Journal of Zhejiang University-SCIENCE C (Computers & Electronics)
ISSN 1869-1951 (Print); ISSN 1869-196X (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Tracking control of the linear differential inclusion*

Jun HUANG, Zheng-zhi HAN

(*School of Electronic, Information and Electrical Engineering, Shanghai Jiao Tong University, Shanghai 200240, China*)

E-mail: {cauchyhot, zzhan}@sjtu.edu.cn

Received July 7, 2010; Revision accepted Oct. 29, 2010; Crosschecked May 5, 2011

Abstract: The tracking control of linear differential inclusion is discussed. First, the definition of uniformly ultimate boundedness for linear differential inclusion is given. Then, a feedback law is constructed by using the convex hull Lyapunov function. The sufficient condition is given to guarantee the tracking error system uniformly ultimately bounded. Finally, a numerical example is simulated to illustrate the effectiveness of this control design.

Key words: Linear differential inclusions, Tracking control, Convex hull Lyapunov functions, Uniformly ultimate boundedness

doi:10.1631/jzus.C1000240 **Document code:** A **CLC number:** TP13

1 Introduction

Differential inclusion (DI) has been widely studied because of its practical and theoretical significance (Aubin and Cellina, 1984; Smirnov, 2002). Recently, many works have focused on linear differential inclusion (LDI), since it is much simpler than nonlinear DI and can include a class of nonlinear systems with uncertainties (Boyd *et al.*, 1994; Botchkarev and Tripakis, 2000; Chen, 2001; Goebel *et al.*, 2004). To study the stability of LDI, Hu and Lin (2004) defined the convex hull Lyapunov function (CHLF), and it is verified that the CHLF holds less conservative than the traditional Lyapunov function. By using the CHLF, Hu (2007) designed a nonlinear feedback law to stabilize LDI, Cai *et al.* (2009) presented saturated control design for LDI, Liu *et al.* (2010) discussed LDI with time delay, Huang *et al.* (2010) investigated time-delayed LDI with stochastic disturbance, and Huang *et al.* (2011) studied LDI with affine uncertainty.

Tracking control is always a hot topic in the field of control system design, and many papers have been published to present various designing methods of the tracking, such as the composite tracking control method (Sun, 2008), H^∞ control method (Gao and Chen, 2008), sliding mode control method (Slotine and Sastry, 1983), robust control method (Qu and Dawson, 1995), and adaptive control method (Li *et al.*, 2003). We know that, for tracking control of the uncertain system, it is not easy to realize a completely perfect tracking, and making the tracking error uniformly ultimately bounded (UUB) is thus meaningful (Pogromsky *et al.*, 2003; Lee and Zak, 2004; Hayakawa *et al.*, 2005; Wu and Shi, 2010). However, to the best of the authors' knowledge, little attention has been paid to the tracking control of LDI.

Motivated by the above discussion, this paper considers the tracking control of LDI. It gives a new control design by the CHLF and makes the tracking error system UUB.

2 Problem formulation and preliminaries

Let us consider LDI described by

$$\dot{\boldsymbol{x}} \in \mathrm{co}\{\boldsymbol{A}_i\boldsymbol{x}+\boldsymbol{B}_i\boldsymbol{u}\} : i \in I[1,N], \quad \boldsymbol{x}(0)=\boldsymbol{x}_0, \quad (1)$$

where co$\{\cdot\}$ denotes the convex hull of a set. $\boldsymbol{x} \in \mathbb{R}^n$ is the system state, and $\boldsymbol{u} \in \mathbb{R}^m$ is the control input. $\boldsymbol{A}_i$, $\boldsymbol{B}_i$ are determined real matrices of compatible

* Project (No. 61074003) supported by the National Natural Science Foundation of China

dimensions, and $I[1,N]$ denotes the set $\{1,2,\cdots,N\}$. $\boldsymbol{x}_0 \in \mathbb{R}^n$ is the initial state.

The desired state $\bar{\boldsymbol{x}}$ is bounded and satisfies the following equation:

$$\dot{\bar{\boldsymbol{x}}} = \boldsymbol{A}\bar{\boldsymbol{x}}, \quad \bar{\boldsymbol{x}}(0) = \bar{\boldsymbol{x}}_0, \tag{2}$$

where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is a known matrix, and $\bar{\boldsymbol{x}}_0 \in \mathbb{R}^n$ is the initial state.

Denote $\boldsymbol{e} = \boldsymbol{x} - \bar{\boldsymbol{x}}$ and $\boldsymbol{e}_0 = \boldsymbol{x}_0 - \bar{\boldsymbol{x}}_0$. Then the tracking error system is

$$\begin{aligned}&\dot{\boldsymbol{e}} \in \mathrm{co}\{\boldsymbol{A}_i\boldsymbol{e} + \boldsymbol{B}_i\boldsymbol{u} + (\boldsymbol{A}_i - \boldsymbol{A})\bar{\boldsymbol{x}}\},\\ &i \in I[1,N], \boldsymbol{e}(0) = \boldsymbol{e}_0.\end{aligned} \tag{3}$$

We now introduce the definition of UUB. For simplicity, let us consider the system (3) without the input $\boldsymbol{u}$, i.e.,

$$\begin{aligned}&\dot{\boldsymbol{e}} \in \mathrm{co}\{\boldsymbol{A}_i\boldsymbol{e} + (\boldsymbol{A}_i - \boldsymbol{A})\bar{\boldsymbol{x}}\},\\ &i \in I[1,N], \boldsymbol{e}(0) = \boldsymbol{e}_0.\end{aligned} \tag{4}$$

Definition 1 Let $\boldsymbol{e}(t)$ be a solution of the system (4). The system (4) is said to be UUB, if there exist positive constants r, $T(r)$, and b such that $||\boldsymbol{e}_0|| \le r$ implies that $||\boldsymbol{e}(t)|| \le b$ for all $t > T(r)$, where $||\cdot||$ denotes the Euclidean norm.

