

Fig. 1 Stationary probability density $p(q)$ of displacement of 2-mode jump equation (19) with $\Lambda=\Lambda_1$, $\Lambda=\Lambda_2$, and $\Lambda=\Lambda_3$ in Eq. (22), and with $s(t)$=1 and $s(t)$=2

The lines are obtained from numerical solution of Eq. (15) while the dots are obtained from direct simulation of original Eq. (19)

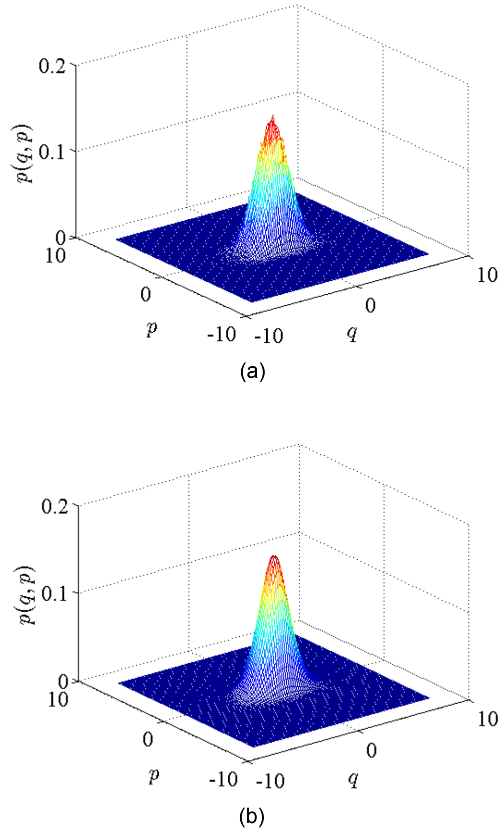

Fig. 2 Joint stationary probability densities $p(q, p)$ of the displacement and velocity of 2-mode jump equation (19) with the transition rate $\Lambda=\Lambda_2$

(a) Numerical solution of Eq. (15); (b) Direct simulation of original Eq. (19)

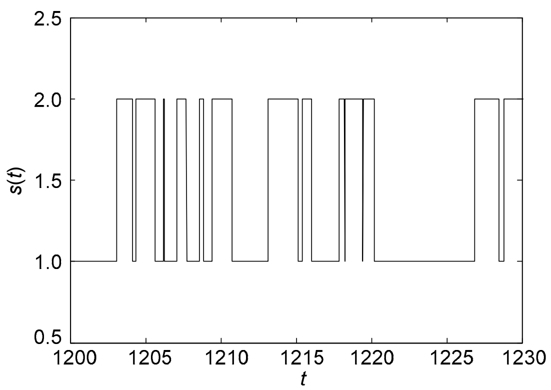

Fig. 3 Sample of jump process $s(t)$ of 2-mode jump equation (19)

demonstrates the validity and accuracy of the proposed method. Finally, a sample of 2-mode independent Markov jump process $s(t)$ is shown in Fig. 3.

5.2 Three-mode system

In this case, l=3 and S={1, 2, 3}. The numerical results shown in Figs. 4–6 are for system with parameters ω=1.0, α=1.0, D=0.02, $\beta^{(1)}$=0.01, $\beta^{(2)}$=0.025, $\beta^{(3)}$=0.04, $h^{(1)}$=2.0, $h^{(2)}$=1.0, and $h^{(3)}$=1.0. Three special cases with

$$\Lambda_1=\begin{bmatrix}-1.4 & 0.7 & 0.7\\ 1.5 & -3.0 & 1.5\\ 1.5 & 1.5 & -3.0\end{bmatrix},\quad \Lambda_2=\begin{bmatrix}-3.0 & 1.5 & 1.5\\ 0.7 & -1.4 & 0.7\\ 1.5 & 1.5 & -3.0\end{bmatrix},$$
$$\Lambda_3=\begin{bmatrix}-3.0 & 1.5 & 1.5\\ 1.5 & -3.0 & 1.5\\ 0.7 & 0.7 & -1.4\end{bmatrix} \tag{23}$$

are considered. The stationary probability density $p(q)$ of displacement is shown in Fig. 4 for different cases. The joint stationary probability densities $p(q, p)$ of displacement and velocity are exhibited in Figs. 5a and 5b. Again, the analytical results obtained from solving the FPK equation (15) match closely with those from digital simulation of the original Eq. (19), which indicates that the proposed method is very effective for solving the vibration problem of a nonlinear Markov jump system under stochastic excitation. Finally, a sample of 3-mode jump process $s(t)$ is shown in Fig. 6.

