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Modified residue harmonic balance solution for coupled integrable dispersionless equations with disturbance terms#

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This is a supplementary study of the solution method previously proposed by the author (Lee, 2017; Rahman and Lee, 2017). The proposed method is used for solving three-coupled integrable dispersionless equations with disturbance terms. It has only been adopted for solving (1) a nonlinear beam problem, and (2) a nonlinear vibro-acoustic problem. In the solution process, the three-coupled nonlinear equations can be transformed into only one Duffing equation. The higher-level nonlinear solutions, which were ignored in the previous method, can be generated using the proposed approach. Hence, in each step in the solution, only one independent nonlinear algebraic equation need be solved. As in the previous method, the proposed method has the advantage that the periodic solutions are represented by Fourier functions rather than the tedious implicit functions. The solutions from the proposed method agree reasonably well with those obtained from the classical harmonic balance method.

In recent decades, many solution methods have been developed and adopted to solve various differential equations and conduct numerical simulations for structural dynamic, fluid dynamic, wave propagation problems, etc. (Hu and Wang, 2008; Li et al., 2011; Huang et al., 2012; Leung et al., 2012; Yin et al., 2012; Zhu et al., 2013; Zhong et al., 2014; Zhou et al., 2014; Wang et al., 2016; Zhang et al., 2016; Zheng et al., 2016; Lee, 2017; Lu et al., 2018). For example, Guner and Bekir (2018) adopted the solitary wave ansatz method to find exact analytical solutions of the space-time fractional Zakharov-Kuznetsov-Benjamin-Bona-Mahony equation, the space-time fractional Klein-Gordon equation, and the space-time fractional modified regularized long wave equation. Hashemi and Akgul (2018) obtained the analytical solution of a nonlinear Schrodinger equation in both time and space fractional terms. Two analytical approaches, Nucci's reduction method and the simplest equation method, were used to extract analytical solutions especially of soliton kinds. Tasbozan et al. (2018) focused on the exact solution sets of a nonlinear conformable time-fractional coupled Drinfeld-Sokolov-Wilson equation using the Sine-Gordon expansion method. They also studied an analytical approximate method, namely a perturbation-iteration algorithm for the system.

Among various solution methods, the approach of harmonic balance is the one of significant attention from researchers in the areas of nonlinear vibration and nonlinear structural dynamics. There have been many different versions of harmonic balance methods (e.g. spreading residue harmonic balance method (Qian et al., 2017), incremental harmonic balance method (Huang et al., 2011, 2018; Huang and Zhu, 2017), residue harmonic balance method (Leung et al., 2012; Mohammadian and Shariati, 2017), and modified residue harmonic balance method (Lee, 2017, 2018; Rahman and Lee, 2017)). To solve the nonlinear problem in this study, the author proposes

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the modified harmonic balance method from Leung and his co-authors' research work (Leung et al., 2012), in which their method was newly applied to solving three-coupled non-homogenous integrable dispersionless equations without disturbance terms. As mentioned, the proposed method was previously adopted in (Rahman and Lee, 2017; Lee, 2018). In the solution process, the transformation is adopted to convert the coupled nonlinear partial differential equations with disturbance terms into a nonlinear ordinary differential equation. An order parameter is introduced into the residual of the Fourier truncation series in the harmonic balance processes. The proposed method is based on a primary idea in (Leung et al., 2012; Lee, 2017, 2018; Rahman and Lee, 2017).