Remark 1 It is obvious that if $\lim_{t\to\infty} ||\boldsymbol{e}(t)|| \le b$, then the system (4) is UUB. For convenience, in this paper we call b the ultimate bound of the system (4).

The following lemma gives the condition, under which the system (4) is UUB:

Lemma 1 Let $\boldsymbol{e}$ be a solution of the system (4). If there exists a positive definite function $V(\boldsymbol{e}) = \boldsymbol{e}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{e}$ (λ_1 and λ_2 are the minimal and maximal eigenvalues of $\boldsymbol{Q}$ respectively) such that

$$\dot{V}(\boldsymbol{e}) \le -\beta V(\boldsymbol{e}) + \varepsilon, \tag{5}$$

where $\beta > 0$, $\varepsilon > 0$ are real constants, then the system (4) is UUB with ultimate bound $\sqrt{\varepsilon/(\lambda_1\beta)}$.

Proof From Eq. (5), we can obtain

$$\begin{aligned}V(\boldsymbol{e}) &\le V(\boldsymbol{e}_0)\exp(-\beta t) + \varepsilon\int_0^t \exp(-\beta(t-\tau))\mathrm{d}\tau\\ &= V(\boldsymbol{e}_0)\exp(-\beta t) + \frac{\varepsilon}{\beta}(1-\exp(-\beta t))\\ &\le V(\boldsymbol{e}_0)\exp(-\beta t) + \frac{\varepsilon}{\beta}.\end{aligned} \tag{6}$$

Eq. (6) deduces to

$$||\boldsymbol{e}|| \le \sqrt{\frac{\lambda_2}{\lambda_1}||\boldsymbol{e}_0||^2\exp(-\beta t) + \frac{\varepsilon}{\lambda_1\beta}}. \tag{7}$$

Eq. (7) implies that $\lim_{t\to\infty}||\boldsymbol{e}(t)|| \le \sqrt{\varepsilon/(\lambda_1\beta)}$. From Definition 1 and Remark 1, we complete the proof.

We now give some basic material of the CHLF. Detailed presentation can be referred to Hu and Lin (2004).

Define a function $V(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$, where $\boldsymbol{P}$ is a positive definite matrix. Let ρ be a positive number. The ρ-level set of $V(\boldsymbol{x})$ is given by $L_V(\rho) = \{\boldsymbol{x} : V(\boldsymbol{x}) \le \rho\}$, and $L_V(\rho)$ is a compact set of $\mathbb{R}^n$. In what follows, we denote $L_V(1)$ by L_V, and then $\sqrt{\rho}L_V = L_V(\rho)$.

Let $\boldsymbol{Q}_j \in \mathbb{R}^{n\times n}$ be positive definite matrices for $j \in I[1,J]$. Then a CHLF $V_{\mathrm{c}}(\boldsymbol{x})$ is defined by

$$V_{\mathrm{c}}(\boldsymbol{x}) = \min_{\boldsymbol{s}\in S}\left(\boldsymbol{x}^{\mathrm{T}}\left(\sum_{j=1}^{J} s_j\boldsymbol{Q}_j\right)^{-1}\boldsymbol{x}\right), \tag{8}$$

where $S = \{s = (s_1, s_2, \cdots, s_J)^{\mathrm{T}} \in \mathbb{R}^J : s_j \ge 0, \sum_{j=1}^{J} s_j = 1\}$. A function s^* on S is defined by

$$s^*(\boldsymbol{x}) = \arg\min_{\boldsymbol{s}\in S}\left(\boldsymbol{x}^{\mathrm{T}}\left(\sum_{j=1}^{J} s_j\boldsymbol{Q}_j\right)^{-1}\boldsymbol{x}\right). \tag{9}$$

Eq. (8) is well-defined and $s^*(\boldsymbol{x})$ is always a continuous function of $\boldsymbol{x}$ except for some degenerated cases. Eq. (8) is equivalent to

$$V_{\mathrm{c}}(\boldsymbol{x}) = \min_{\boldsymbol{s}\in S}\alpha$$

$$\text{s.t.}\begin{bmatrix}\alpha & \boldsymbol{x}^{\mathrm{T}}\\ \boldsymbol{x} & \sum_{j=1}^{J} s_j\boldsymbol{Q}_j\end{bmatrix} \ge 0, \quad \boldsymbol{s} \in S,$$

which is easier to compute than Eq. (8). It is obvious that

$$L_{V_{\mathrm{c}}} = \mathrm{co}\{\varepsilon(\boldsymbol{Q}_j^{-1}), j \in I[1,J]\},$$

where $\varepsilon(\boldsymbol{Q}_j^{-1}) = \{\boldsymbol{x} : \boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}_j^{-1}\boldsymbol{x} \le 1\}$. For a compact convex set W, let ∂W denote the boundary of W. From the theory of convex analysis, we know that a point $\boldsymbol{x} \in \partial W$ that cannot be represented as a convex combination of other points in W is an extreme point and that a compact convex set is characterized by its extreme points. Denote $E_k = \partial L_{V_{\mathrm{c}}} \bigcap \varepsilon(\boldsymbol{Q}_k^{-1}) = \{\boldsymbol{x} : V_{\mathrm{c}}(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}_k^{-1}\boldsymbol{x} = 1\}$.