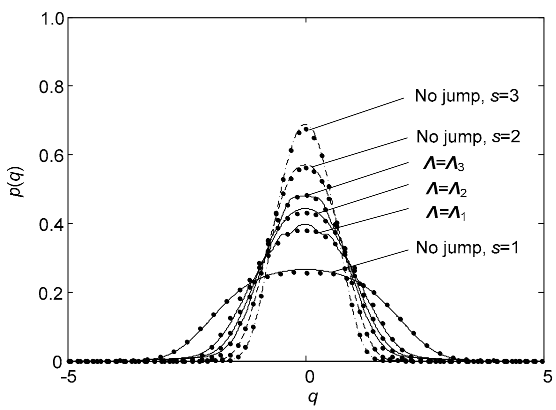

Fig. 4 Stationary probability density $p(q)$ of displacement of 3-mode jump equation (19) with $\Lambda=\Lambda_1$, $\Lambda=\Lambda_2$, and $\Lambda=\Lambda_3$ in Eq. (23), and with $s(t)$=1, $s(t)$=2, and $s(t)$=3

The lines are obtained from numerical solution of Eq. (15) while the dots are obtained from the direct simulation of original Eq. (19)

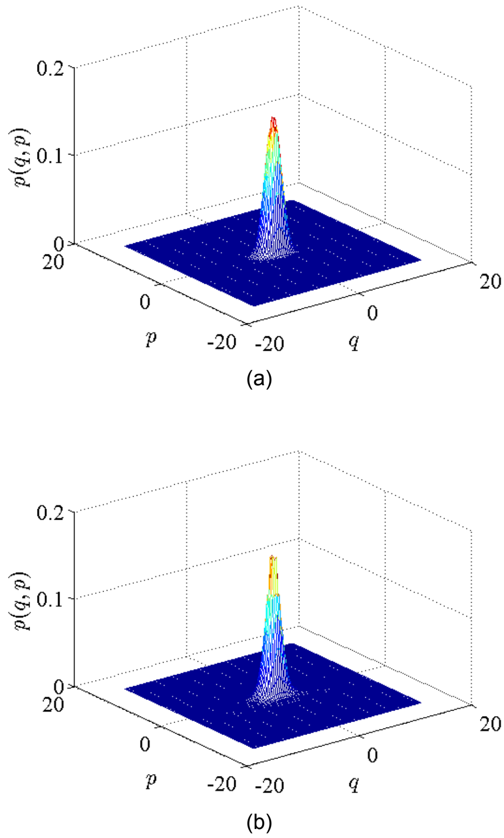

Fig. 5 Joint probability density $p(q, p)$ of the displacement and momentum of 3-mode jump equation (19) with transition rate $\Lambda=\Lambda_2$

(a) Numerical solution of Eq. (15); (b) Direct simulation of original Eq. (19)

