Journal of Zhejiang University-SCIENCE C (Computers & Electronics)
ISSN 1869-1951 (Print); ISSN 1869-196X (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Degree elevation of unified and extended spline curves*

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Received Mar. 6, 2014; Revision accepted Aug. 25, 2014; Crosschecked Nov. 13, 2014

Abstract: Unified and extended splines (UE-splines), which unify and extend polynomial, trigonometric, and hyperbolic B-splines, inherit most properties of B-splines and have some advantages over B-splines. The interest of this paper is the degree elevation algorithm of UE-spline curves and its geometric meaning. Our main idea is to elevate the degree of UE-spline curves one knot interval by one knot interval. First, we construct a new class of basis functions, called bi-order UE-spline basis functions which are defined by the integral definition of splines. Then some important properties of bi-order UE-splines are given, especially for the transformation formulae of the basis functions before and after inserting a knot into the knot vector. Finally, we prove that the degree elevation of UE-spline curves can be interpreted as a process of corner cutting on the control polygons, just as in the manner of B-splines. This degree elevation algorithm possesses strong geometric intuition.

Key words: Degree elevation, Unified and extended splines (UE-splines), Bi-order UE-splines, Corner cutting, Geometric explanation
doi:10.1631/jzus.C1400076 **Document code:** A **CLC number:** TP391.7

1 Introduction

B-spline is very popular in generating and representing curves and surfaces due to its properties, such as local support and shape preservation. However, B-splines do suffer from some shortcomings. For example, many transcendental curves which play important roles in engineering cannot be represented exactly by polynomial B-splines. Though the non-uniform rational B-spline (NURBS) can represent some analysis curves exactly, its derivative and integral are very complex. To avoid these flaws of (rational) B-splines, many new splines are constructed. The original C-B-splines which are generated over the space spanned by $\{\sin t, \cos t, t, 1|0 \le t \le \alpha\}$ were introduced by Zhang (1996; 1997). Wang *et al.* (2004) proposed non-uniform algebraic-trigonometric B-splines (NUAT B-splines) generated over the space spanned by $\{1, t, ..., t^{k-3}, \sin t, \cos t\}$ $(k \ge 3)$. The NUAT B-splines, which can represent the cycloid and the helix exactly, not only share the same properties with polynomial B-splines but also preserve the optimal shape (Min and Wang, 2004). Li and Wang (2005) and Lü *et al.* (2002) constructed the H-Bézier curves and the non-uniform hyperbolic polynomial B-splines (NUAH B-splines) in the space spanned by $\{1, t, ..., t^{k-3}, \sinh t, \cosh t\}$ $(k \ge 3)$. This kind of curve can exactly represent the catenary. Recently, Wang and Fang (2008) proposed the unified and extended (UE) splines over the space spanned by $\{1, t, ..., t^{k-3}, \sin(\omega t), \cos(\omega t)\}$ which unify all splines listed above by introducing a frequency sequence $\{\omega_i = \sqrt{\alpha_i}\}_{-\infty}^{+\infty}$. For all ω_i:

1. If $\omega_i \equiv 0$, the UE-splines reduce to B-splines.
2. If $\alpha_i > 0$, the UE-splines turn to be trigonometric B-splines.
3. If $\alpha_i < 0$, the UE-splines are hyperbolic B-splines.

UE-splines not only unify the above mentioned

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splines but also include much more forms (Wang and Fang, 2008). A UE-spline can include a hyperbolic segment, a segment of usual B-splines, etc. in a uniform spline curve without using several piecewise curves. Furthermore, UE-splines inherit most of the outstanding properties of usual B-splines.

It is well known that the degree elevation of Bézier curves is a corner cutting process. However, the degree elevation of B-spline curves, which has been studied by many researchers (Forrest, 1972; Cohen *et al.*, 1980; 1985; Prautzsch, 1984; Barry and Goldman, 1988), is somewhat complex and is not interpreted as a corner cutting process until Wang and Deng (2007) proposed a new degree elevation algorithm of B-spline curves and pointed out that the degree elevation also can be interpreted as corner cutting. Zhu *et al.* (2010) and Zhang and Wang (2008) gave the degree elevation algorithm of C-B-splines and algebraic hyperbolic B-splines, which also can be interpreted as corner cutting. All of these degree elevation algorithms use a new kind of spline called bi-order splines which put strong geometric meaning on the process of degree elevation. These methods inspired the degree elevation algorithm of UE-splines and revealed its connection to corner cutting. In this paper, we explore the degree elevation algorithm of UE-splines and show that the degree elevation algorithm of UE-splines can also be interpreted as corner cutting.

2 UE-spline and its degree elevation

The UE-splines, whose basis functions are constructed based on the integral definition of splines, provide a unified form over a common space for those existing splines, such as B-splines, C-B-splines, and H-B-splines.

2.1 UE-spline basis functions

Definition 1 Let $\boldsymbol{T}$ be a given knot sequence $\{t_i\}_{-\infty}^{+\infty}$ with $t_i \le t_{i+1}$ and $\boldsymbol{W}$ be a given frequency sequence $\{\omega_i = \sqrt{\alpha_i}\}_{-\infty}^{+\infty}$, where $\alpha_i \in \mathbb{R}$ and $\alpha_i \le \min_{j=i,i+1}(\pi/(t_{j+1}-t_j))^2$. A UE-spline of order k is defined over the space spanned by $\{1, t, ..., t^{k-3}, \sin(\omega t), \cos(\omega t)\}$ in which k is an arbitrary integer larger than or equal to 3. A set of basis functions $N_{i,k}(t)$ of the space is defined as follows (Wang and Fang, 2008):

$$N_{i,2}(t) = \begin{cases} \dfrac{\sin(\omega_i(t-t_i))}{\sin(\omega_i(t_{i+1}-t_i))}, & t_i < t \le t_{i+1}, \\ \dfrac{\sin(\omega_{i+1}(t_{i+2}-t))}{\sin(\omega_{i+1}(t_{i+2}-t_{i+1}))}, & t_{i+1} < t \le t_{i+2}, \\ 0, & \text{otherwise}, \end{cases}$$

in which if $\omega_i = 0$, the value of the function is evaluated by the limit as ω_i approaches 0.