The theory is given in Data S1, including all governing equations, solution steps, and explanations of variables. After the implementation of the proposed method into the nonlinear problem, the results are generated and shown in Tables1–4 and Figs. 1 and 2, respectively. Tables 1–3 show the comparisons between the results from the proposed method and from the classical harmonic balance method (Lee, 2002) for various disturbance magnitudes and frequencies. v is a time dependent variable in the integrable dispersionless equations; k_1 and k_2 are the integration constants; ω is the excitation frequency; κ is the excitation magnitude; $\alpha=2k_1/c$; $\beta=2/c^2$; c is a constant representing the speed of the propagating waves. The zero-, first-, and second-level periodic solutions of v are computed using Eqs. (12), (16), and (21), respectively. Hence, the amplitudes of the periodic responses can be found. In the case of small disturbance (κ=1), the first-level solutions can achieve an error rate of less than 4% for various frequencies. In general, the first- and second-level solutions are very close. In the other excitation cases, the maximum difference between the second-level and harmonic balance solutions is less than 4.3%. Table 4 shows the results of the two coupled nonlinear differential equations Eqs. (5a) and (5b) solved by the classical harmonic balance method. It should be noted that k_2=1.

Fig. 1 shows the amplitude of v plotted against the frequency. The solid line represents the first-level results from the proposed method while the dashed lines represent the results from the old residue harmonic balance method (Leung et al., 2012; Hasan et al., 2016; Lee, 2017). It can be seen that the zero-order solutions from the two methods overlap with each other. There are two zero-order solution types, one is linear and the other one is nonlinear. The zero-order nonlinear solution curves are always higher that the linear one. That agrees with a well-known concept in structural dynamics. For the linear vibration of a structural dynamic system, the vibration amplitude is small, while for the nonlinear vibration, the vibration amplitude is high. The main difference between the new and old methods is that there is no higher-order nonlinear solution from the old method. It should be noted that the first-order nonlinear solution curves from the new method are inclined, while the two curves from the old method are vertical. The higher-order solutions are linear because all terms in the higher-level solution processes are linear in the old method. That agrees with observations in nonlinear vibration and acoustic research (Rahman and Lee, 2017; Lee, 2018). Figs. 2a–2c show the phase plots for various frequencies. In Fig 2a, the phase plot, which comes from the first-level nonlinear solution, is a larger circle containing two small ellipses inside it. It is implied that at low frequency (i.e. $\omega/\alpha^{1/2}$=1), the nonlinear response contains significant higher harmonic components (or the response is not simple harmonic and contains more than one dominant frequency).

Table 1 Comparison of amplitude $|v|$ (k_1=0.5, $k_2=\alpha=\beta$=1, $\omega/\alpha^{1/2}$=1)

κ	$\lvert v\rvert$			
	Zero-level	First-level	Second-level	Lee, 2002
1	1.3867	1.3530	1.3474	1.3020
2	1.7472	1.6755	1.6487	1.5788
4	2.2013	2.1016	1.9146	1.8864
8	1.3867	1.3530	1.3474	1.3020

Table 2 Comparison of amplitude $|v|$ (k_1=0.5, $k_2=\alpha=\beta$=1, $\omega/\alpha^{1/2}$=2)

κ	$\lvert v\rvert$			
	Zero-level	First-level	Second-level	Lee, 2002
1	2.1495	2.1122	2.1096	2.0482
2	2.2743	2.2314	2.2280	2.1592
4	2.4800	2.4261	2.4206	2.3383
8	2.7956	2.7199	2.7092	2.6035

Table 3 Comparison of amplitude |*v*| (k_1=0.5, k_2=α=β=1, $\omega/\alpha^{1/2}$=6)

κ	|*v*|			
	Zero-level	First-level	Second-level	Lee, 2002
1	6.8455	6.7053	6.6942	6.4684
2	6.8597	6.7188	6.7076	6.4809
4	6.8877	6.7455	6.7341	6.5058
8	6.9428	6.7980	6.7862	6.5546

Table 4 Comparison of the two amplitudes, |*v*| and |*w*| in Eqs. (5a) and (5b) using the classical harmonic method (α=3/4, β=2/3, ω=1)

κ	1 harmonic component		2 harmonic components		3 harmonic components	
	|*v*|	|*w*|	|*v*|	|*w*|	|*v*|	|*w*|
1	1.3653	1.3653	1.4322	1.4322	1.4352	1.4352
2	1.6380	1.6380	1.7724	1.7724	1.7814	1.7814
4	1.9517	1.9517	2.2521	2.2521	2.2845	2.2845
8	2.2013	2.2013	2.9324	2.9324	3.0575	3.0575