Lemma 2 (Hu *et al.*, 2006) For each $k \in I[1, J]$, $E_k = \{\boldsymbol{x} \in \partial L_{V_c} : \boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}_k^{-1}(\boldsymbol{Q}_j - \boldsymbol{Q}_k)\boldsymbol{Q}_k^{-1}\boldsymbol{x} \le 0, j \in I[1, J]\}$.

Lemma 3 (Hu and Lin, 2004) Let $\boldsymbol{x} \in \mathbb{R}^n$. For simplicity and without loss of generality, assume $s_k^*(\boldsymbol{x}) > 0$ for $k \in I[1, J_0]$ and $s_k^* = 0$ for $k \in I[J_0 + 1, J]$. Denote

$$\boldsymbol{Q}(s^*) = \sum_{k=1}^{J_0} s_k^*\boldsymbol{Q}_k, \boldsymbol{x}_k = \boldsymbol{Q}_k\boldsymbol{Q}(s^*)^{-1}\boldsymbol{x}, k \in I[1, J_0].$$

Then $V_c(\boldsymbol{x}_k) = V_c(\boldsymbol{x}) = \boldsymbol{x}_k^{\mathrm{T}}\boldsymbol{Q}_k^{-1}\boldsymbol{x}_k$, and $\boldsymbol{x}_k \in (V_c(\boldsymbol{x}))^{1/2}E_k$ for $k \in I[1, J_0]$. Moreover, $\boldsymbol{x} = \sum_{k=1}^{J_0} s_k^*\boldsymbol{x}_k$, and $\forall k \in I[1, J_0]$,

$$\nabla V_c(\boldsymbol{x}) = \nabla V_c(\boldsymbol{x}_k) = 2\boldsymbol{Q}_k^{-1}\boldsymbol{x}_k = 2\boldsymbol{Q}(s^*)^{-1}\boldsymbol{x}, \tag{10}$$

where $\nabla V_c(\boldsymbol{x})$ denotes the gradient of V_c at $\boldsymbol{x}$.

Lemma 4 is a direct result of the Schwartz inequality, and its proof is omitted.

Lemma 4 For any vectors $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^n$, and a constant $\varepsilon > 0$, the following inequality holds:

$$2\boldsymbol{v}_1^{\mathrm{T}}\boldsymbol{v}_2 \le \frac{1}{\varepsilon}\boldsymbol{v}_1^{\mathrm{T}}\boldsymbol{v}_1 + \varepsilon\boldsymbol{v}_2^{\mathrm{T}}\boldsymbol{v}_2. \tag{11}$$

3 Main results

In this section, we design a nonlinear feedback law to make the error system (3) UUB.

Theorem 1 Let $\boldsymbol{Q}_k \in \mathbb{R}^{n\times n}$, $\boldsymbol{Q}_k = \boldsymbol{Q}_k^{\mathrm{T}} > 0$, $k \in I[1, J]$, and $V_c(\boldsymbol{e})$ be the function as defined in Eq. (8). If there exist $\boldsymbol{F}_k \in \mathbb{R}^{m\times n}$, $\lambda_{ijk} \ge 0$, $i \in I[1, N]$, $j, k \in I[1, J]$, and $\beta > 0$, $\varepsilon > 0$ such that

$$\begin{aligned}&\boldsymbol{Q}_k\boldsymbol{A}_i^{\mathrm{T}} + \boldsymbol{A}_i\boldsymbol{Q}_k + \boldsymbol{F}_k^{\mathrm{T}}\boldsymbol{B}_i^{\mathrm{T}} + \boldsymbol{B}_i\boldsymbol{F}_k \\ &\quad + \frac{1}{\varepsilon}(\boldsymbol{A}_i - \boldsymbol{A})(\boldsymbol{A}_i - \boldsymbol{A})^{\mathrm{T}} \\ &\le \sum_{j=1}^{J}\lambda_{ijk}(\boldsymbol{Q}_j - \boldsymbol{Q}_k) - \beta\boldsymbol{Q}_k, \quad \forall\, i, k,\end{aligned} \tag{12}$$

then define

$$\boldsymbol{F}(s^*) = \sum_{k=1}^{J} s_k^*\boldsymbol{F}_k, \quad \boldsymbol{Q}(s^*) = \sum_{k=1}^{J} s_k^*\boldsymbol{Q}_k, \tag{13}$$

where $s^* \in S$ is as defined in Eq. (9). Then, by the nonlinear feedback law

$$\boldsymbol{u} = \boldsymbol{F}(s^*)\boldsymbol{Q}(s^*)^{-1}\boldsymbol{e}, \tag{14}$$

the system (3) is UUB with the ultimate bound $\sqrt{\varepsilon/(\beta\kappa)}||\bar{\boldsymbol{x}}||$, where κ is the minimal eigenvalue of $\boldsymbol{Q}(s^*)^{-1}$. Moreover, $\boldsymbol{u}$ is a continuous function of $\boldsymbol{e}$.