For $k \ge 3$, $N_{i,k}(t)$ is defined recursively by

$$N_{i,k}(t) = \int_{-\infty}^{t} \left(\frac{N_{i,k-1}(s)}{\sigma_{i,k-1}} - \frac{N_{i+1,k-1}(s)}{\sigma_{i+1,k-1}} \right) \mathrm{d}s, \tag{1}$$

where

$$\sigma_{i,k} = \int_{-\infty}^{+\infty} N_{i,k}(t)\mathrm{d}t.$$

This is a UE-spline basis with a simple knot sequence. If there are multiple knots in the knot sequence, we set $0/0 = 0$ and $N_{i,k}/\sigma_{i,k} = 0$ when $N_{i,k}(t) = 0$. In addition, $\sigma_{i,k}$ and $N_{i,k}$ have to satisfy

$$\int_{-\infty}^{t} \frac{N_{i,k}(t)}{\sigma_{i,k}}\mathrm{d}t = \begin{cases} 1, & t \ge t_{i+k}, \\ 0, & t < t_{i+k}. \end{cases}$$

The UE-spline basis function $N_{i,k}(t)$ is C^{k-r_i-1} at the knot t_i if t_i has multiplicity r_i.

Like the basis functions of polynomial B-splines, the UE-spline basis functions $N_{i,k}(t)$ have properties such as positivity, local support, partition of unity, and linear independence.

It is apparent that each ω_i corresponds to a knot interval $[t_i, t_{i+1})$. This is important for our study.

2.2 UE-spline curve

Definition 2 A UE-spline curve of order k is defined as follows:

$$\boldsymbol{P}(t) = \sum_{i=0}^{m} N_{i,k}(t)\boldsymbol{P}_i,\ t_{k-1} \le t \le t_{m+1}, \tag{2}$$

where $\boldsymbol{P}_i$ $(i = 0, 1, ..., m)$ are the control points and $\{N_{i,k}(t)\}_{i=0}^{m}$ is the UE-spline basis of order k corresponding to the knot vector $\boldsymbol{T} = \{t_i\}_{i=0}^{m+k}$ and the frequency sequence $\boldsymbol{W} = \{\omega_i = \sqrt{\alpha_i}\}_{i=0}^{m+k-1}$ where $\alpha_i \in \mathbb{R}$ and $\alpha_i \le \min_{j=i,i+1}(\pi/(t_{j+1}-t_j))^2$ (Wang and Fang, 2008).

When all the frequencies $\omega_i \equiv 0, 1$, or -1, the UE-splines reduce to B-splines, NUAT B-splines, or

NUAH B-splines, respectively. UE-spline curves not only include the above-listed spline curves, but also unify the above splines in a single curve using a unified formula.

2.3 Degree elevation of UE-spline

Similar to B-splines, the UE-splines of order k in the space $\Gamma_k = \{1, t, ..., t^{k-3}, \sin\omega t, \cos\omega t\}$ can be represented exactly in the space $\Gamma_{k+1} = \{1, t, ..., t^{k-2}, \sin\omega t, \cos\omega t\}$. This means that any UE-spline of order k can be represented as a UE-spline of order $k+1$ without changing the shape of the original curves. Assume that the UE-spline $\boldsymbol{P}(t)$ of order k is defined the curve as given in Eq. (2). To elevate the degree of $\boldsymbol{P}(t)$, we should find the control points $\hat{\boldsymbol{P}}$, a frequency sequence $\hat{\boldsymbol{W}}$, and a knot vector $\hat{\boldsymbol{T}}$ such that

$$\boldsymbol{P}(t) = \hat{\boldsymbol{P}}(t) = \sum_{i=0}^{\hat{m}} \hat{N}_{i,k+1}(t)\hat{\boldsymbol{P}}_i, \tag{3}$$

where $\hat{\boldsymbol{P}}_i$ are new control points whose basis functions are $\hat{N}_{i,k+1}(t)$, and $\hat{m}$ is the number of the control points.

Since the UE-splines before and after degree elevation share the same continuity at the knots, the multiplicities of the knots in the knot vector increase along with the degree elevation. In fact, knot vectors can be classified as clamped or unclamped ones. We rewrite the knot vector $\boldsymbol{T}$ as

$$\boldsymbol{T} = \{\underbrace{t_1, ..., t_1}_{z_1}, ..., \underbrace{t_j, ..., t_j}_{z_j}, ..., \underbrace{t_n, ..., t_n}_{z_n}\},$$

where z_i denotes the multiplicity of the knot t_i and $t_i < t_j$ with $i < j$. Similarly, $\boldsymbol{W}$ could be clamped as

$$\boldsymbol{W} = \{\underbrace{\omega_1, ..., \omega_1}_{z_1}, ..., \underbrace{\omega_j, ..., \omega_j}_{z_j}, ..., \underbrace{\omega_n, ..., \omega_n}_{z_n - 1}\}.$$