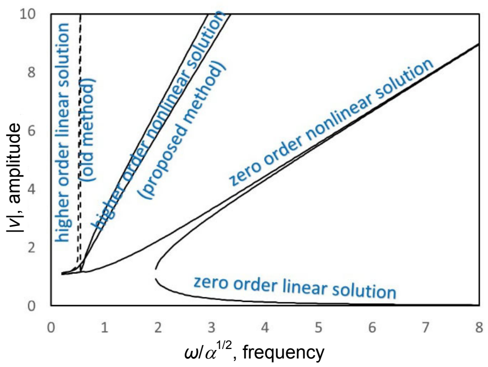

Fig. 1 Amplitude of *v* against frequency (k_1=0.5, k_2=α=β=1, κ=10)

In Figs. 2b and 2c, the two-phase plots, which come from the zero-level solution, are a deformed ellipse and an almost prefect ellipse. The nonlinear response of the deformed ellipse case contains some higher harmonic components only, while the nonlinear response of the prefect ellipse case contains few higher harmonic components. The nonlinear responses shown in Figs. 2a–2c are obtained from the Duffing equation transformed from the three-coupled integrable dispersionless equations. In some nonlinear structural dynamic problems, the same Duffing equation is used for computing the nonlinear responses of beams/plates with large amplitude vibrations.

In final conclusion, the modified residue harmonic balance method has been introduced to solve the three-coupled integrable dispersionless equations with harmonic disturbance. The former version of this method was used for solving the governing equations of quadratic nonlinear beam problems, cubic nonlinear beam problems, nonlinear vibro-acoustic problems, problems of nonlinear delay differential systems, etc. The proposed method can generate the higher-level nonlinear solutions ignored in the old method. The zero-, first-, and second-level solutions have been found and compared with those from the classical harmonic balance method. The second-level solutions from the proposed method and those from the classical harmonic balance method are generally consistent.

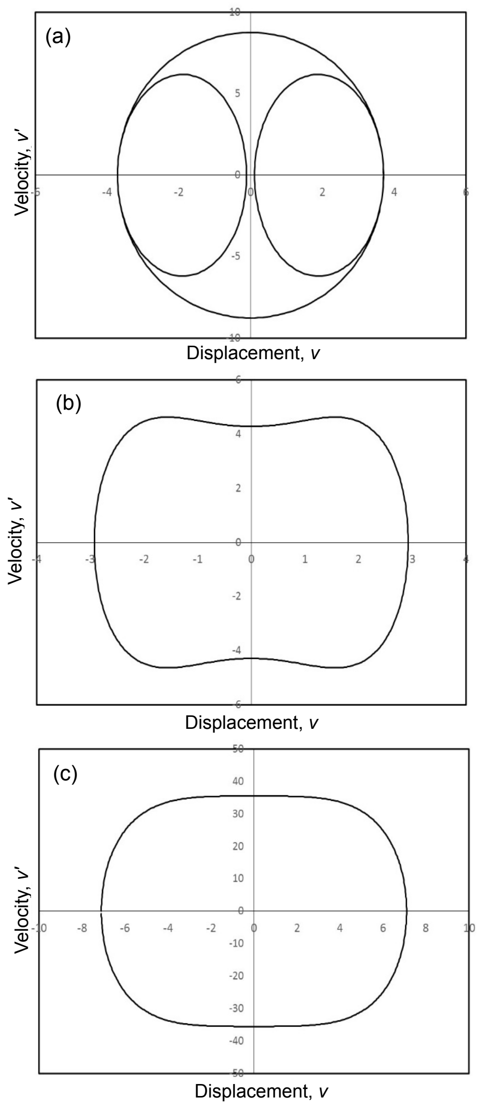

Fig. 2 Phase plot (k_1=0.5, k_2=α=β=1, κ=8, $\omega/\alpha^{1/2}$=1) (a); Phase plot (k_1=0.5, k_2=α=β=1, κ=8, $\omega/\alpha^{1/2}$=2) (b); Phase plot (k_1=0.5, k_2=α=β=1, κ=8, $\omega/\alpha^{1/2}$=6) (c)

References

Guner O, Bekir A, 2018. Solving nonlinear space-time fractional differential equations via ansatz method. *Computational Methods for Differential Equations*, 6(1):1-11.