Proof Multiplying Eq. (12) from both sides by $\boldsymbol{Q}_k^{-1}$, we can obtain

$$\begin{aligned}&\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{Q}_k^{-1} + \boldsymbol{Q}_k^{-1}\boldsymbol{A}_i + \boldsymbol{Q}_k^{-1}\boldsymbol{F}_k^{\mathrm{T}}\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{Q}_k^{-1} + \boldsymbol{Q}_k^{-1}\boldsymbol{B}_i\boldsymbol{F}_k\boldsymbol{Q}_k^{-1} \\ &\quad + \frac{1}{\varepsilon}\boldsymbol{Q}_k^{-1}(\boldsymbol{A}_i - \boldsymbol{A})(\boldsymbol{A}_i - \boldsymbol{A})^{\mathrm{T}}\boldsymbol{Q}_k^{-1} \\ &\le \sum_{j=1}^{J}\lambda_{ijk}\boldsymbol{Q}_k^{-1}(\boldsymbol{Q}_j - \boldsymbol{Q}_k)\boldsymbol{Q}_k^{-1} - \beta\boldsymbol{Q}_k^{-1}, \quad i \in [1, N].\end{aligned} \tag{15}$$

By the feedback law (14), the closed-loop system (3) is

$$\dot{\boldsymbol{e}} \in \mathrm{co}\{(\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{F}(s^*)\boldsymbol{Q}(s^*)^{-1})\boldsymbol{e} + (\boldsymbol{A}_i - \boldsymbol{A})\bar{\boldsymbol{x}}\}, \tag{16}$$

where $i \in I[1, N]$. In the following, we prove the conclusion in two aspects.

1. Let $\boldsymbol{e} \in E_k$ for some $k \in I[1, J]$. Then, $V_c(\boldsymbol{e}) = \boldsymbol{e}^{\mathrm{T}}\boldsymbol{Q}_k^{-1}\boldsymbol{e} = 1$ and $s^*(\boldsymbol{e})$ is a vector in which the kth element is 1 and the rest are zeros. We have $\boldsymbol{F}(s^*)\boldsymbol{Q}(s^*)^{-1} = \boldsymbol{F}_k\boldsymbol{Q}_k^{-1}$ and $\nabla V(\boldsymbol{e}) = 2\boldsymbol{Q}_k^{-1}\boldsymbol{e}$, and then

$$\begin{aligned}&\nabla V_c(\boldsymbol{e})^{\mathrm{T}}[(\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{F}(s^*)\boldsymbol{Q}(s^*)^{-1})\boldsymbol{e} + (\boldsymbol{A}_i - \boldsymbol{A})\bar{\boldsymbol{x}}] \\ &= 2\boldsymbol{e}^{\mathrm{T}}\boldsymbol{Q}_k^{-1}(\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{F}_k\boldsymbol{Q}_k^{-1})\boldsymbol{e} + 2\boldsymbol{e}^{\mathrm{T}}\boldsymbol{Q}_k^{-1}(\boldsymbol{A}_i - \boldsymbol{A})\bar{\boldsymbol{x}}.\end{aligned} \tag{17}$$

In view of Lemma 4, we have

$$\begin{aligned}&2\boldsymbol{e}^{\mathrm{T}}\boldsymbol{Q}_k^{-1}(\boldsymbol{A}_i - \boldsymbol{A})\bar{\boldsymbol{x}} \\ &\le \frac{1}{\varepsilon}\boldsymbol{e}^{\mathrm{T}}\boldsymbol{Q}_k^{-1}(\boldsymbol{A}_i - \boldsymbol{A})(\boldsymbol{A}_i - \boldsymbol{A})^{\mathrm{T}}\boldsymbol{Q}_k^{-1}\boldsymbol{e} + \varepsilon\bar{\boldsymbol{x}}^{\mathrm{T}}\bar{\boldsymbol{x}}.\end{aligned} \tag{18}$$

Then, for $i \in I[1, N]$,

$$\begin{aligned}&\nabla V_c(\boldsymbol{e})^{\mathrm{T}}[(\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{F}(s^*)\boldsymbol{Q}(s^*)^{-1})\boldsymbol{e} + (\boldsymbol{A}_i - \boldsymbol{A})\bar{\boldsymbol{x}}] \\ &\le 2\boldsymbol{e}^{\mathrm{T}}\boldsymbol{Q}_k^{-1}(\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{F}_k\boldsymbol{Q}_k^{-1})\boldsymbol{e} \\ &\quad + \frac{1}{\varepsilon}\boldsymbol{e}^{\mathrm{T}}\boldsymbol{Q}_k^{-1}(\boldsymbol{A}_i - \boldsymbol{A})(\boldsymbol{A}_i - \boldsymbol{A})^{\mathrm{T}}\boldsymbol{Q}_k^{-1}\boldsymbol{e} + \varepsilon||\bar{\boldsymbol{x}}||^2.\end{aligned} \tag{19}$$