Considering that the multiplicities of the knots in the knot vector increase along with the degree elevation, the multiplicities of the knots t_i increase by 1 in $\hat{\boldsymbol{P}}(t)$ compared with $\boldsymbol{P}(t)$. Therefore, the knot vector $\hat{\boldsymbol{T}}$ must be

$$\hat{\boldsymbol{T}} = \{\underbrace{t_1, ..., t_1}_{z_1+1}, ..., \underbrace{t_j, ..., t_j}_{z_j+1}, ..., \underbrace{t_n, ..., t_n}_{z_n+1}\}.$$

From the definition of the degree elevation for UE-spline curves in Eq. (3), we find that each knot interval $[t_i, t_{i+1})$ corresponds to an ω_i. So, the frequency sequence $\hat{\boldsymbol{W}}$ must take the form

$$\hat{\boldsymbol{W}} = \{\underbrace{\omega_1, ..., \omega_1}_{z_1+1}, ..., \underbrace{\omega_j, ..., \omega_j}_{z_j+1}, ..., \underbrace{\omega_n, ..., \omega_n}_{z_n}\}.$$

The degree elevation for UE-spline curves is transformed to find the new control points $\hat{\boldsymbol{P}}$ in Eq. (3) such that $\boldsymbol{P}(t) = \hat{\boldsymbol{P}}(t)$. The intuitive way is to obtain and solve the linear equations of $\hat{\boldsymbol{P}}_i$ through computing $N_{i,k}(t)$ and $\hat{N}_{i,k}(t)$ at some $\hat{m}+1$ suitable values. Then we can obtain the transformation formulae between the UE-spline basis functions of orders k and $k+1$. In this method, we insert all the knots t_i at the same time. Thus, it is difficult to find the geometric meaning.

In fact, to elevate the degree of UE-splines we can insert the required knots one by one, rather than inserting all at the same time. In each step, we elevate the degree in one knot interval by inserting only one knot. This means that the degree of basis functions will be elevated in the former knot intervals after inserting one knot. As a result, the degree of the curves will be elevated in the knot interval one by one. In effect, the splines are of different degrees in different knot intervals during the process. Thus, we introduce a new kind of UE-splines called bi-order UE-splines.

3 Bi-order UE-splines

Let

$$\begin{aligned}\boldsymbol{T}^j &= \{\underbrace{t_1, ..., t_1}_{z_1+1}, ..., \underbrace{t_j, ..., t_j}_{z_j+1}, \underbrace{t_{j+1}, ..., t_{j+1}}_{z_{j+1}}, ..., \underbrace{t_n, ..., t_n}_{z_n}\} \\ &= \{t_0^j, t_1^j, ..., t_{m_j}^j\},\end{aligned}$$

where $m_j = z_1 + z_2 + ... + z_n + j - 1$. We define bi-order UE-spline basis function $N_{i,k}^j(t)$ whose order is $k+1$ in $[t_1, t_{j+1})$ and k in $[t_{j+1}, t_n]$ on $\boldsymbol{T}^j$.

Note that $\boldsymbol{T}^j = \boldsymbol{T}^{j-1} + \{t_j\}$. We insert t_j into $\boldsymbol{T}^{j-1}$. The order of $N_{i,k}^{j-1}(t)$ is $k+1$ in $[t_1, t_j)$ and k in $[t_j, t_n]$. After the knot insertion, the order of $N_{i,k}^j(t)$ turns to be $k+1$ in $[t_j, t_{j+1})$. The order of the basis functions will be elevated one knot interval by one knot interval since the knots are inserted one by one. Meanwhile, we can obtain the transformation formulae between $N_{i,k}^j(t)$ and $N_{i,k}^{j-1}(t)$ through knot insertion. Based on the basis functions $N_{i,k}^j(t)$, we define bi-order UE-spline curves $\boldsymbol{P}^j(t)$ over $\boldsymbol{T}^j$.

According to the transformation formulae, the control points of $\boldsymbol{P}^j(t)$ can be obtained from $\boldsymbol{P}^{j-1}(t)$. This process can be interpreted as corner cutting on the control polygons of $\boldsymbol{P}^{j-1}(t)$.

3.1 Bi-order UE-spline basis functions

Definition 3 First, we give a non-decreasing knot vector $\boldsymbol{T}^j$ and a frequency sequence $\boldsymbol{W}^j$ as follows:

$$\boldsymbol{T}^j = \{\underbrace{t_1,...,t_1}_{z_1+1},...,\underbrace{t_j,...,t_j}_{z_j+1},\underbrace{t_{j+1},...,t_{j+1}}_{z_{j+1}},...,\underbrace{t_n,...,t_n}_{z_n}\} = \{t_0^j, t_1^j, ..., t_{m_j}^j\}$$

and

$$\boldsymbol{W}^j = \{\underbrace{\omega_1,...,\omega_1}_{z_1+1},...,\underbrace{\omega_j,...,\omega_j}_{z_j+1},\underbrace{\omega_{j+1},...,\omega_{j+1}}_{z_{j+1}},...,\underbrace{\omega_n,...,\omega_n}_{z_n-1}\} = \{\omega_0^j,\omega_1^j,...,\omega_{m_j-1}^j\},$$

where $m_j = z_1 + z_2 + ... + z_n + j - 1$.

Assume

$$A_{i,2}^j(t) = \begin{cases} \dfrac{\sin(\omega_i^j(t-t_i^j))}{\sin(\omega_i^j(t_{i+1}^j - t_i^j))}, & t_i^j < t \le t_{i+1}^j, \\ \dfrac{\sin(\omega_{i+1}^j(t_{i+2}^j - t))}{\sin(\omega_{i+1}^j(t_{i+2}^j - t_{i+1}^j))}, & t_{i+1}^j < t \le t_{i+2}^j, \\ 0, & \text{otherwise}, \end{cases}$$

and $D_{i,2}^j = \int_{-\infty}^{+\infty} A_{i,2}^j(t)\mathrm{d}t$.