Hasan ASMZ, Lee YY, Leung AYT, 2016. The multi-level residue harmonic balance solutions of multi-mode nonlinearly vibrating beams on an elastic foundation. *Journal of Vibration and Control*, 22(14):3218-3235. https://doi.org/10.1177/1077546314562225

Hashemi MS, Akgul A, 2018. Solitary wave solutions of time-space nonlinear fractional Schrödinger's equation: two analytical approaches. *Journal of Computational and Applied Mathematics*, 339:147-160. https://doi.org/10.1016/j.cam.2017.11.013

Hu YF, Wang B, 2008. Solution of two-dimensional scattering problem in piezoelectric/piezomagnetic media using a polarization method. *Applied Mathematics and Mechanics-English Edition*, 29(12):1535-1552. https://doi.org/10.1007/s10483-008-1202-x

Huang JB, Xiao ZX, Liu J, et al., 2012. Simulation of shock wave buffet and its suppression on an OAT15A supercritical airfoil by IDDES. *Science China Physics, Mechanics and Astronomy*, 55(2):260-271. https://doi.org/10.1007/s11433-011-4601-9

Huang JL, Zhu WD, 2017. An incremental harmonic balance method with two timescales for quasiperiodic motion of nonlinear systems whose spectrum contains uniformly spaced sideband frequencies. *Nonlinear Dynamics*, 90(2): 1015-1033. https://doi.org/10.1007/s11071-017-3708-6

Huang JL, Chen SH, Su RKL, et al., 2011. Nonlinear analysis of forced responses of an axially moving beam by incremental harmonic balance method. *Mechanics of Advanced Materials and Structures*, 18(8):611-616. https://doi.org/10.1080/15376494.2011.621845

Huang JL, Su KLR, Lee YYR, et al., 2018. Various bifurcation phenomena in a nonlinear curved beam subjected to base harmonic excitation. *International Journal of Bifurcation and Chaos*, 28(7):1830023. https://doi.org/10.1142/S0218127418300239

Lee YY, 2002. Structural-acoustic coupling effect on the nonlinear natural frequency of a rectangular box with one flexible plate. *Applied Acoustics*, 63(11):1157-1175. https://doi.org/10.1016/S0003-682X(02)00033-6

Lee YY, 2017. Large amplitude free vibration of a flexible panel coupled with a leaking cavity. *Journal of Zhejiang University-SCIENCE A (Applied Physics & Engineering)*, 18(1):75-82. https://doi.org/10.1631/jzus.A1600145

Lee YY, 2018. Nonlinear structure-extended cavity interaction simulation using a new version of harmonic balance method. *PLoS One*, 13(7):e0199159. https://doi.org/10.1371/journal.pone.0199159

Leung AYT, Yang HX, Guo ZJ, 2012. Periodic wave solutions of coupled integrable dispersionless equations by residue harmonic balance. *Communications in Nonlinear Science and Numerical Simulation*, 17(11):4508-4514. https://doi.org/10.1016/j.cnsns.2012.03.005

Li W, Yang Y, Sheng DR, et al., 2011. Nonlinear dynamic analysis of a rotor/bearing/seal system. *Journal of Zhejiang University-SCIENCE A (Applied Physics & Engineering)*, 12(1):46-55. https://doi.org/10.1631/jzus.A1000130