Substituting Eq. (15) into Eq. (19) yields

$$\begin{aligned}&\nabla V_c(\boldsymbol{e})^{\mathrm{T}}[(\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{F}(s^*)\boldsymbol{Q}(s^*)^{-1})\boldsymbol{e} + (\boldsymbol{A}_i - \boldsymbol{A})\bar{\boldsymbol{x}}] \\ &\le \sum_{j=1}^{J}\lambda_{ijk}\boldsymbol{e}^{\mathrm{T}}\boldsymbol{Q}_k^{-1}(\boldsymbol{Q}_j - \boldsymbol{Q}_k)\boldsymbol{Q}_k^{-1}\boldsymbol{e} \\ &\quad - \beta\boldsymbol{e}^{\mathrm{T}}\boldsymbol{Q}_k^{-1}\boldsymbol{e} + \varepsilon||\bar{\boldsymbol{x}}||^2.\end{aligned} \tag{20}$$

By Lemma 2, for $i \in I[1,N]$, $k \in I[1,J]$, we have

$$\sum_{j=1}^{J} \lambda_{ijk} \boldsymbol{e}^{\mathrm{T}} \boldsymbol{Q}_k^{-1} (\boldsymbol{Q}_j - \boldsymbol{Q}_k) \boldsymbol{Q}_k^{-1} \boldsymbol{e} \leq 0. \tag{21}$$

Then, for any $i \in I[1,N]$,

$$\nabla V_{\mathrm{c}}(\boldsymbol{e})^{\mathrm{T}}[(\boldsymbol{A}_i + \boldsymbol{B}_i \boldsymbol{F}(s^*) \boldsymbol{Q}(s^*)^{-1})\boldsymbol{e} + (\boldsymbol{A}_i - \boldsymbol{A})\bar{\boldsymbol{x}}]$$
$$\leq -\beta \boldsymbol{e}^{\mathrm{T}} \boldsymbol{Q}_k^{-1} \boldsymbol{e} + \varepsilon ||\bar{\boldsymbol{x}}||^2 = -\beta V_{\mathrm{c}}(\boldsymbol{e}) + \varepsilon ||\bar{\boldsymbol{x}}||^2. \tag{22}$$

Eq. (22) implies that

$$\max_{i \in [1,N]} \{\nabla V_{\mathrm{c}}(\boldsymbol{e})^{\mathrm{T}}[(\boldsymbol{A}_i + \boldsymbol{B}_i \boldsymbol{F}(s^*) \boldsymbol{Q}(s^*)^{-1})\boldsymbol{e}$$
$$+ (\boldsymbol{A}_i - \boldsymbol{A})\bar{\boldsymbol{x}}]\} \leq -\beta V_{\mathrm{c}}(\boldsymbol{e}) + \varepsilon ||\bar{\boldsymbol{x}}||^2. \tag{23}$$

2. Let $\boldsymbol{e}_0 \in \partial L_{V_{\mathrm{c}}}$. From Lemma 3, it is obvious that $\boldsymbol{e}_0$ is a convex combination of a set of $\boldsymbol{e}_k$'s and $\boldsymbol{e}_k \in E_k$ for $k \in [1,J]$. To simplify the following proof, we suppose that $s_k^*(\boldsymbol{e}_0) > 0$ for $k \in I[1,J_0]$ and $s_k^*(\boldsymbol{e}_0) = 0$ for $k \in I[J_0+1,J]$. Then, $\boldsymbol{e}_0 = \sum_{k=1}^{J_0} s_k^* e_k$. Since $\nabla V_{\mathrm{c}}(\boldsymbol{e}_0) = 2\boldsymbol{Q}(s^*)^{-1}\boldsymbol{e}_0$,

$$\boldsymbol{Q}(s^*)^{-1}\boldsymbol{e}_0 = \boldsymbol{Q}_k^{-1}\boldsymbol{e}_k,\ V_{\mathrm{c}}(\boldsymbol{e}_0) = V_{\mathrm{c}}(\boldsymbol{e}_k), \tag{24}$$

we have

$$\boldsymbol{F}(s^*)\boldsymbol{Q}(s^*)^{-1}\boldsymbol{e}_0 = \sum_{k=1}^{J_0} s_k^* \boldsymbol{F}_k \boldsymbol{Q}_k^{-1} \boldsymbol{e}_k. \tag{25}$$