When $k=2$, the bi-order UE-spline basis functions over $\boldsymbol{T}^j$ and $\boldsymbol{W}^j$ are defined as follows:

$$N_{i,2}^j(t) = \begin{cases} \displaystyle\int_{-\infty}^{t}\left(\frac{A_{i,2}^j(s)}{D_{i,2}^j} - \frac{A_{i+1,2}^j(s)}{D_{i+1,2}^j}\right)\mathrm{d}s, \ 0 \le i < s_j - 1, \\ \dfrac{\cos\left(\omega_{i+1}^j(t_{i+2}^j - t)\right) - 2 + \cos\left(\omega_{i+1}^j(t - t_{i+1}^j)\right)}{1 - \cos\left(\omega_{i+1}^j(t_{i+2}^j - t_{i+1}^j)\right)} + 1, \\ \qquad t_i^j \le t < t_{i+2}^j \text{ and } i = s_j - 1, \\ \dfrac{1 - \cos\left(\omega_i^j(t - t_i^j)\right)}{1 - \cos\left(\omega_i^j(t_{i+1}^j - t_i^j)\right)}, \ t_i^j \le t < t_{i+1}^j \text{ and } i = s_j, \\ \dfrac{\sin\left(\omega_{i+1}^j(t_{i+2}^j - t)\right)}{\sin\left(\omega_{i+1}^j(t_{i+2}^j - t_{i+1}^j)\right)}, \ t_{i+1}^j \le t < t_{i+2}^j \text{ and } i = s_j, \\ A_{i,2}^j(t), \qquad i > s_j, \\ 0, \qquad \text{otherwise}, \end{cases}$$

where $s_j = z_1 + z_2 + ... + z_j + j - 1$.

For $k \ge 3$, $N_{i,k}(t)$ is defined recursively by

$$N_{i,k}^j(t) = \int_{-\infty}^{t}\left(\frac{N_{i,k-1}^j(s)}{\sigma_{i,k-1}^j} - \frac{N_{i+1,k-1}^j(s)}{\sigma_{i+1,k-1}^j}\right)\mathrm{d}s, \quad (4)$$

where $i = 0, 1, ..., m_j - k$ and

$$\sigma_{i,k}^j = \int_{-\infty}^{+\infty} N_{i,k}^j(t)\mathrm{d}t.$$

Here we set $0/0 = 0$. If $N_{i,k}^j(t) \equiv 0$, we define

$$\int_{-\infty}^{t} \frac{N_{i,k}^j(t)}{\sigma_{i,k}^j}\mathrm{d}t = \begin{cases} 1, & t \ge t_{i+k+1}^j (i \le s_j - k) \\ & \text{or } t \ge t_{i+k}^j (i > s_j - k), \\ 0, & t < t_{i+k+1}^j (i \le s_j - k) \\ & \text{or } t < t_{i+k}^j (i > s_j - k). \end{cases}$$

From the definition above, we can find that when $0 \le i \le s_j - 1$ the order of $N_{i,2}^j(t)$ is 3, and when $i > s_j$ the order of $N_{i,2}^j(t)$ is 2. When $i = s_j$, $N_{i,2}^j(t)$ is a bi-order basis function (Fig. 1).

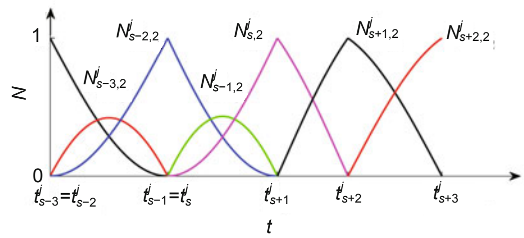

Fig. 1 The initial basis functions $\{N_{i,2}^j\}$, where $s = s_j$

Thus, $N_{i,k}^j(t)$ $(i = 0, 1, ..., m_j - k)$ are bi-order UE-spline basis functions for the orders are $k+1$ in $[t_1, t_{j+1})$ and are k in $[t_{j+1}, t_n]$ on $\boldsymbol{T}^j$.

Since t_n is the last knot, the basis functions $\{N_{i,k}^{n-1}(t)\}$ on $\boldsymbol{T}^{n-1}$ and $\{N_{i,k}^n(t)\}$ on $\boldsymbol{T}^n$ are equal. It can be easily inferred that $\{N_{i,k}^0(t)\}$ and $\{N_{i,k}^{n-1}(t)\}$ are UE-spline basis functions of order k and $k+1$, respectively.

3.2 Transformation formulae of bi-order UE-spline basis functions

It is easy to find that $\boldsymbol{T}^0 = \boldsymbol{T}$ and $\boldsymbol{T}^n = \hat{\boldsymbol{T}}$. Then we have the following theorem:

Theorem 1 Assume that $\{N_{i,k}(t)\}$ and $\{\hat{N}_{i,k+1}(t)\}$ are UE-spline basis functions defined over knot vectors $\boldsymbol{T}$ and $\hat{\boldsymbol{T}}$, respectively. $\{N_{i,k}^0(t)\}$ and $\{N_{i,k}^{n-1}(t)\}$ are the bi-order UE-spline basis functions defined on $\boldsymbol{T}^0 = \boldsymbol{T}$ and $\boldsymbol{T}^{n-1}$, respectively. Then we can obtain $N_{i,k}(t) = N_{i,k}^0(t)$ and $\hat{N}_{i,k+1}(t) = N_{i,k}^{n-1}(t)$.

Proof When $k=2$, $N_{i,k}(t) = N_{i,k}^0(t)$ for $\boldsymbol{T}^0 = \boldsymbol{T}$.