Lu QQ, Shao W, Wu YF, et al., 2018. Vibration analysis of an axially moving plate based on sound time-frequency analysis. *International Journal of Acoustics and Vibration*, 23(2):226-233. https://doi.org/10.20855/ijav.2018.23.21429

Mohammadian M, Shariati M, 2017. Approximate analytical solutions to a conservative oscillator using global residue harmonic balance method. *Chinese Journal of Physics*, 55(1):47-58. https://doi.org/10.1016/j.cjph.2016.11.007

Qian YH, Pan JL, Chen SP, et al., 2017. The spreading residue harmonic balance method for strongly nonlinear vibrations of a restrained cantilever beam. *Advances in Mathematical Physics*, 2017:5214616. https://doi.org/10.1155/2017/5214616

Rahman MS, Lee YY, 2017. New modified multi-level residue harmonic balance method for solving nonlinearly vibrating double-beam problem. *Journal of Sound and Vibration*, 406:295-327. https://doi.org/10.1016/j.jsv.2017.06.017

Tasbozan O, Şenol M, Kurt A, et al., 2018. New solutions of fractional Drinfeld-Sokolov-Wilson system in shallow water waves. *Ocean Engineering*, 161:62-68. https://doi.org/10.1016/j.oceaneng.2018.04.075

Wang B, Guo JF, Feng JG, et al., 2016. Nonlinear dynamics and coupling effect of libration and vibration of tethered space robot in deorbiting process. *Journal of Central South University*, 23(5):1095-1105. https://doi.org/10.1007/s11771-016-0359-6

Yin XL, Ma J, Wang XG, et al., 2012. Spin squeezing under non-Markovian channels by the hierarchy equation method. *Physical Review A*, 86(1):012308. https://doi.org/10.1103/PhysRevA.86.012308

Zhang CL, Wang XY, Chen WQ, et al., 2016. Propagation of extensional waves in a piezoelectric semiconductor rod. *AIP Advances*, 6(4):045301. https://doi.org/10.1063/1.4945752

Zheng XH, Zhang BF, Jiao ZX, et al., 2016. Tunable, continuous-wave single-resonant optical parametric oscillator with output coupling for resonant wave. *Chinese Physics B*, 25(1):014208. https://doi.org/10.1088/1674-1056/25/1/014208

Zhong W, Ma J, Liu J, et al., 2014. Derivation of quantum Chernoff metric with perturbation expansion method. *Chinese Physics B*, 23(9):090305. https://doi.org/10.1088/1674-1056/23/9/090305

Zhou WJ, Wei XS, Wei XZ, et al., 2014. Numerical analysis of a nonlinear double disc rotor-seal system. *Journal of*

Zhejiang University-SCIENCE A (Applied Physics & Engineering), 15(1):39-52. https://doi.org/10.1631/jzus.A1300230

Zhu ZQ, Jin XL, Hu SD, et al., 2013. The approximate response of real order nonlinear oscillator using Homotopy analysis method. *Advances in Vibration Engineering*, 12(2):123-134.

List of electronic supplementary materials

Data S1 Theory

中文概要

题　目：一种针对有扰动项的耦合可积非色散方程的修正残差谐波平衡求解方法

目　的：本文将改进残余谐波平衡方法用于求解有扰动项的耦合可积非色散方程，并简化取得破解方案的过程。

创新点：1. 在取得每一阶段破解方案的过程中，只需处理一条非线性代数方程式及一组线性代数方程式；2. 能找出旧方法不能找出的非线性答案。

方　法：1. 使用理论推导、方程式替换及残余谐波平衡方法；2. 通过仿真模拟，推导震动位移与频率之间的关系（图 1）以及位移与速度之间的关系（图 2）。

结　论：1. 成功将改进残余谐波平衡方法应用于有扰动项的耦合可积非色散方程；2. 通过与其他方法产生的数据进行比较，验证了所提方法的可行性和有效性（表 1–3）。

关键词：大幅自主震动；残余谐波平衡；有扰动项的耦合可积非色散方程