Then,

$$\begin{aligned}
&\nabla V_{\mathrm{c}}(\boldsymbol{e}_0)^{\mathrm{T}}[(\boldsymbol{A}_i + \boldsymbol{B}_i \boldsymbol{F}(s^*) \boldsymbol{Q}(s^*)^{-1})\boldsymbol{e}_0 + (\boldsymbol{A}_i - \boldsymbol{A})\bar{\boldsymbol{x}}] \\
&= 2\boldsymbol{e}_0^{\mathrm{T}} \boldsymbol{Q}(s^*)^{-1} \left[\boldsymbol{A}_i \boldsymbol{e}_0 + \boldsymbol{B}_i \boldsymbol{F}(s^*) \boldsymbol{Q}(s^*)^{-1} \boldsymbol{e}_0 \right. \\
&\qquad \left. + (\boldsymbol{A}_i - \boldsymbol{A})\bar{\boldsymbol{x}}\right] \\
&= 2\boldsymbol{e}_k^{\mathrm{T}} \boldsymbol{Q}_k^{-1} \left[\boldsymbol{A}_i \sum_{k=1}^{J_0} s_k^* \boldsymbol{e}_k + (\boldsymbol{A}_i - \boldsymbol{A})\bar{\boldsymbol{x}} \right. \\
&\qquad \left. + \boldsymbol{B}_i \sum_{k=1}^{J_0} s_k^* \boldsymbol{F}_k \boldsymbol{Q}_k^{-1} \boldsymbol{e}_k \right] \\
&= \sum_{k=1}^{J_0} s_k^* \left[2\boldsymbol{e}_k^{\mathrm{T}} \boldsymbol{Q}_k^{-1} \boldsymbol{A}_i \boldsymbol{e}_k + 2\boldsymbol{e}_k^{\mathrm{T}} \boldsymbol{Q}_k^{-1} \boldsymbol{B}_i \boldsymbol{F}_k \boldsymbol{Q}_k^{-1} \boldsymbol{e}_k \right. \\
&\qquad \left. + 2\boldsymbol{e}_k^{\mathrm{T}} \boldsymbol{Q}_k^{-1} (\boldsymbol{A}_i - \boldsymbol{A})\bar{\boldsymbol{x}}\right].
\end{aligned} \tag{26}$$

By the same arguments as that used in step 1, Eq. (26) deduces to

$$\max_{i \in [1,N]} \{\nabla V_{\mathrm{c}}(\boldsymbol{e}_0)^{\mathrm{T}}[(\boldsymbol{A}_i + \boldsymbol{B}_i \boldsymbol{F}(s^*) \boldsymbol{Q}(s^*)^{-1})\boldsymbol{e}_0$$
$$+ (\boldsymbol{A}_i - \boldsymbol{A})\bar{\boldsymbol{x}}]\} \leq \sum_{k=1}^{J_0} s_k^* (-\beta \boldsymbol{e}_k^{\mathrm{T}} \boldsymbol{Q}_k^{-1} \boldsymbol{e}_k) + \varepsilon ||\bar{\boldsymbol{x}}||^2$$
$$= -\beta V_{\mathrm{c}}(\boldsymbol{e}_0) + \varepsilon ||\bar{\boldsymbol{x}}||^2. \tag{27}$$

By Lemma 1, combining Eq. (23) with Eq. (27), we can conclude that the system (3) is UUB under the feedback law (14), and that the ultimate bound of the system (3) is $\sqrt{\varepsilon/(\beta\kappa)}||\bar{\boldsymbol{x}}||$. Moreover, $\boldsymbol{F}(s^*)$ and $\boldsymbol{Q}(s^*)$ are continuous in s^*, and $\boldsymbol{Q}(s^*) > 0$. The continuity of $\boldsymbol{u}$ follows from that of $s^*(\boldsymbol{e})$. We thus complete the proof.

4 Example

In this section, we consider a second-order LDI system described by

$$\dot{\boldsymbol{x}} \in \mathrm{co}\{\boldsymbol{A}_1 \boldsymbol{x} + \boldsymbol{B}_1 u, \boldsymbol{A}_2 \boldsymbol{x} + \boldsymbol{B}_2 u\},\ \ \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{28}$$

where

$$\boldsymbol{A}_1 = \begin{bmatrix} 0.25 & -0.9 \\ 1 & 0.2 \end{bmatrix},\ \boldsymbol{B}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\boldsymbol{A}_2 = \begin{bmatrix} 0.1 & -0.7 \\ 1 & 0.15 \end{bmatrix},\ \boldsymbol{B}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ \boldsymbol{x}_0 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

The desired state $\bar{\boldsymbol{x}}$ satisfies

$$\dot{\bar{\boldsymbol{x}}} = \boldsymbol{A}\bar{\boldsymbol{x}}, \quad \bar{\boldsymbol{x}}(0) = \bar{\boldsymbol{x}}_0, \tag{29}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \bar{\boldsymbol{x}}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Let $V_{\mathrm{c}}(\boldsymbol{e})$ be composed by $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$, and $\varepsilon = 0.01$. There exist $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$, $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, β, λ_{121}, λ_{112}, λ_{221}, and λ_{212} as the following:

$$\boldsymbol{Q}_1 = \begin{bmatrix} 24.2632 & 42.3931 \\ 42.3931 & 76.1995 \end{bmatrix},$$

$$\boldsymbol{Q}_2 = \begin{bmatrix} 23.1692 & 39.8241 \\ 39.8241 & 71.1436 \end{bmatrix},$$

$$\boldsymbol{F}_1 = \begin{bmatrix} -76.1204 & -207.6789 \end{bmatrix},$$

$$\boldsymbol{F}_2 = \begin{bmatrix} -70.6421 & -196.9910 \end{bmatrix}, \quad \beta = 2,$$

$$\lambda_{121} = 0.1765,\ \lambda_{112} = 0.1245,$$

$$\lambda_{221} = 0.5676,\ \lambda_{212} = 0.2326,$$

such that Eq. (12) holds for $i = 1,2$ and $k = 1,2$. Let

$$\boldsymbol{F}(s^*) = s^* \boldsymbol{F}_1 + (1-s^*)\boldsymbol{F}_2,\ \boldsymbol{Q}(s^*) = s^* \boldsymbol{Q}_1 + (1-s^*)\boldsymbol{Q}_2,$$

where s^* is defined by

$$s^*(\boldsymbol{e}) = \arg\min_{s\in S} \left(\boldsymbol{e}^{\mathrm{T}}(s\boldsymbol{Q}_1 + (1-s)\boldsymbol{Q}_2)^{-1}\boldsymbol{e}\right).$$