Assume that $N_{i,k}(t) = N_{i,k}^0(t)$ when $k>1$. Then we have $\sigma_{i,k}(t) = \sigma_{i,k}^0(t)$ and $N_{i,k}(t) = N_{i,k}^0(t)$. Thus,

$$N_{i,k+1}^0(t) = \int_{-\infty}^{t} \left(\frac{N_{i,k}^0(t)}{\sigma_{i,k}^0} - \frac{N_{i+1,k}^0(t)}{\sigma_{i+1,k}^0} \right) \mathrm{d}t = N_{i,k+1}(t).$$

It is apparent that $\hat{N}_{i,3}(t) = N_{i,2}^{n-1}(t)$ over $\hat{\boldsymbol{T}}$. Similarly we can obtain $\hat{N}_{i,k+1}(t) = N_{i,k}^{n-1}(t)$ by mathematical induction.

In fact, $\{N_{i,k}^0(t)\}$ and $\{N_{i,k}^{n-1}(t)\}$ are UE-spline basis functions of orders k and $k+1$, respectively. However, we also call them bi-order UE-spline basis functions.

Relationships between $\{N_{i,2}^{j-1}(t)\}$ and $\{N_{i,2}^{j}(t)\}$ are (Fig. 2)

$$N_{i,2}^{j-1}(t) = \begin{cases} N_{i,2}^j(t), & i<s_j-2, \\ N_{i,2}^j(t) + \dfrac{N_{i+1,2}^j(t)}{2\cos^2\left(\omega_{i+2}^j(t_{i+3}^j - t_{i+2}^j)/2\right)}, & i=s_j-2, \\ \dfrac{N_{i,2}^j(t)}{2\cos^2\left(\omega_{i+1}^j(t_{i+2}^j - t_{i+1}^j)/2\right)} + N_{i+1,2}^j(t), & i=s_j-1, \\ N_{i+1,2}^j(t), & i\geq s_j. \end{cases}$$

When $k=3$, $\{N_{i,k}^{j-1}(t)\}$ can be represented as the linear combination of $\{N_{i,k}^j(t)\}$:

$$N_{i,3}^{j-1}(t) = (1-\alpha_{i,3}^j)N_{i,3}^j(t) + \alpha_{i+1,3}^j N_{i+1,3}^j(t),$$

where $\alpha_{i,k}^j$ satisfies

$$\alpha_{i,3}^j = \begin{cases} 0, & i \leq s_j-3, \\ 1-\sigma_{i,2}^j/\sigma_{i,2}^{j-1}, & i=s_j-2, \\ \sigma_{i+1,2}^j/\sigma_{i,2}^{j-1}, & i=s_j-1, \\ 1, & i\geq s_j. \end{cases}$$

Knowing that we can obtain $\hat{N}_{i,k+1}(t) = N_{i,k}^{n-1}(t)$ by inserting knots one by one on $\boldsymbol{T}$, we deduce the transformation formulae of bi-order UE-spline basis functions $\{N_{i,k}^{j-1}(t)\}$ and $\{N_{i,k}^j(t)\}$.

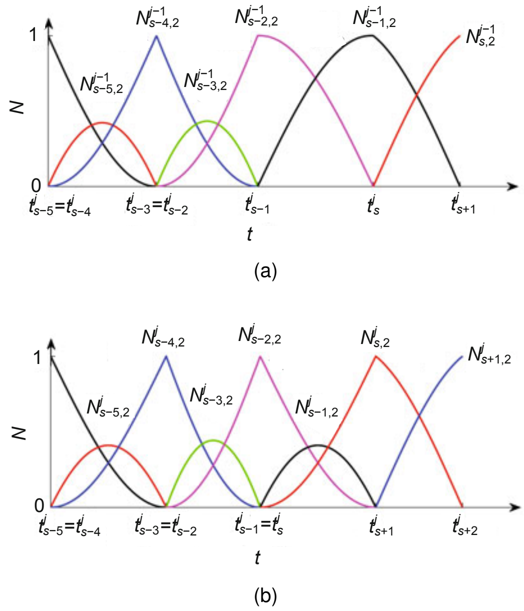

Fig. 2 The basis functions $\{N_{i,2}^{j-1}\}$ (a) and $\{N_{i,2}^j\}$ (b), where $s=s_j$

Theorem 2 For the bi-order UE-spline basis functions $\{N_{i,k}^{j-1}(t)\}$ and $\{N_{i,k}^{j-1}(t)\}$ defined over $\boldsymbol{T}^{j-1}$ and $\boldsymbol{T}^j$ by Eq. (4), we have

$$N_{i,k}^{j-1}(t) = \begin{cases} N_{i,k}^j(t), & i<s_j-k, \\ (1-\alpha_{i,k}^j)N_{i,k}^j(t) + \alpha_{i+1,k}^j N_{i+1,k}^j(t), & s_j-k\leq i, \text{ and } i\leq s_j-1, \\ N_{i+1,k}^j(t), & i\geq s_j, \end{cases} \tag{5}$$

where $\alpha_{i,k}^j$ satisfies

$$\alpha_{i,h+1}^j = \begin{cases} 0, & i\leq s_j-h-1, \\ \alpha_{i+1,h}^j \sigma_{i+1,h}^j/\sigma_{i,h}^{j-1}, & s_j-h\leq i\leq s_j-1, \\ 1, & i\geq s_j, \end{cases}$$

when $h\geq 3$ and

$$\alpha_{i,3}^j = \begin{cases} 0, & i \leq s_j-3, \\ 1-\sigma_{i,2}^j/\sigma_{i,2}^{j-1}, & i=s_j-2, \\ \sigma_{i+1,2}^j/\sigma_{i,2}^{j-1}, & i=s_j-1, \\ 1, & i\geq s_j. \end{cases}$$

Proof It is trivial that Theorem 2 holds for $k=3$.