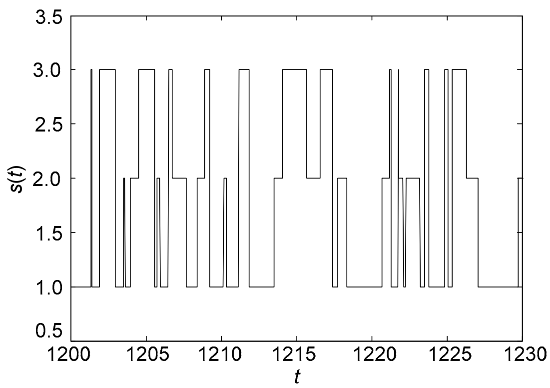

Fig. 6 Sample of jump process $s(t)$ of 3-mode jump system

6 Conclusions

Study of Markov jump systems is of practical significance because of their wide applications in industry and economy. In this paper, an approximate method for predicting the stationary response of stochastically excited nonlinear systems with continuous-time Markov jump has been proposed. In the case of small transition rate, the original system was reduced to one governed by a 1D averaged Itô equation with the Markov jump process as parameter using the stochastic averaging method. The FPK equation governing the probability density of the total energy has been derived. One example has been worked out in detail. The comparison of the analytical results obtained using the proposed method with those from digital simulation of the original system indicates that the proposed method is feasible and effective for solving the random vibration problem of a nonlinear Markov jump system.

Note that the proposed method has the potential to be extended to the Multi-DOF Markov jump systems. However, in the case of Multi-DOF system, the solution of the FPK equation is more difficult to obtain, even for a stationary solution. It will be the topic of our future research.

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Appendix A

Without jump, the transition probability density $p(H, s, t+\Delta t \mid H', s, t)$ satisfies the following FPK equation:

$$\begin{aligned}&\frac{\partial}{\partial t}p(H,s,t+\Delta t\,|\,H',s,t)\\&=-\frac{\partial}{\partial H}[m(H,s)p(H,s,t+\Delta t\,|\,H',s,t)]\\&+\frac{1}{2}\frac{\partial^2}{\partial H^2}\left[\{\bar{\sigma}(H,s)\}^2\,p(H,s,t+\Delta t\,|\,H',s,t)\right].\end{aligned}\tag{A1}$$

For a short time interval, FPK equation (A1) can be rewritten as

$$\begin{aligned}&p(H,s,t+\Delta t\,|\,H',s,t)\\&=\delta(H-H')-\Delta t\frac{\partial}{\partial H}[m(H,s)\delta(H-H')]\\&+\frac{1}{2}\Delta t\frac{\partial^2}{\partial H^2}\left[\{\bar{\sigma}(H,s)\}^2\,\delta(H-H')\right]+o(\Delta t),\end{aligned}\tag{A2}$$

where $\delta(H-H')=p(H, s, t \mid H', s, t)$. This equation will be used in the following steps.

Recall from Eq. (2) that $P(s, t+\Delta t \mid r, t)$ denotes the probability that the system takes the mode s at time $t+\Delta t$ given that it has the mode r at t. In order that $s(t+\Delta t)=s$, the system either remains in the mode s or it jumps from mode r (r=1, 2, …, $s-1$, $s+1$, …, l) to mode s in the interval $[t, t+\Delta t]$. Hence, the following equation is obtained (Wu, 2007; Fang *et al.*, 2012):

$$\begin{aligned}&p(H,s,t+\Delta t)=\int_0^\infty p(H^*,s,t)\\&\quad\times p(H,s,t+\Delta t\,|\,H^*,s,t)P(s,t+\Delta t\,|\,s,t)\mathrm{d}H^*\\&\quad+\sum_{\substack{r=1\\r\neq s}}^{l}\int_0^\infty p(H',r,t)q(H,s,t+\Delta t\,|\,H',r,t)\\&\quad\times P(s,t+\Delta t\,|\,r,t)\mathrm{d}H'.\end{aligned}\tag{A3}$$

In Eq. (A3), the first term in the right hand side denotes the probability density where the system stays in mode s in time interval Δt. The second term represents the probability density where the system jumps to mode s from mode r. $q(H, s, t+\Delta t \mid H', r, t)$ is the transition probability density of H for system jumping from mode r at time t to mode s at time $t+\Delta t$. Then, using Eqs. (2) and (3), Eq. (A3) can be rewritten as

$$\begin{aligned}&p(H,s,t+\Delta t)=\int_0^\infty p(H^*,s,t)\\&\quad\times p(H,s,t+\Delta t\,|\,H^*,s,t)\left[1-\Delta t\sum_{\substack{r=1\\r\neq s}}^{l}\lambda_{sr}\right]\mathrm{d}H^*\\&\quad+\Delta t\sum_{\substack{r=1\\r\neq s}}^{l}\lambda_{rs}\int_0^\infty p(H',r,t)q(H,s,t+\Delta t\,|\,H',r,t)\mathrm{d}H'.\end{aligned}\tag{A4}$$