We can design the nonlinear feedback law as

$$u = \boldsymbol{F}(s^*)\boldsymbol{Q}(s^*)^{-1}\boldsymbol{e}. \tag{30}$$

Then the closed-loop system (3) is UUB, and the ultimate bound is $\sqrt{\varepsilon/(\beta\kappa)}||\bar{\boldsymbol{x}}|| \approx 0.1$.

By the theory of DI (Smirnov, 2002), the system (28) can be written as

$$\dot{\boldsymbol{x}} = \alpha(\boldsymbol{A}_1\boldsymbol{x} + \boldsymbol{B}_1 u) + (1-\alpha)(\boldsymbol{A}_2\boldsymbol{x} + \boldsymbol{B}_2 u), \tag{31}$$

where $0 \le \alpha \le 1$. We use the Simulink in Matlab to do the simulation. The values of parameter α are taken as 0.1 and 0.5, respectively. Denote $\boldsymbol{x} = (x_1 \quad x_2)^{\mathrm{T}}$, and $\bar{\boldsymbol{x}} = (x_{\mathrm{d1}} \quad x_{\mathrm{d2}})^{\mathrm{T}}$. Figs. 1–4 show the state trajectories $\boldsymbol{x}(t)$ under the feedback law (30) for $\alpha = 0.1$, $\alpha = 0.5$ respectively and the desired state trajectories $\bar{\boldsymbol{x}}(t)$ of the system (29). We can conclude that the designed nonlinear feedback law is effective.

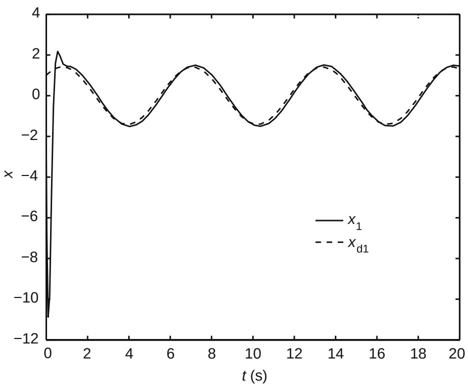

Fig. 1 The state trajectory $\boldsymbol{x}_1$ of the closed-loop system (31) when $\alpha = 0.1$ and the desired state trajectory $\boldsymbol{x}_{\mathrm{d1}}$

5 Conclusions

In this paper, we deal with the problem of tracking control of linear differential inclusion. By using the convex hull Lyapunov function, we design a nonlinear feedback law to make the tracking error uniformly ultimately bounded and give a set of bilinear matrix inequalities to guarantee the existence of the feedback law. A simulation illustrates the effectiveness of the proposed method.

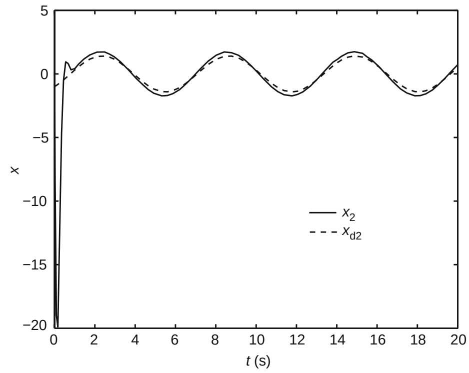

Fig. 2 The state trajectory $\boldsymbol{x}_2$ of the closed-loop system (31) when $\alpha = 0.1$ and the desired state trajectory $\boldsymbol{x}_{\mathrm{d2}}$

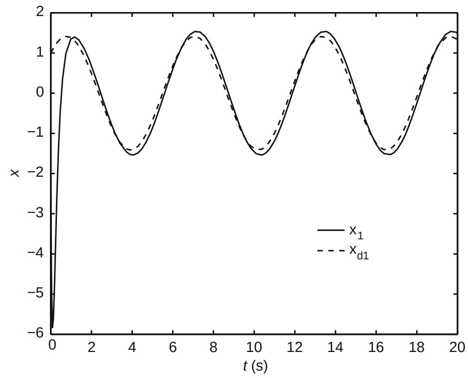

Fig. 3 The state trajectory $\boldsymbol{x}_1$ of the closed-loop system (31) when $\alpha = 0.5$ and the desired state trajectory $\boldsymbol{x}_{\mathrm{d1}}$

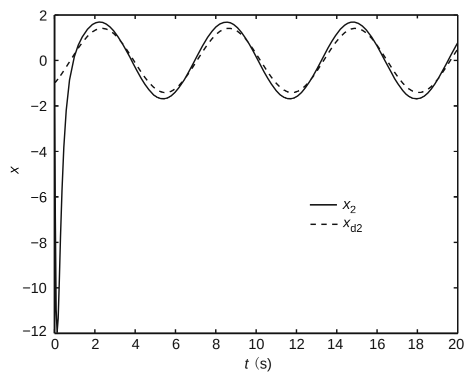

Fig. 4 The state trajectory $\boldsymbol{x}_2$ of the closed-loop system (31) when $\alpha = 0.5$ and the desired state trajectory $\boldsymbol{x}_{\mathrm{d2}}$

References

Aubin, J., Cellina, A., 1984. Differential Inclusions: Set-Valued Maps and Viability Theory. Springer Verlag, Berlin.