Assume that it holds for $k\geq 3$. Since $N_{i,k}^{j-1}(t) = (1-\alpha_{i,k}^j)N_{i,k}^j(t) + \alpha_{i+1,k}^j N_{i+1,k}^j(t)$, we have $\sigma_{i,k}^{j-1} = (1-\alpha_{i,k}^j)\sigma_{i,k}^j + \alpha_{i+1,k}^j \sigma_{i+1,k}^j$.

Considering the definitions of $N_{i,k}^{j-1}(t)$ and $N_{i,k}^{j}(t)$ we have

$$
\begin{aligned}
N_{i,k+1}^{j-1}(t) &= \int_{-\infty}^{t} \left(\frac{N_{i,k}^{j-1}(t)}{\sigma_{i,k}^{j-1}} - \frac{N_{i+1,k}^{j-1}(t)}{\sigma_{i+1,k}^{j-1}} \right) \mathrm{d}t \\
&= \int_{-\infty}^{t} \frac{(1-\alpha_{i,k}^{j})N_{i,k}^{j}(t) + \alpha_{i+1,k}^{j} N_{i+1,k}^{j}(t)}{(1-\alpha_{i,k}^{j})\sigma_{i,k}^{j} + \alpha_{i+1,k}^{j}\sigma_{i+1,k}^{j}} \mathrm{d}t \\
&\quad - \int_{-\infty}^{t} \frac{(1-\alpha_{i+1,k}^{j})N_{i+1,k}^{j}(t) + \alpha_{i+2,k}^{j} N_{i+2,k}^{j}(t)}{(1-\alpha_{i+1,k}^{j})\sigma_{i+1,k}^{j} + \alpha_{i+2,k}^{j}\sigma_{i+2,k}^{j}} \mathrm{d}t.
\end{aligned}
$$

For

$$
\begin{aligned}
&\frac{\alpha_{i+1,k}^{j} N_{i+1,k}^{j}(t)}{(1-\alpha_{i,k}^{j})\sigma_{i,k}^{j} + \alpha_{i+1,k}^{j}\sigma_{i+1,k}^{j}} \\
&= \left(1 - \frac{(1-\alpha_{i,k}^{j})\sigma_{i,k}^{j}}{(1-\alpha_{i,k}^{j})\sigma_{i,k}^{j} + \alpha_{i+1,k}^{j}\sigma_{i+1,k}^{j}} \right) \frac{N_{i+1,k}^{j}(t)}{\sigma_{i+1,k}^{j}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{(1-\alpha_{i+1,k}^{j}) N_{i+1,k}^{j}(t)}{(1-\alpha_{i+1,k}^{j})\sigma_{i+1,k}^{j} + \alpha_{i+2,k}^{j}\sigma_{i+2,k}^{j}} \\
&= \left(1 - \frac{\alpha_{i+2,k}^{j}\sigma_{i+2,k}^{j}}{(1-\alpha_{i+1,k}^{j})\sigma_{i+1,k}^{j} + \alpha_{i+2,k}^{j}\sigma_{i+2,k}^{j}} \right) \frac{N_{i+1,k}^{j}(t)}{\sigma_{i+1,k}^{j}},
\end{aligned}
$$

$$
\begin{aligned}
N_{i,k+1}^{j-1}(t) =& \frac{(1-\alpha_{i,k}^{j})\sigma_{i,k}^{j}}{(1-\alpha_{i,k}^{j})\sigma_{i,k}^{j} + \alpha_{i+1,k}^{j}\sigma_{i+1,k}^{j}} \\
&\cdot \int_{-\infty}^{t} \left(\frac{N_{i,k}^{j}(t)}{\sigma_{i,k}^{j}} - \frac{N_{i+1,k}^{j}(t)}{\sigma_{i+1,k}^{j}} \right) \mathrm{d}t \\
&+ \frac{\alpha_{i+2,k}^{j}\sigma_{i+2,k}^{j}}{(1-\alpha_{i+1,k}^{j})\sigma_{i+1,k}^{j} + \alpha_{i+2,k}^{j}\sigma_{i+2,k}^{j}} \\
&\cdot \int_{-\infty}^{t} \left(\frac{N_{i+1,k}^{j}(t)}{\sigma_{i+1,k}^{j}} - \frac{N_{i+2,k}^{j}(t)}{\sigma_{i+2,k}^{j}} \right) \mathrm{d}t \\
=& \left(1 - \alpha_{i+1,k}^{j} \frac{\sigma_{i+1,k}^{j}}{\sigma_{i,k}^{j-1}} \right) N_{i,k+1}^{j}(t) \\
&+ \alpha_{i+2,k}^{j} \frac{\sigma_{i+2,k}^{j}}{\sigma_{i+1,k}^{j-1}} N_{i+1,k+1}^{j}(t).
\end{aligned}
$$

Thus, $\alpha_{i,k+1}^{j} = \alpha_{i+1,k}^{j} \dfrac{\sigma_{i+1,k}^{j}}{\sigma_{i,k}^{j-1}}$.

This means that the theorem holds for $k+1$. By induction, Theorem 2 holds.

3.3 Properties of bi-order UE-spline basis functions

The bi-order UE-spline basis functions have good properties like normal UE-splines:

1. Positivity: $N_{i,k}^{j}(t) > 0$ for $t_i^j < t < t_{i+k}^j$ $(i > s_j - k)$ or $t_i^j < t < t_{i+k+1}^j$ $(i \leq s_j - k)$.