Substituting Eq. (A2) into Eq. (A4) yields

$$\begin{aligned}&p(H,s,t+\Delta t)=p(H,s,t)\\&\quad-\Delta t\frac{\partial}{\partial H}[m(H,s)p(H,s,t)]\\&\quad+\frac{1}{2}\Delta t\frac{\partial^2}{\partial H^2}\left[\{\bar{\sigma}(H,s)\}^2\,p(H,s,t)\right]\\&\quad-\Delta t\sum_{\substack{r=1\\r\neq s}}^{l}\left[\lambda_{sr}p(H,s,t)\right.\\&\quad\left.-\lambda_{rs}\int_0^\infty p(H',r,t)q(H,s,t+\Delta t\,|\,H',r,t)\mathrm{d}H'\right]+o(\Delta t),\\&s,r\in S.\end{aligned}\tag{A5}$$

Dividing both sides of Eq. (A5) by Δt and letting $\Delta t \to 0$ lead to the following FPK equation:

$$\begin{aligned}\frac{\partial}{\partial t}p(H,s,t) &= -\frac{\partial}{\partial H}[m(H,s)p(H,s,t)] \\ &+\frac{1}{2}\frac{\partial^2}{\partial H^2}\left[\{\bar{\sigma}(H,s)\}^2 p(H,s,t)\right]-\sum_{\substack{r=1\\ r\neq s}}^{l}\Big[\lambda_{sr}p(H,s,t) \\ &-\lambda_{rs}\int_0^{\infty}p(H',r,t)q(H,s,t\,|\,H',r,t)\,\mathrm{d}H'\Big].\end{aligned} \tag{A6}$$

Appendix B

According to the probability theorem, the probability distribution $P(H)$ of H satisfies

$$P(H)=\int_{-\infty}^{H}\sum_{s\in S}\left[p(H\,|\,s)P(s)\right]\mathrm{d}H=\int_{-\infty}^{H}p(H)\mathrm{d}H, \tag{B1}$$

where $p(H\,|\,s)$ is the conditional probability density of H when s is known to be a particular value. $P(s)$ is the probability distribution of the Markov process $s(t)$ when $s(t)=s$. From Eq. (B1), one obtains

$$p(H)=\sum_{s\in S}\left[p(H\,|\,s)P(s)\right]. \tag{B2}$$

On the other hand, the joint probability density $p(H, s)$ satisfies

$$\begin{aligned}p(H,s)&=\frac{\partial P(H,s)}{\partial H}=\frac{\partial}{\partial H}\left[P(s)\cdot P(H\,|\,s)\right]\\ &=P(s)p(H\,|\,s).\end{aligned} \tag{B3}$$

Substituting Eq. (B3) into Eq. (B2) leads to

$$p(H)=\sum_{s\in S}p(H,s). \tag{B4}$$

中文概要

题　目：随机激励下连续时间马尔科夫跳变非线性系统的平稳响应研究

目　的：提出一种预测随机激励下连续时间马尔科夫跳变非线性系统的平稳响应的近似方法。

创新点：1. 得到了含有马尔科夫跳变参数的关于能量的平均 Itô 方程；2. 建立了含有马尔科夫跳变参数的平均 Itô 方程相应的 FPK 方程。

方　法：1. 将一个随机激励的马尔科夫跳变非线性系统由状态方程转化为等价的 Itô 方程，并根据 Itô 微分法则给出哈密顿量（系统总能量）的 Itô 方程；2. 通过随机平均法，得到关于系统能量的平均 Itô 方程；3. 推导并求解相应的 FPK 方程。

结　论：1. 跳变规律对马尔科夫跳变非线性系统随机响应具有重要影响；2. 理论结果与数字模拟结果吻合验证了理论方法的准确性。

关键词：非线性系统；连续时间马尔科夫跳变；随机激励；随机平均