Botchkarev, O., Tripakis, S., 2000. Verification of hybrid systems with linear differential inclusions using ellipsoidal approximations. *LNCS*, **1790**:73-88. [doi:10.1007/3-540-46430-1_10]

Boyd, S., Ghaoui, L., Feron, E., Balakrishnan, V., 1994. Linear Matrix Inequlities in System and Control Theory. SIAM, Philadelphia.

Cai, X., Liu, L., Zhang, W., 2009. Saturated control design for linear differential inclusions subject to disturbance. *Nonl. Dynam.*, **58**(3):487-496. [doi:10.1007/s11071-009-9494-z]

Chen, J.W., 2001. Asymptotic stability for tracking control of nonlinear uncertain dynamical systems described by differential inclusions. *J. Math. Anal. Appl.*, **261**(1):369-389. [doi:10.1006/jmaa.2001.7531]

Gao, H., Chen, T., 2008. Network-based H^{∞} output tracking control. *IEEE Trans. Automat. Control*, **53**(3):655-667. [doi:10.1109/TAC.2008.919850]

Goebel, R., Teel, A., Hu, T., Lin, Z., 2004. Dissipativity for Dual Linear Differential Inclusions Through Conjugate Storage Functions. 43rd IEEE Conf. on Decision and Control, p.2700-2705.

Hayakawa, T., Haddad, W., Hovakimyan, N., Chellaboina, V., 2005. Neural network adaptive control for nonlinear nonnegative dynamical systems. *IEEE Trans. Neur. Networks*, **16**(2):399-413. [doi:10.1109/TNN.2004.841791]

Hu, T., 2007. Nonlinear control design for linear differential inclusions via convex hull of quadratics. *Automatica*, **43**(4):685-692. [doi:10.1016/j.automatica.2006.10.015]

Hu, T., Lin, Z., 2004. Properties of the composite quadratic Lyapunov functions. *IEEE Trans. Automat. Control*, **49**(7):1162-1167. [doi:10.1109/TAC.2004.831132]

Hu, T., Teel, A., Zaccarian, L., 2006. Stability and performance for saturated systems via quadratic and nonquadratic Lyapunov functions. *IEEE Trans. Automat. Control*, **51**(11):1770-1786. [doi:10.1109/TAC.2006.884942]

Huang, J., Han, Z., Cai, X., Liu, L., 2010. Control of time-delayed linear differential inclusions with stochastic disturbance. *J. Franklin Inst.*, **347**(10):1895-1906. [doi:10.1016/j.jfranklin.2010.10.008]

Huang, J., Han, Z., Cai, X., Liu, L., 2011. Robust stabilization of linear differential inclusions with affine uncertainty. *Circ. Syst. Signal Process.*, in press. [doi:10.1007/s00034-011-9268-y]

Lee, Y., Zak, S., 2004. Uniformly ultimately bounded fuzzy adaptive tracking controllers for uncertain systems. *IEEE Trans. Fuzzy Syst.*, **12**(6):797-811. [doi:10.1109/TFUZZ.2004.836087]

Li, Z., Chen, G., Shi, S., Han, C., 2003. Robust adaptive tracking control for a class of uncertain chaotic systems. *Phys. Lett. A*, **310**(1):40-43. [doi:10.1016/S0375-9601(03)00115-4]

Liu, L., Han, Z., Cai, X., Huang, J., 2010. Robust stabilization of linear differential inclusion system with time delay. *Math. Comput. Simul.*, **80**(5):951-958. [doi:10.1016/j.matcom.2009.10.006]

Pogromsky, A., Santoboni, G., Nijmeijer, H., 2003. An ultimate bound on the trajectories of the Lorenz system and its applications. *Nonlinearity*, **16**(5):1597-1605. [doi:10.1088/0951-7715/16/5/303]

Qu, Z., Dawson, D., 1995. Robust Tracking Control of Robot Manipulators. Piscataway, NJ.

Slotine, J., Sastry, S., 1983. Tracking control of nonlinear systems using sliding surfaces, with application to robot manipulators. *Int. J. Control*, **38**(2):465-492. [doi:10.1080/00207178308933088]

Smirnov, G., 2002. Introduction to the Theory of Differential Inclusions. SIAM, Philadelphia.

Sun, Y., 2008. Composite tracking control for a class of uncertain nonlinear control systems with uncertain deadzone nonlinearities. *Chaos Sol. Fract.*, **35**(2):383-389. [doi:10.1016/j.chaos.2006.05.034]

Wu, H., Shi, P., 2010. Adaptive variable structure state estimation for uncertain systems with persistently bounded disturbances. *Int. J. Robust Nonlinear Control*, **20**(17):2003-2015. [doi:10.1002/rnc.1567]