2. Partition of unity: $\sum_i N_{i,k}^{j}(t) \equiv 1$.

3. Linear independence: $\{N_{i,k}^{j}(t)\}$ are linearly independent of $\boldsymbol{T}^j$ if there is no zero function in $\{N_{i,k}^{j}(t)\}$.

In fact, we define bi-order UE-spline basis functions based on the properties above.

3.4 Bi-order UE-spline curves

Based on the definition of bi-order UE-spline basis functions, we construct the bi-order UE-spline curves.

Definition 4 A bi-order UE-spline of order k defined on $\boldsymbol{T}^j$ is defined as

$$
\boldsymbol{P}^j(t) = \sum_{i=0}^{m_j-k} N_{i,k}^{j}(t)\boldsymbol{P}_i^j, \tag{6}
$$

where $\{\boldsymbol{P}_i^j\}$ are the control points and $\{N_{i,k}^{j}(t)\}$ are bi-order UE-spline basis functions.

$\boldsymbol{P}^0(t)$ is the original UE-spline of order k defined on the knot vector $\boldsymbol{T}$ and frequency vector $\boldsymbol{W}$. $\boldsymbol{P}^{n-1}(t)$ is the UE-spline of order $k+1$ elevated from $\boldsymbol{P}(t)$ defined on the knot vector $\boldsymbol{T}^{n-1}$ and the frequency vector $\boldsymbol{W}^{n-1}$. Since the bi-order UE-spline basis functions $\{N_{i,k}^{j-1}(t)\}$ can be represented by $\{N_{i,k}^{j}(t)\}$, we can derive $\{\boldsymbol{P}_i^j\}$ from $\{\boldsymbol{P}_i^{j-1}\}$.

Theorem 3 If $\boldsymbol{P}^{j-1}(t)$ is a bi-order UE-spline curve defined over $\boldsymbol{T}^{j-1}$ while $\boldsymbol{P}^j(t)$ is a bi-order UE-spline curve defined over $\boldsymbol{T}^j$, their control points $\{\boldsymbol{P}_i^j\}$ and $\{\boldsymbol{P}_i^{j-1}\}$ satisfy

$$
\boldsymbol{P}_i^j = \begin{cases} \boldsymbol{P}_i^{j-1}, & i \leq s_j - k, \\ (1-\alpha_{i,k}^{j})\boldsymbol{P}_i^{j-1} + \alpha_{i,k}^{j}\boldsymbol{P}_{i-1}^{j-1}, & \\ \qquad s_j - k + 1 \leq i \text{ and } i \leq s_j - 1, & \\ \boldsymbol{P}_{i-1}^{j-1}, & i \geq s_j, \end{cases} \tag{7}
$$

where $\alpha_{i,k}^{j}$ is defined in Eq. (5).

Proof Based on the fact that the knot insertion does not change the shape of the curve, we obtain $\boldsymbol{P}^j(t)$ by inserting a knot t_j into knot vector $\boldsymbol{T}^{j-1}$.

According to $N_{i,k}^{j-1}(t) = (1-\alpha_{i,k}^{j})N_{i,k}^{j}(t) + \alpha_{i+1,k}^{j}N_{i+1,k}^{j}(t)$, we have

$$
\boldsymbol{P}_i^j = (1-\alpha_{i,k}^{j})\boldsymbol{P}_i^{j-1} + \alpha_{i,k}^{j}\boldsymbol{P}_{i-1}^{j-1}.
$$

This completes the proof.

According to Theorem 3, we can obtain $\{\boldsymbol{P}_i^j\}$ by corner cutting on control points $\{\boldsymbol{P}_i^{j-1}\}$. Applying this method one knot interval by one knot interval, the order of UE-splines will be elevated from order k to $k+1$ using a corner cutting process.

Theorem 4 The degree elevation of UE-spline curves can be interpreted as a corner cutting process for control polygons.

Proof Assume that a UE-spline of order k on $\boldsymbol{T}=\{t_0^0, ..., t_m^0, ..., t_{m+k}^0\}$ is defined as

$$\boldsymbol{P}(t)=\sum_{i=0}^{m} N_{i,k}(t)\boldsymbol{P}_i,\ t_{k-1}^0 \le t \le t_{m+1}^0. \quad (8)$$

Rewrite the knot vectors $\boldsymbol{T}$ and $\boldsymbol{W}$ as

$$\boldsymbol{T}=\{\underbrace{t_1,...,t_1}_{z_1},...,\underbrace{t_a,...,t_a}_{z_a},...,\underbrace{t_b,...,t_b}_{z_b},...,\underbrace{t_n,...,t_n}_{z_n}\}$$

and

$$\boldsymbol{W}=\{\underbrace{\omega_1,...,\omega_1}_{z_1},...,\underbrace{\omega_a,...,\omega_a}_{z_a},...,\underbrace{\omega_b,...,\omega_b}_{z_b},..., \underbrace{\omega_n,...,\omega_n}_{z_n-1}\},$$

where $t_{k-1}^0=t_a$, $t_{m+1}^0=t_b$, and $t_i<t_j$ when $i<j$. Inserting all the diverse knots t_i in the knot vector $\boldsymbol{T}$, we can find that the order of the basis functions defined on $\boldsymbol{T}$ can be elevated as $k+1$. There are $n-1$ steps to obtain the degree elevation of basis functions.

Thus, we can obtain the degree elevation process through the following formula:

$$\begin{aligned}\boldsymbol{P}(t)&=\sum_{i=0}^{m} N_{i,k}(t)\boldsymbol{P}_i=\sum_{i=1}^{m+1} N_{i,k}^1(t)\boldsymbol{P}_i^1=...\\&=\sum_{i=a-1}^{m+a-1} N_{i,k}^{a-1}(t)\boldsymbol{P}_i^{a-1}=\sum_{i=a-1}^{m+j} N_{i,k}^{j}(t)\boldsymbol{P}_i^{j}\\&=\sum_{i=a-1}^{m+b-1} N_{i,k}^{b-1}(t)\boldsymbol{P}_i^{b-1}=\sum_{i=a-1}^{m+b-1} N_{i,k}^{n-1}(t)\boldsymbol{P}_i^{n-1},\end{aligned}$$

where $t_a=t_{k-1}^0 \le t \le t_{m+1}^0=t_b$ and the $\{\boldsymbol{P}_i^j\}$ can be inferred from Eq. (7). It means that the degree elevation of UE-splines can be interpreted as a corner cutting process for control polygons.

Without loss of generality, Fig. 3 shows the corner cutting process of the degree elevation of UE-spline defined as Eq. (8).

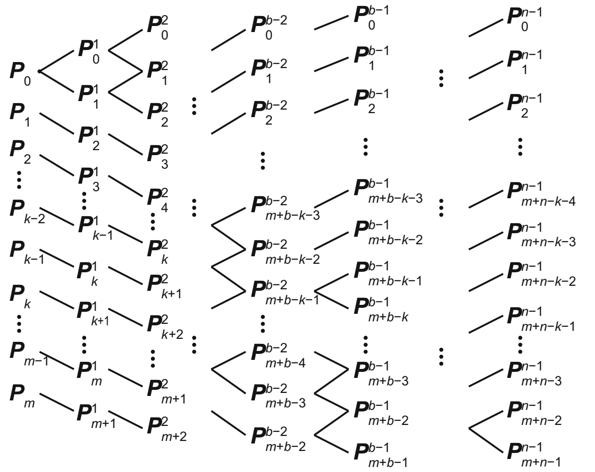

Fig. 3 The corner cutting process of the degree elevation of UE-splines

Fig. 4 presents the corner cutting process of the degree elevation of the UE-spline whose original order is four. The knot vector is $\boldsymbol{T}=\{t_0^0, t_1^0, ..., t_9^0\}$ where $t_0^0=t_1^0$, $t_4^0=t_5^0$, and other knots are simple knots.

4 Conclusions

We have shown how to elevate the degree of UE-spline curves in this paper. First, we propose the definition of bi-order UE-splines. Then we give the properties of bi-order UE-spline basis functions, especially for the transformation formulae between the bi-order UE-spline basis functions. Finally, the degree elevation of UE-spline curves is proved to be a corner cutting process.

After elevating a UE-spline curve from order k to $k+1$, we can obtain a new control polygon by corner cutting. When the elevated order goes to infinity, we wonder whether the new control polygon sequences converge to the initial UE-spline curve. If the conjecture can be proved, the degree elevation algorithm can be used to construct the splines by geometric methods. Future work includes proving that the UE-spline curve can be obtained by the degree elevation algorithm, and using the integral definitions of spline to investigate more properties and applications of the bi-order UE-splines to construct multi-degree splines (Wang and Deng, 2007; Shen and Wang, 2010; Cao and Wang, 2011; Shen *et al.*, 2013). In addition, iso-geometric analysis may be studied using UE-splines (Xu *et al.*, 2011; 2013).

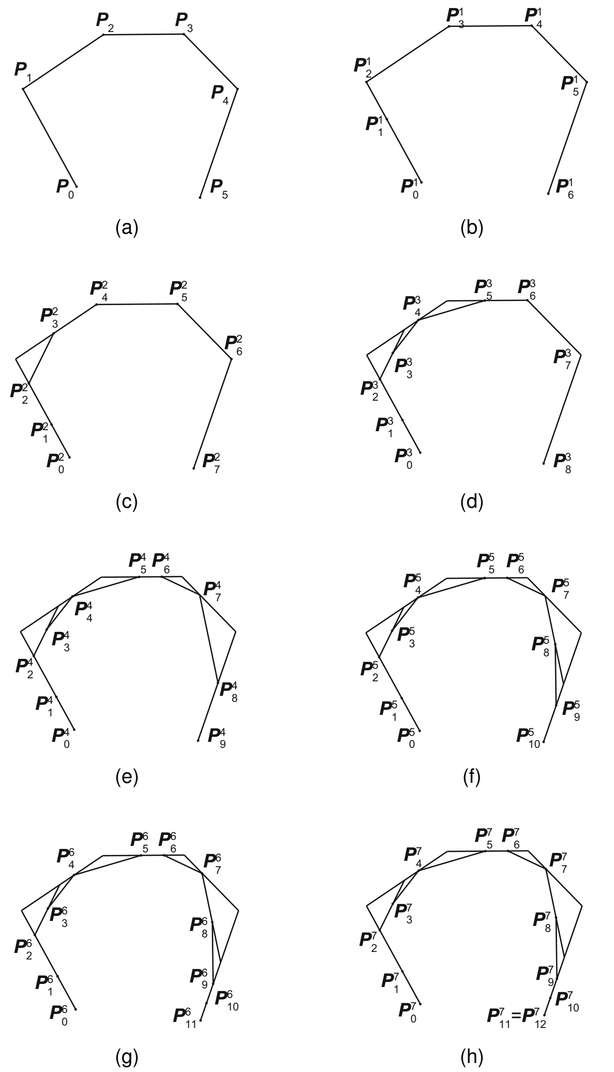

Fig. 4 A sample graph of degree elevation on control polygon (a) can be interpreted as corner cutting after inserting knots t_0^0 (b), t_2^0 (c), t_3^0 (d), t_4^0 (e), t_6^0 (f), t_7^0 (g), and t_8^0 (h) one by one

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